Range-Consistent Forbidden Regions of Allen's Relations

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Abstract. For all 8192 combinations of Allen's 13 relations between one task with origin o_i and fixed length ℓ_i and another task with origin o_j and fixed length ℓ_j, this paper shows how to systematically derive a formula $F(\underline{o_j}, \overline{o_j}, \ell_i, \ell_j)$, where $\underline{o_j}$ and $\overline{o_j}$ respectively denote the earliest and the latest origin of task j, evaluating to a set of integers which are infeasible for o_i for the given combination. Such forbidden regions allow maintaining range-consistency for an Allen constraint.

1 Introduction

More than 30 years ago Allen proposed 13 basic mutually exclusive relations [1] to exhaustively characterise the relative position of two tasks. By considering all potential disjunctions of these 13 basic relations one obtains 8192 general relations. While most of the work has been focussed on qualitative reasoning [8,5] with respect to these general relations, and more specifically on the identification and use of the table of transitive relations [11], or on logical combinators involving Allen constraints [4,10], no systematic study was done for explicitly characterising the set of infeasible/feasible values of task origin/length with respect to known consistencies. In the context of range consistency the contribution of this paper is to derive from the structure of basic Allen Relations the *exact formulae for the lower and upper bounds of the intervals of infeasible values* for the 8192 general relations and to synthesised a corresponding data base [2].

After recalling the definition of Basic Allen's relations, Section 2.1 gives the forbidden regions for these basic Allen's relation, Section 2.2 unveils a regular structure on the limits of those forbidden regions, and Section 2.3 shows how to systematically compute a compact normal form for the forbidden regions of all the 8192 general relations.

Definition 1 (Basic Allen's relations). *Given two tasks i, j respectively defined by their origin o_i, o_j and their length $\ell_i > 0$, $\ell_j > 0$, the following 13 basic Allen's relations systematically describe relationships between the two tasks, i.e. for two fixed tasks only one basic Allen's relation holds.*

- b : $o_i + \ell_i < o_j$
- m : $o_i + \ell_i = o_j$
- o : $o_i < o_j \wedge$ $o_i + \ell_i > o_j \wedge$ $o_i + \ell_i < o_j + \ell_j$
- s : $o_i = o_j \wedge$ $o_i + \ell_i < o_j + \ell_j$
- d : $o_j < o_i \wedge$ $o_i + \ell_i < o_j + \ell_j$
- f : $o_j < o_i \wedge$ $o_i + \ell_i = o_j + \ell_j$
- e : $o_i = o_j \wedge$ $o_i + \ell_i = o_j + \ell_j$

The basic relations bi, mi, oi, si, di *and* fi *are respectively derived from* b, m, o, s, d *and* f *by permuting task i and task j. The expression i r j denotes that the basic relation* r *holds between task i and task j.*

Definition 2 (Allen's constraint). *Given two tasks i, j respectively defined by their origin o_i, o_j and their length $\ell_i > 0$, $\ell_j > 0$, and a basic relation* r*, the* $\textsc{Allen}(\mathsf{r}, o_i, \ell_i, o_j, \ell_j)$ *constraint holds if and only if the condition i r j holds.*

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if r = b then
   propagate o_i + ℓ_i < o_j
else if r = m then
   propagate o_i + ℓ_i = o_j
else if ... then
   ...
end if
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Note that o_i, o_j, ℓ_i, ℓ_j are integer variables. Similarly, the basic relation r is an integer variable r, whose initial domain is included in $\{\mathsf{b}, \mathsf{bi}, \mathsf{m}, \mathsf{mi}, \mathsf{o}, \mathsf{oi}, \mathsf{s}, \mathsf{si}, \mathsf{d}, \mathsf{di}, \mathsf{f}, \mathsf{fi}, \mathsf{e}\}$. This constraint could be decomposed as on the right-hand side, but such a decomposition would propagate nothing until r has been fixed, whereas our formulae capture perfect constructive disjunction for all the 8192 general relations, e.g. for use in a range-consistency propagator.

2 Range Consistency

Given an integer variable o, $D(o)$, $\underline{o}$, $\overline{o}$ respectively denote the *set of values*, the *smallest value*, the *largest value* that can be assigned to o. The *range* of a variable o is the interval $[\underline{o}..\overline{o}]$ and is denoted by $R(o)$. A constraint CTR is *range consistent* (RC) [3] if and only if, when a variable o of CTR is assigned any value in its domain $D(o)$, there exist values in the ranges of all the other variables of CTR such that the constraint CTR holds.

