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Incompatibility of efficiency and strategyproofness in the random assignment setting with indifferences

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ABSTRACT

A fundamental resource allocation setting is the random assignment problem in which agents express preferences over objects that are then randomly allocated to the agents. In 2001, Bogomolnaia and Moulin presented the probabilistic serial (PS) mechanism that is an anonymous, Pareto optimal, and weak strategyproof mechanism when the preferences are considered with respect to stochastic dominance. The result holds when agents have strict preferences over individual objects. It has been an open problem whether there exists a mechanism that satisfies the same properties when agents may have indifference among the objects. We show that for this more general domain, there exists no extension of PS that is ex post efficient and weak strategyproof. The result is surprising because it does not even require additional symmetry or fairness conditions such as anonymity, neutrality, or equal treatment of equals. Our result further demonstrates that the lack of weak SD-strategyproofness of the extended PS mechanism of Katta and Sethuraman (2006) is not a design flaw but is due to an inherent incompatibility of efficiency and strategyproofness of PS in the full preference domain.

1. Introduction

In the assignment problem, agents express a complete and transitive set of preferences over objects and objects are divided among agents according to these preferences. The problem models one of the most fundamental settings in computer science and economics with numerous applications (Gärdenfors, 1973; Young, 1995; Svensson, 1994, 1999; Bouveret et al., 2010; Abraham et al., 2005; Mennle and Seuken, 2014; Nesterov, 2016). Depending on the application setting, the objects could be car-park spaces, dormitory rooms, kidneys, school seats, etc. The assignment problem is also referred to as *house allocation* (Abraham et al., 2005; Abdulkadiroğlu and Sönmez, 1999).

How do we identify desirable assignment rules for the problem? A natural way is to consider efficiency and strategyproofness with respect to the stochastic dominance (SD) relation. SD is a fundamental way to extend ordinal preferences over individual objects to random allocation because one allocation is SD preferred over another if it yields more utility with respect to all cardinal utility functions consistent with the ordinal preferences. We consider two requirements — ex post efficiency and weak SD-strategyproofness. SD-efficiency is Pareto optimality with respect to the SD relation. It is a weak property since it is only violated if the improving agent gets more utility with respect to all utility functions consistent with the ordinal preferences. Similarly, weak SD-strategyproofness is also a weak property since it is only violated if an agent can misreport their preference and get more utility with respect to all utility functions consistent with the ordinal preferences.

For the assignment problem, many existing papers assume that the agents have strict preferences over individual objects. Although strictness of preferences is a natural restriction, it cannot model preferences in which an agent is completely indifferent among some objects because they have the same quality that the agent cares about. The most famous mechanism for the problem is *random serial dictatorship (RSD)*: a permutation of the agents is chosen uniformly at random and then agents in the permutation are given the most preferred object that is still not allocated. Although RSD is strategyproof in the strongest sense and also ex post efficient (the outcome can be represented as convex combination of deterministic Pareto optimal outcomes), Bogomolnaia and Moulin (2001) showed that RSD is not SD-efficient even for strict preferences where SD-efficiency is a stronger property than ex post efficiency. They proposed a rival mechanism called *probabilistic serial (PS)* that is anonymous, neutral, SD-efficient, and weak SD-strategyproof.

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Under PS, agents 'eat' the most favored available object at an equal rate until all the objects are consumed. When a most preferred object is completely eaten, agents eat their next most preferred object that is not completely eaten. The fraction of object consumed by an agent is the probability of the agent getting that object.[1]

Bogomolnaia and Moulin (2001) left open the problem of generalizing PS for the full domain in which agents may express indifference between objects. The full domain is a generalization of strict preferences that can also capture other well-studied preferences restrictions such as dichotomous or trichotomous preferences in which agents puts the objects in two or three preference classes. Katta and Sethuraman (2006) proposed an extension of PS called EPS that is anonymous, neutral, and SD-efficient. However they showed that EPS is not weak SD-strategyproof. Since the work of Katta and Sethuraman (2006), it has been open whether there exists an anonymous, SD-efficient, and weak SD-strategyproof random assignment mechanism. Since PS is the only known random assignment rule that satisfies the three properties for strict preferences, it is tempting to think that a rule that satisfies anonymity, SD-efficiency, and weak SD-strategyproofness would be some other interesting extension of PS. A random assignment rule is *an extension of PS* if it returns the same assignment as PS for all preference profiles in which the preferences of each agent are strict. However we prove the following which is the main result of this paper.

For $n \geq 3$, there exists no extension of the probabilistic serial rule that is ex post efficient and weak SD-strategyproof.

The impossibility is startling because it does not even require any fairness conditions such as anonymity, neutrality, or even equal treatment of equals.