2.1 Forbidden Regions Normal Form of Basic Allen's Relations

For each of the 13 basic Allen's relations column **RC** of Tab. 1 provides the corresponding normalised forbidden regions.

Lemma 1. *(a) The correct and complete forbidden region for $o_i + \ell < o_j$ is $o_i \notin [\overline{o_j} - \ell..+\infty)$. (b) The correct and complete forbidden region for $o_j + \ell < o_i$ is $o_i \notin (-\infty..\underline{o_j} + \ell]$.*

Table 1. Inconsistent values for RC for the 13 basic Allen's relations between two tasks i and j respectively defined by their origin o_i, o_j and their length ℓ_i and ℓ_j subject to Allen's relation $i \; \mathsf{r} \; j$ with $\mathsf{r} \in \{\mathsf{b}, \mathsf{bi}, \ldots, \mathsf{e}\}$ (for reasons of symmetry we only show the filtering of task i).

rel	**RC** parameter cases	inconsistent values
b		$o_i \notin [\overline{o_j} - \ell_i .. +\infty)$
bi		$o_i \notin (-\infty .. \underline{o_j} + \ell_j]$
m		$o_i \notin (-\infty .. \underline{o_j} - \ell_i - 1] \cup [\overline{o_j} - \ell_i + 1 .. +\infty)$
mi		$o_i \notin (-\infty .. \underline{o_j} + \ell_j - 1] \cup [\overline{o_j} + \ell_j + 1 .. +\infty)$
o	$\ell_i > 1 \wedge \ell_j > 1 \wedge \ell_i \leq \ell_j$:	$o_i \notin (-\infty .. \underline{o_j} - \ell_i] \cup [\overline{o_j} .. +\infty)$
	$\ell_i > 1 \wedge \ell_j > 1 \wedge \ell_i > \ell_j$:	$o_i \notin (-\infty .. \underline{o_j} - \ell_i] \cup [\overline{o_j} + \ell_j - \ell_i .. +\infty)$
	$\ell_i = 1 \vee \ell_j = 1$:	$o_i \notin (-\infty .. +\infty)$
oi	$\ell_i > 1 \wedge \ell_j > 1 \wedge \ell_i \leq \ell_j$:	$o_i \notin (-\infty .. \underline{o_j} + \ell_j - \ell_i] \cup [\overline{o_j} + \ell_j .. +\infty)$
	$\ell_i > 1 \wedge \ell_j > 1 \wedge \ell_i > \ell_j$:	$o_i \notin (-\infty .. \underline{o_j}] \cup [\overline{o_j} + \ell_j .. +\infty)$
	$\ell_i = 1 \vee \ell_j = 1$:	$o_i \notin (-\infty .. +\infty)$
s	$\ell_i < \ell_j$:	$o_i \notin (-\infty .. \underline{o_j} - 1] \cup [\overline{o_j} + 1 .. +\infty)$
	$\ell_i \geq \ell_j$:	$o_i \notin (-\infty .. +\infty)$
si	$\ell_j < \ell_i$:	$o_i \notin (-\infty .. \underline{o_j} - 1] \cup [\overline{o_j} + 1 .. +\infty)$
	$\ell_j \geq \ell_i$:	$o_i \notin (-\infty .. +\infty)$
d	$\ell_i + 1 < \ell_j$:	$o_i \notin (-\infty .. \underline{o_j}] \cup [\overline{o_j} + \ell_j - \ell_i .. +\infty)$
	$\ell_i + 1 \geq \ell_j$:	$o_i \notin (-\infty .. +\infty)$
di	$\ell_j + 1 < \ell_i$:	$o_i \notin (-\infty .. \underline{o_j} + \ell_j - \ell_i] \cup [\overline{o_j} .. +\infty)$
	$\ell_j + 1 \geq \ell_i$:	$o_i \notin (-\infty .. +\infty)$
f	$\ell_i < \ell_j$:	$o_i \notin (-\infty .. \underline{o_j} + \ell_j - \ell_i - 1] \cup [\overline{o_j} + \ell_j - \ell_i + 1 .. +\infty)$
	$\ell_i \geq \ell_j$:	$o_i \notin (-\infty .. +\infty)$
fi	$\ell_j < \ell_i$:	$o_i \notin (-\infty .. \underline{o_j} + \ell_j - \ell_i - 1] \cup [\overline{o_j} + \ell_j - \ell_i + 1 .. +\infty)$
	$\ell_j \geq \ell_i$:	$o_i \notin (-\infty .. +\infty)$
e	$\ell_j = \ell_i$:	$o_i \notin (-\infty .. \underline{o_j} - 1] \cup [\overline{o_j} + 1 .. +\infty)$
	$\ell_j \neq \ell_i$:	$o_i \notin (-\infty .. +\infty)$

Proof. (a) Given $o_i + \ell < o_j$ then clearly $o_i < \overline{o_j} - \ell$ and hence $o_i \notin [\overline{o_j} - \ell.. + \infty)$. Given $v < \overline{o_j} - \ell$ then $o_i = v, o_j = \overline{o_j}$ is a solution of the constraint. (b) Given $o_j + \ell < o_i$ then clearly $o_i > \underline{o_j} + \ell$ and hence $o_i \notin (-\infty..\underline{o_j} + \ell]$. Given $v < \underline{o_j} + \ell$ then $o_i = v, o_j = \underline{o_j}$ is a solution of the constraint. ◻

Lemma 2. *Given constraint $c \equiv c_1 \wedge c_2$ if $o_i \notin R_1$ is a correct forbidden region of c_1 and $o_i \notin R_2$ is a correct forbidden region of c_2 then $o_i \notin R_1 \cup R_2$ is a correct forbidden region of c.*

Proof. Since there can be no solution of c_1 with $o_i \in R_1$ and no solution of c_2 with $o_i \in R_2$ there can be no solution of c with $o_i \in R_1 \cup R_2$. ◻

Theorem 1. *Forbidden intervals of consecutive values shown in column **RC** of Table 1 are correct and complete.*

Proof. We proceed by cases and omit relations bi, mi, oi, si, di, fi for which the reasoning is analogous to b, m, o, s, d, f.

- **b** Follows from Lem. 1(a).
- **m** Correctness: given i meets j then $o_i + \ell_i = o_j$ thus $o_i + \ell_i \leq o_j \wedge o_i + \ell_i \geq o_j$ thus $o_i + (\ell_i - 1) < o_j \wedge o_j + (-\ell_i - 1) < o_i$. From Lem. 1 we have that $[\overline{o_j} - \ell_i + 1.. + \infty)$ and $(-\infty..\underline{o_j} - \ell_i - 1]$ are correct forbidden regions and by Lem. 2 correctness holds. Completeness: choose $v \in [\underline{o_j} - \ell_i..\overline{o_j} - \ell_i]$ then $o_i = v, o_j = v + \ell_i$ is a solution.
- **o** Correctness: given i overlaps with j we have $o_i < o_j \wedge o_i + \ell_i > o_j \wedge o_i + \ell_i < o_j + \ell_j$. Suppose $\ell_i = 1$ then this implies $o_i < o_j \wedge o_i + 1 > o_j$ contradiction, or suppose $\ell_j = 1$ then this implies $o_i + \ell_i > o_j \wedge o_i + \ell_i < o_j + 1$ contradiction hence $(-\infty.. + \infty)$ is a correct forbidden region. Lemma 1 gives us correct forbidden regions $(-\infty..\underline{o_j} - \ell_i]$, $[\overline{o_j} + \ell_j - \ell_i.. + \infty)$, $[\overline{o_j}.. + \infty)$. If $\ell_i \leq \ell_j$ this is equivalent to $(-\infty..\underline{o_j} - \ell_i] \cup [\overline{o_j}.. + \infty)$. If $\ell_i > \ell_j$ this is equivalent to $(-\infty..\underline{o_j} - \ell_i] \cup [\overline{o_j} + \ell_i - \ell_j.. + \infty)$. Completeness: when $\ell_i = 1$ or $\ell_j = 1$ then completeness follows from the contradiction. Choose $v \in [\underline{o_j} - \ell_i + 1..\overline{o_j} + \min(0, \ell_j - \ell_i) - 1]$ then $o_i = v, o_j = v + \ell_i - 1$ is a solution.
- **s** Correctness: If $\ell_i \geq \ell_j$ then the constraints are unsatisfiable and $(-\infty.. + \infty)$ is a correct forbidden region. Otherwise from s we have that $o_i < o_j + 1 \wedge o_i > o_j - 1 \wedge o_i + \ell_i < o_j + \ell_j$ and Lem. 1 gives us correct forbidden regions $[\overline{o_j} + 1.. + \infty)$, $(-\infty..\underline{o_j} - 1]$ and $[\overline{o_j} + \ell_j - \ell_i.. + \infty)$. If $\ell_i < \ell_j$ then this gives $(-\infty..\underline{o_j} - 1] \cup [\overline{o_j} + 1.. + \infty)$. Completeness: If $\ell_i \geq \ell_j$ then completeness follows from the unsatisfiability. Otherwise choose $v \in [\underline{o_j}..\overline{o_j}]$ then $o_i = v, o_j = v$ is a solution.
- **d** Correctness: If $\ell_i + 1 \geq \ell_j$ then $o_i + \ell_i \geq o_i + \ell_j - 1 \geq o_j + \ell_j$ but this contradicts $o_i + \ell_i < o_j + \ell_j$ hence $(-\infty.. + \infty)$ is a correct forbidden region. Otherwise Lem. 1 gives us correct forbidden regions $(-\infty..\underline{o_j}]$ and $[\overline{o_j} + \ell_j - \ell_i.. + \infty)$ whose union is the correct forbidden region. Completeness: The contradiction proves completeness when $\ell_i + 1 \geq \ell_j$. Otherwise choose $v \in [\underline{o_j} + 1..\overline{o_j} + \ell_j - \ell_i - 1]$ then $o_i = v, o_j = v - 1$ is a solution.
- **f** Correctness: Suppose $\ell_i \geq \ell_j$ then $o_j < o_i = o_j + \ell_j - \ell_i \leq o_j$, a contradiction, hence $(-\infty.. + \infty)$ is a correct forbidden region. Otherwise $o_j + \ell_j = o_i + \ell_i$ is equivalent to