Related work. Brandl et al. (2016b) proved that for the randomized voting setting, there exists no way to extend random dictatorship to cater for indifferences that is still anonymous, neutral, SD-efficient, and weak SD-strategyproof. Our result is similar in spirit because it shows that extending a well-known rule to the case of indifferences results in a loss of the properties of the rule. Recently, Brandl et al. (2016a) have proved that for the randomized voting setting (Gibbard, 1978) with at least four agents, there exists no anonymous, neutral, SD-efficient, and weak SD-strategyproof rule settling a conjecture of Aziz et al. (2013b). Eberl (2016) presented a fully mechanized proof of the result of Brandl et al. (2016a). Note that the result (Brandl et al., 2016a) is for more than three agents, requires anonymity and neutrality as well as a stronger notion of efficiency. More importantly, it does not imply that there exists no such rule for a random assignment problem since the random assignment problem is more restricted and structured than voting. Brandl et al. (2016a) also discuss the open problem whether there exists an anonymous, neutral, SD-efficient, and weak SD-strategyproof random assignment rule.

2. Preliminaries

The model we consider is the *random assignment problem* which is a triple $(N, O, \succsim)$ where N is the set of n agents $\{1, \dots, n\}$, $O = \{o_1, \dots, o_n\}$ is the set of objects, and $\succsim = (\succsim_1, \dots, \succsim_n)$ is a preference profile specified by a tuple of complete and transitive preference relations $\succsim_i$ of agent i over objects in O. A good reference for this setting is (Bogomolnaia and Moulin, 2001). We will denote by $\mathcal{R}(O)$ the set of all complete and transitive relations over the set of objects O. Note that we require that the number of agents is equal to the number of objects but we do not require that agents have strict preferences over objects.

[1] A formal specification of PS is presented in Bogomolnaia and Moulin (2001, p. 305).

A *random assignment* p is an $n \times n$ matrix $[p(i)(o_j)]_{1\leq i\leq n, 1\leq j\leq n}$ such that for all $i \in N$, and $o_j \in O$, $p(i)(o_j) \in [0, 1]$; $\sum_{i\in N} p(i)(o_j) = 1$ for all $j \in \{1, \dots, n\}$; and $\sum_{o_j\in O} p(i)(o_j) = 1$ for all $i \in N$. The value $p(i)(o_j)$ represents the probability of object o_j being allocated to agent i. Each row $p(i) = (p(i)(o_1), \dots, p(i)(o_n))$ represents the allocation of agent i. The set of columns correspond to probability vectors of the objects $o_1, \dots, o_n$. A random assignment is deterministic if $p(i)(o) \in \{0, 1\}$ for all $i \in N$ and $o \in O$.

A *random assignment rule* is a function that specifies for each preferences profile a random assignment. Two minimal fairness conditions for rules are *anonymity* and *neutrality*. A rule is anonymous if its outcome depends only on the preference profile and does not depend on the identity of the agents. A rule is neutral if its outcome depends only on the preference profile and does not depend on the identity of the objects. A rule satisfies equal treatment of equals if agents with identical preferences get identical allocations. Note that anonymity implies equal treatment of equals.

In order to reason about preferences over random allocations, we extend preferences over objects to preferences over random allocations (see e.g., Cho, 2015). One standard extension is first order *SD (stochastic dominance)*. Given two random assignments p and q, and a preference relation $\succsim_i$, it holds that $p(i) \succsim_i^{SD} q(i)$ i.e., an agent i *SD prefers* allocation $p(i)$ to allocation $q(i)$ if for all $o \in O$:

$$\sum_{o_j\in\{o_k : o_k \succsim_i o\}} p(i)(o_j) \geq \sum_{o_j\in\{o_k : o_k \succsim_i o\}} q(i)(o_j).$$

We say i strictly SD prefers p to q, denoted $p(i) \succ_i^{SD} q(i)$, if $p(i) \succsim_i^{SD} q(i)$ holds and $q(i) \succsim_i^{SD} p(i)$ does not hold. Assignments p and q are SD-equivalent for agent i, $p(i) \sim_i^{SD} q(i)$, if $p(i) \succsim_i^{SD} q(i)$ and $q(i) \succsim_i^{SD} p(i)$. Note that SD is incomplete with respect to allocations, hence, it can be the case that two allocations $p(i)$ and $q(i)$ are *incomparable*: $p(i) \not\succsim_i^{SD} q(i)$ and $q(i) \not\succsim_i^{SD} p(i)$.