$o_j+\ell_j-1 < o_i+\ell_i \wedge o_j+\ell_j+1 > o_i+\ell_i$. From these two inequalities and from $o_j < o_i$, Lem. 1 gives us correct forbidden regions $(-\infty..\underline{o_j}+\ell_j-\ell_i-1]$, $[\overline{o_j}+\ell_j-\ell_i+1..+\infty)$ and $(-\infty..\underline{o_j}]$. Since $\ell_i < \ell_j$ the correct union is $(-\infty..\underline{o_j}+\ell_j-\ell_i-1] \cup [\overline{o_j}+\ell_j-\ell_i+1..+\infty)$. Completeness: If $\ell_i \geq \ell_j$ then the contradiction gives the completeness. Otherwise choose $v \in [\underline{o_j}+\ell_j-\ell_i..\overline{o_j}+\ell_j-\ell_i]$ then $o_i = v, o_j = v+\ell_i-\ell_j$ is a solution.

e Correctness: Suppose $\ell_i \neq \ell_j$ then the constraints $o_i = o_j \wedge o_i+\ell_i = o_j+\ell_j$ contradict and $(-\infty..+\infty)$ is a correct forbidden region. When $\ell_i = \ell_j$ Lemma 1 gives us correct forbidden regions $(-\infty..\underline{o_j}-1]$, $[\overline{o_j}+1..+\infty)$ from both constraints, and their union is the correct answer. Completeness: If $\ell_i \neq \ell_j$ then the contradiction proves completeness, otherwise choose $v \in [\underline{o_j}..\overline{o_j}]$ then $o_i = v, o_j = v$ is a solution. □

2.2 Structure of the Normalised Forbidden Regions

All forbidden regions of the basic Allen's relations given in Section 2.1 consist of one or two intervals of the form $(-\infty..up]$, $[low..+\infty)$ or $(-\infty..+\infty)$. Indeed, only the forbidden regions for b and bi consist of a single (nonuniversal) forbidden region. In the following, we call *upper limit* (resp. *lower limit*) the terms *up* (resp. *low*). In the case of a single universal forbidden region, $up = +\infty$ and $low = -\infty$.

We show that all upper limits (resp. lower limits) can be totally ordered provided we know the relative order between the lengths ℓ_i and ℓ_j of the corresponding tasks. This is because all upper limits (resp. lower limits) correspond to linear expressions involving $+\underline{o_j}$ (resp. $+\overline{o_j}$). Fig. 1 illustrates this for the case $\ell_i < \ell_j$, where each limit is a node mentioning the associated formula, the basic Allen's relation(s) from which it is generated and the restriction on the parameters. We also show that we always have that the k^{th} upper limit is strictly less than the $k+1^{th}$ lower limit. This is because the k^{th} upper limit and the $k+1^{th}$ lower limit are issued from the same basic Allen's relation. Within Fig. 1 a solid arrow from a start node to an end node indicates that the limit attached to the start node is necessarily strictly less than (resp. strictly less by one than) the limit attached to the end node.

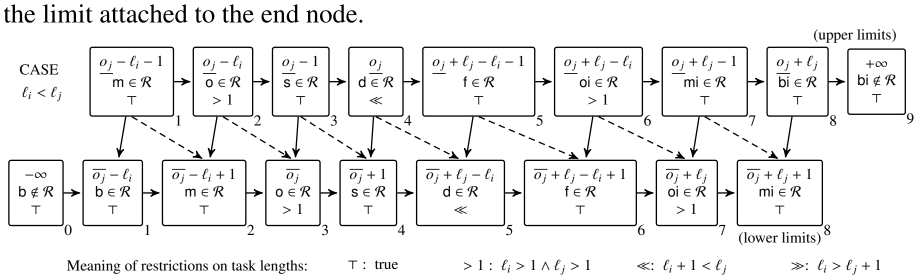

Fig. 1. Ordering the upper limits (resp. lower limits) of the forbidden regions of a general Allen relation $\mathcal{R}$ depending on the relative length of the two tasks i and j when $\ell_i < \ell_j$; a solid arrow from a limit x to a limit y represents an inequality of the form $x < y$, while a dashed arrow represents an inequality of the form $x+1 < y$. Upper (resp. lower) limits of each of the three cases are identified by a unique identifier located on the corresponding lower rightmost corner.

2.3 Normal Form for the Forbidden Regions

Given any general Allen's relation $\mathcal{R}$ we now show how to synthesise a normalised sequence of forbidden regions for this relation under the different cases regarding the relative sizes of the two tasks to which $\mathcal{R}$ applies (i.e., $\ell_i < \ell_j$, $\ell_i = \ell_j$, $\ell_i > \ell_j$). This will lead to a data base [2] of normalised forbidden regions for the 8192 general relations. A typical entry of that data base, for instance for relation {b, bi, d, di, e, f, fi, m, mi, si}, looks like:

$$\begin{cases} [\overline{o_j} - \ell_i + 1..\underline{o_j}] \cup [\overline{o_j} + \ell_j - \ell_i + 1..\underline{o_j} + \ell_j - 1] & \text{if } \ell_i < \ell_j \wedge \ell_i > 1 \\ [\overline{o_j}..\underline{o_j}] & \text{if } \ell_i = 1 \wedge \ell_j > 1 \\ \emptyset & \text{if } \ell_j = 1 \\ [\overline{o_j} - \ell_i + 1..\underline{o_j} + \ell_j - \ell_i - 1] \cup [\overline{o_j} + 1..\underline{o_j} + \ell_j - 1] & \text{if } \ell_i \geq \ell_j \wedge \ell_i > 1 \wedge \ell_j > 1 \end{cases} \quad (1)$$

Each case consists of a normalised sequence of forbidden regions F and of a condition C involving the lengths of the tasks; such a case will be denoted as (F if C). Generating such cases is done by using the normalised forbidden regions of the 13 Allen's basic relations given in column **RC** of Tab. 1, as well as the strong ordering structure between the limits (see Fig. 1) of these forbidden regions we identified in Sect. 2.2 in three steps as follows.