Consider a preference profile $\succsim$. A random assignment p is *SD-efficient* under $\succsim$ if there exists no random assignment q such that $q(i) \succsim_i^{SD} p(i)$ for all $i \in N$ and $q(i) \succ_i^{SD} p(i)$ for some $i \in N$. A deterministic assignment is Pareto optimal under $\succsim$ if and only if it is *SD*-efficient under $\succsim$. A random assignment is *ex post efficient* under $\succsim$ if it can be represented as a probability distribution over the set of deterministic assignments that are each Pareto optimal under $\succsim$. It is known that SD-efficiency implies ex post efficiency (Katta and Sethuraman, 2006).

We say that a random assignment rule is ex post efficient if for each preferences profile $\succsim$, the random assignment it returns is ex post efficient under $\succsim$. We say that a random assignment rule is *SD*-efficient if for each preferences profile $\succsim$, the random assignment it returns is *SD*-efficient under $\succsim$.

A random assignment rule f is *SD-strategyproof* if $f(\succsim)(i) \succsim_i^{SD} f(\succsim_i', \succsim_{-i})(i)$ for all $\succsim_i' \in \mathcal{R}(O)$ and $\succsim_{-i} \in \mathcal{R}(O)^{n-1}$. A random assignment rule f is *weak SD-strategyproof* if $f(\succsim_i', \succsim_{-i})(i) \not\succ_i^{SD} f(\succsim)(i)$ for all $\succsim_i' \in \mathcal{R}(O)$ and $\succsim_{-i} \in \mathcal{R}(O)^{n-1}$. It is easy to see that *SD*-strategyproofness implies weak *SD*-strategyproofness (Bogomolnaia and Moulin, 2001).

RSD is a random assignment rule in which a permutation of agents is chosen uniformly at random and agents in the permutation successively take their most preferred available object (Abdulkadiroğlu and Sönmez, 1998; Bogomolnaia and Moulin, 2001; Aziz et al., 2013a).

3. Result

For strict preferences, PS is the only known random assignment rule that is anonymous, SD-efficient, and weak SD-strategyproof. EPS is the only known generalization of PS (Katta and Sethuraman, 2006) to the case of indifferences and it is anonymous and SD-efficient but not weak SD-strategyproof. In view of these facts, it is natural to wonder whether a rule that satisfies anonymity, SD-efficiency, and weak SD-strategyproofness would be an extension of PS. A random assignment rule is an *extension of PS* if it returns the same random assignment as PS for all preference profiles in which all preferences of all agents are strict. However we prove the following theorem which implies that there is no such extension: For $n \geq 3$, there exists no extension of the probabilistic serial rule that is ex post efficient and weak SD-strategyproof.

We first prove a simple characterization of weak SD-strategyproofness that is helpful.

Lemma 1. *A random assignment rule f is weak SD-strategyproof if and only if for any agent $i \in N$, any preference profile $\succsim$, and any preference $\succsim_i'$, one of the following two conditions holds.*

(i) $$f(\succsim)(i) \sim_i^{SD} f(\succsim_i', \succsim_{-i})(i)$$

(ii) *there exists some $o \in O$ such that*
$$\sum_{o' \succsim_i o} f(\succsim)(i)(o') > \sum_{o' \succsim_i o} f(\succsim_i', \succsim_{-i})(i)(o').$$

Proof. Assume that neither of the two conditions holds. In that case, $\sum_{o' \succsim_i o} f(\succsim)(i)(o') \leq \sum_{o' \succsim_i o} f(\succsim_i', \succsim_{-i})(i)(o')$ for each $o \in O$ and $\sum_{o' \succsim_i o} f(\succsim)(i)(o') < \sum_{o' \succsim_i o} f(\succsim_i', \succsim_{-i})(i)(o')$ for some $o \in O$. This means that $f(\succsim_i', \succsim_{-i})(i) \succ_i^{SD} f(\succsim)(i)$ which contradicts weak SD-strategyproofness of f.

We prove the other direction. Assume that one of the two conditions holds. Suppose the first condition holds, weak SD-strategyproofness is not violated because agent i is indifferent between the two outcomes. Now suppose the second condition holds, i is not indifferent between the two outcomes, but because for some o, $\sum_{o' \succsim_i o} f(\succsim)(i)(o')$ is strictly greater than $\sum_{o' \succsim_i o} f(\succsim_i', \succsim_{-i})(i)(o')$, the following holds: $f(\succsim_i', \succsim_{-i})(i) \not\succ_i^{SD} f(\succsim)(i)$. Hence f does not violate weak SD-strategyproofness when i's truthful preference is $\succsim_i$ and misreport is $\succsim_i'$. □

We now prove the central result of the paper.

Theorem 1. *For $n \geq 3$, there exists no extension of PS that is ex post efficient and weak SD-strategyproof.*

Proof. Let us consider a random assignment rule f that is an extension of PS that is ex post efficient, and weak SD-strategyproof for $n = 3$.

We focus on the following three preference profiles $\succsim$, $\succsim'$ and $\succsim''$. The first two involve only strict preferences.

$\succsim_1$: a, b, c
$\succsim_2$: a, c, b
$\succsim_3$: a, b, c

$\succsim_1'$: a, b, c
$\succsim_2'$: a, c, b
$\succsim_3'$: b, c, a

$\succsim_1''$: a, b, c
$\succsim_2''$: a, c, b
$\succsim_3''$: $\{a, b\}, c$

It can be ascertained that the PS outcomes of profiles $\succsim$ and $\succsim'$ are as follows:

$$PS(\succsim) = \begin{pmatrix} 1/3 & 1/2 & 1/6 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/2 & 1/6 \end{pmatrix}.$$

$$PS(\succsim') = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 0 & 3/4 & 1/4 \end{pmatrix}.$$

Since f is an extension of PS, it follows that $f(\succsim) = PS(\succsim)$ and $f(\succsim') = PS(\succsim')$. Let us assume that the outcome of $f(\succsim'')$ is as follows.

$$f(\succsim'') = \begin{pmatrix} C_{1a} & C_{1b} & C_{1c} \\ C_{2a} & C_{2b} & C_{2c} \\ C_{3a} & C_{3b} & C_{3c} \end{pmatrix}.$$

We first claim that ex post efficiency of f requires that $C_{3a} = 0$. Assume for contradiction that $C_{3a} > 0$. Then there must be some deterministic assignment p that is Pareto optimal under profile $\succsim''$ and gives object a to agent 3. Then some other agent i in $\{1, 2\}$ must get object b. But then p is not Pareto optimal because 3 and i can exchange a and b to get a Pareto improvement. We have established that $C_{3a} = 0$.

Since we have established that $C_{3a} = 0$, it follows that:

$$C_{3b} + C_{3c} = 1. \tag{1}$$

The conditions of weak SD-strategyproofness in Lemma 1 imply the following constraints for agent 3:

(i) $f(\succsim)(3) \not\succ_3''^{SD} f(\succsim'')(3)$:

(a) $C_{3a} + C_{3b} \geq 1/2 + 1/3 = 5/6$

Since $C_{3a} = 0$, it follows that

$$C_{3b} \geq 5/6. \tag{2}$$

(ii) $f(\succsim'')(3) \not\succ_3'^{SD} f(\succsim')(3)$

(a) $(3/4 = C_{3b}$ and $1/4 = C_{3c})$ or
(b) $3/4 > C_{3b}$ or
(c) $1 > C_{3b} + C_{3c}$

The conditions (a), (b), (c) are derived using the characterization of weak *SD*-strategyproofness in Lemma 1. Condition (a) corresponds to $f(\succsim'')(3) \sim_3'^{SD} f(\succsim')(3)$. The first condition (a) cannot hold because of (2). Conditions (b) and (c) correspond to the following condition of Lemma 1: there exists some $o \in O$ such that $\sum_{o' \succsim_i o} f(\succsim)(i)(o') > \sum_{o' \succsim_i o} f(\succsim_i', \succsim_{-i})(i)(o')$. The second condition (b) also cannot hold because of (2). The third condition cannot hold because of (1). This leads to a contradiction.

Hence for $n = 3$, if f is an extension of PS, and is SD-efficient, then f is not weak SD-strategyproof. For $n > 3$, we can adapt the same argument by adding agents and their corresponding objects. Say we add k more agents $a_1, \ldots, a_k$ and k more objects $o_1, \ldots, o_k$. In that case, for each profile, agents 1, 2, 3 prefer each objects in $\{a, b, c\}$ strictly more than each object in $\{o_1, \ldots, o_k\}$. Each a_i strictly prefers o_i to each other object. Then ex post efficiency requires that agents 1, 2, 3 will not get any object in $\{o_1, \ldots, o_k\}$ with non-zero probability and each agent a_i will get o_i with probability 1. □

In this paper, we proved that there exists no extension of the PS rule that is ex post efficient and weak SD-strategyproof. Previously, Katta and Sethuraman (2006) presented EPS which is a particular extension of PS and showed that it is not weak SD-strategyproof. Our result further demonstrates that the lack of weak SD-strategyproofness is not a design flaw of EPS but is due to an inherent incompatibility of efficiency and strategyproofness of PS in the full preference domain. Moreover, our result is not restricted to rules that employ eating after tie-breaking in case there are indifferences but to any rule that only coincides with PS over the strict preference profiles. It still remains to be settled whether there exists an anonymous, SD-efficient, and weak SD-strategyproof random assignment rule.

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