1. Extracting the lower/upper limits of forbidden regions of basic Allen's Relations in $\mathcal{R}$
   (a) First, we filter from the considered general Allen's relation $\mathcal{R}$ those basic relations which are neither mentioned in the upper nor in the lower limits of the forbidden regions attached to the relevant case (i.e., $\ell_i < \ell_j$, $\ell_i = \ell_j$, $\ell_i > \ell_j$). This is because such Allen's basic relations generate one single forbidden region of the form $(-\infty..+\infty)$ and can be therefore removed from the disjunction. For the same reason, we also filter from $\mathcal{R}$ those basic relations for which the parameter restriction does not hold.
   (b) Second, we group together the set of restrictions attached to the remaining basic Allen's relations. This leads to a set of restrictions in $\{\top, \ell_i > 1 \wedge \ell_j > 1, \ell_i + 1 < \ell_j, \ell_i > \ell_j + 1\}$. For those restrictions different from $\top$ we consider all possible combinations where each relation holds or does not hold. When the relation does not hold we remove the corresponding Allen's basic relation for the same reason as before. This gives us a number of cases for which we will generate the forbidden regions using the next steps. Since to each lower limit correspond an upper limit we remain with n lower limits low_{α_k} (with $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq n$) and n upper limits up_{β_k} (with $1 \leq \beta_1 < \beta_2 < \cdots < \beta_n \leq n+1$). A case for which $n = 0$ means a full forbidden region $(-\infty.. + \infty)$.
2. Combining the limits of forbidden regions of basic Allen's relations to get the forbidden regions of $\mathcal{R}$
   Second, the forbidden regions of the considered general Allen's relation $\mathcal{R}$ are given by $\bigcup_{k\in[1,n]|\alpha_k\neq\beta_k}[low_{\alpha_k}..up_{\beta_k}]$.
3. Removing empty forbidden regions of $\mathcal{R}$
   Using the following steps, we eliminate from $\bigcup_{k\in[1,n]|\alpha_k\neq\beta_k}[low_{\alpha_k}..up_{\beta_k}]$ the intervals that are necessarily empty when the origin of task j is fixed.

(a) if $\ell_i < \ell_j \wedge \ell_i = 1$ then eliminate [*low*..*up*] such that *low* (resp. *up*) is attached to a limit associated with m (resp. s). In the following, for simplicity, we just say *eliminate* [m, s]. Similarly we eliminate [f, mi].
(b) if $\ell_i < \ell_j \wedge \ell_i + 1 = \ell_j$ then eliminate [s, f].
(c) if $\ell_i = \ell_j \wedge \ell_i = 1$ then eliminate [m, e] and [e, mi].
(d) if $\ell_i > \ell_j \wedge \ell_j = 1$ then eliminate [m, fi] and [si, mi].
(e) if $\ell_i > \ell_j \wedge \ell_j + 1 = \ell_i$ then eliminate [fi, si].

We now show that the previous three steps procedure generates a symbolic normal form for the forbidden regions of a general relation $\mathcal{R}$.

Lemma 3. *For a general relation $\mathcal{R}$ by systematically combining the three cases $\ell_i < \ell_j$, $\ell_i = \ell_j$, $\ell_i > \ell_j$ with all possible restrictions from $\{\top, \ell_i > 1 \wedge \ell_j > 1, \ell_i + 1 < \ell_j, \ell_i > \ell_j + 1\}$ we generate all possible cases for that relation $\mathcal{R}$.*

Proof. The Cartesian product of $\{\ell_i < \ell_j, \ell_i = \ell_j, \ell_i > \ell_j\} \times \{\top\} \times \{\ell_i > 1 \wedge \ell_j > 1\} \times \{\ell_i + 1 < \ell_j\} \times \{\ell_i > \ell_j + 1\}$ is considered. □

Lemma 4. *For a general relation $\mathcal{R}$ consider one of its case generated in step 1 and the corresponding limits low_{α_k} and up_{β_k}. The forbidden regions of $\mathcal{R}$ are given by $\bigcup_{k\in[1,n]|\alpha_k\neq\beta_k}[low_{\alpha_k}..up_{\beta_k}]$.*

Proof. A forbidden region of $\mathcal{R}$ is an interval of consecutive values that are forbidden for *all* basic relations of $\mathcal{R}$. Since both the lower limits low_{α_k} and the upper limits up_{β_k} are sorted in increasing order, and since $[low_p..up_q] = \emptyset$ for all $p \geq q$ we pick up for each start of a forbidden region low_{α_k} the smallest end up_{β_k} of the forbidden region that was starting before low_{α_k}. □

Lemma 5. *When o_j is fixed the intervals removed by step 3 are the only empty intervals $[low_p..up_q]$ where $p < q$.*

Proof. The other cases being similar we only show the proof for the lower limit $\overline{o_j}-\ell_i+1$ that was generated from m when $\ell_i < \ell_j$.

- Within the case $\ell_i < \ell_j$, $\overline{o_j}-\ell_i+1$ is the lower limit of index 2 in Fig. 1. Consequently we first look at the upper limit of index 3, namely $\underline{o_j} - 1$ that was generated from s. Since we want to check when $\overline{o_j} - \ell_i + 1$ will be strictly greater than $\underline{o_j} - 1$ when o_j is fixed, we get $o_j - \ell_i + 1 > o_j - 1$, which simplifies to $-\ell_i + 1 > -1$ and to $\ell_i \leq 1$, which means that we can eliminate [m, s] when $\ell_i = 1$.
- We now need to compare $\overline{o_j}-\ell_i+1$ with the next upper limit, namely the upper limit of index 4, i.e. $\underline{o_j}$. We get $o_j - \ell_i + 1 > o_j$, which simplifies to $\ell_i < 1$ which is never true. Consequently the interval [m, d] is not empty when o_j is fixed. This implies that the other intervals [m, f], [m, oi], [m, mi], [m, bi] are also not empty when o_j is fixed since their upper limit are located after the upper limit of index 4. □

Example 1. Assuming $\ell_i < \ell_j$ we successively illustrate how to generate the normalised forbidden regions for the relation $\mathcal{R}_1 = \{\text{b}, \text{m}, \text{mi}, \text{bi}\}$ (i.e. nonoverlapping), for $\mathcal{R}_2 = \{\text{b}, \text{m}\}$, and for $\mathcal{R}_3 = \{\text{b}, \text{s}, \text{bi}\}$.

1. By keeping the limits related to the basic relations b, m, mi, bi of $\mathcal{R}_1$ we get $\alpha_1 = 1$, $\alpha_2 = 2, \alpha_3 = 8$ and $\beta_1 = 1, \beta_2 = 7, \beta_3 = 8$. Since $\alpha_1 = \beta_1$ and $\alpha_3 = \beta_3$ we only keep α_2 and β_2 and get the interval $[low_{\alpha_2}..up_{\beta_2}] = [low_2..up_7] = [\overline{o_j} - \ell_i + 1..\underline{o_j} + \ell_j - 1]$, the expected result for a nonoverlapping constraint between two tasks.
2. By keeping the limits related to the basic relations b, m of $\mathcal{R}_2$ we get $\alpha_1 = 1$, $\alpha_2 = 2$ and $\beta_1 = 1, \beta_2 = 9$. Since $\alpha_1 = \beta_1$ we only keep α_2 and β_2 and get the interval $[low_{\alpha_2}..up_{\beta_2}] = [low_2..up_9] = [\overline{o_j} - \ell_i + 1.. + \infty)$.
3. By keeping the limits related to the basic relations b, s, bi of $\mathcal{R}_3$ we get $\alpha_1 = 1$, $\alpha_2 = 4$ and $\beta_1 = 3, \beta_2 = 8$, which leads to $[low_{\alpha_1}..up_{\beta_1}] \cup [low_{\alpha_2}..up_{\beta_2}] = [low_1..up_3] \cup [low_4..up_8] = [\overline{o_j} - \ell_i..\underline{o_j} - 1] \cup [\overline{o_j} + 1..\underline{o_j} + \ell_j]$.

Merging Similar Cases For a given Allen's general relation $\mathcal{R}$, two cases (D_1 if C_1) and (D_2 if C_2) can be merged to a single case (D_{12} if C_{12}) if the following conditions all hold:

- C_{12} is equivalent to $C_1 \vee C_2$ and can be expressed as a conjunction of primitive restrictions.
- D_1, D_2, and D_{12} consist of the same number of intervals.
- For every interval $[b_1, u_1] \in D_1$ there are intervals $[b_2, u_2] \in D_2$ and $[b_{12}, u_{12}] \in D_{12}$ at the same position such that:
  - $b_1 = b_{12}$ and $u_1 = u_{12}$, for any values taken by ℓ_i and ℓ_j such that C_1 holds.
  - $b_2 = b_{12}$ and $u_2 = u_{12}$, for any values taken by ℓ_i and ℓ_j such that C_2 holds.

We used a semi-automatic approach to discover such endpoint generalisation rules. For every Allen's general relation, using these rules, we identified and merged pairs and triples of cases until no more merging was possible. As the result of this process, the data base [2] consists of 32396 cases covering all the 8192 general relations. In this data base, the maximum number of intervals for a case is 5, the average number of intervals is 2.14 and the median is 2.

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3 Conclusion

This work belongs to the line of work that tries to synthesise in a systematic way constraint propagators for specific classes of constraints [6,7,9]. Future work may generalise this for getting a similar normal form for other families of qualitative constraints.

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