Lazy Clause Generation: Combining the best of SAT and CP (and MIP?) solving

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Propagation Based Constraint Solving

- Repeatedly run *propagators*
- Propagators change variable domains by:
  - removing values
  - changing upper and lower bounds
  - fixing to a value
- Run until fixpoint.

KEY INSIGHT:

- Changes in domains are really the fixing of Boolean variables representing domains.
- Propagation is just the generation of clauses on these variables.
- FD solving is just SAT solving: conflict analysis for FREE!

Outline

Outline

1. Finite Domain Propagation
   - FD Example
2. SAT Solving
   - SAT Example
3. Lazy Clause Generation
   - Original Lazy Clause Generation
   - Lazier Clause Generation
   - Global Constraints
   - Search
4. Related Work
5. Conclusion

Terminology

- **domain** D maps variable x to set of possible values $D(x)$
- **propagator** $f_c : D \mapsto D$ for constraint c
  - monotonic decreasing function
  - removes values from the domain which cannot be part of a solution.
- **Problem** set of propagators F and initial domain D_0
- **propagation solver** $solv(F, D) = D'$ where D' is the greatest mutual fixpoint of all $f \in F$.
- **FD solving** interleaves propagation with search: (for simplicity binary)
  - Add new search constraint c. $D' = solv(F \cup \{f_c\}, D)$
  - On failure add backtrack and add $\neg c$. $D' = solv(F \cup \{f_{\neg c}\}, D)$
  - Repeat until all variables fixed

Outline

Finite Domain Propagation Example

Consider the problem with:

Domain D_0:

$$D_0(x_1) = D_0(x_2) = D_0(x_3) = D_0(x_4) = D_0(x_5) = [1 \,..\, 4]$$

F propagators for:

$$x_2 \leq x_5,\ \texttt{alldifferent}([x_1, x_2, x_3, x_4]),\ x_1 + x_2 + x_3 + x_4 \leq 9.$$

x_1		
x_2		
x_3		
x_4		
x_5		

Finite Domain Propagation Example

Consider the problem with:

Domain D_0:

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F propagators for:

$$x_2 \leq x_5,\ \texttt{alldifferent}([x_1, x_2, x_3, x_4]),\ x_1 + x_2 + x_3 + x_4 \leq 9.$$

	$\mathbf{x_1 = 1}$	
x_1	1	
x_2	$[1..4]$	
x_3	$[1..4]$	
x_4	$[1..4]$	
x_5	$[1..4]$	

Finite Domain Propagation Example

Consider the problem with:

Domain D_0:

$$D_0(x_1) = D_0(x_2) = D_0(x_3) = D_0(x_4) = D_0(x_5) = [\,1\,..\,4\,]$$

F propagators for:

$$x_2 \leq x_5,\ \texttt{alldifferent}([x_1, x_2, x_3, x_4]),\ x_1 + x_2 + x_3 + x_4 \leq 9.$$

	$\mathbf{x_1 = 1}$	*alldiff*	
x_1	1	1	
x_2	$[\,1\,..\,4\,]$	$[\,2\,..\,4\,]$	
x_3	$[\,1\,..\,4\,]$	$[\,2\,..\,4\,]$	
x_4	$[\,1\,..\,4\,]$	$[\,2\,..\,4\,]$	
x_5	$[\,1\,..\,4\,]$	$[\,1\,..\,4\,]$	

Finite Domain Propagation Example

Consider the problem with:

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F propagators for:

$$x_2 \leq x_5,\ \texttt{alldifferent}([x_1, x_2, x_3, x_4]),\ x_1 + x_2 + x_3 + x_4 \leq 9.$$

	$\mathbf{x_1 = 1}$	*alldiff*	$x_2 \leq x_5$
x_1	1	1	1
x_2	$[\,1\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$
x_3	$[\,1\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$
x_4	$[\,1\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$
x_5	$[\,1\,..\,4\,]$	$[\,1\,..\,4\,]$	$[\,2\,..\,4\,]$
			D_1

Finite Domain Propagation Example

Consider the problem with:

Domain D_0:

$$D_0(x_1) = D_0(x_2) = D_0(x_3) = D_0(x_4) = D_0(x_5) = [\,1\,..\,4\,]$$

F propagators for:

$$x_2 \leq x_5,\ \texttt{alldifferent}([x_1, x_2, x_3, x_4]),\ x_1 + x_2 + x_3 + x_4 \leq 9.$$

	$\mathbf{x_1 = 1}$	*alldiff*	$x_2 \leq x_5$	$\mathbf{x_5 \leq 2}$
x_1	1	1	1	1
x_2	$[\,1\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$
x_3	$[\,1\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$
x_4	$[\,1\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$
x_5	$[\,1\,..\,4\,]$	$[\,1\,..\,4\,]$	$[\,2\,..\,4\,]$	2
			D_1	

Finite Domain Propagation Example

Consider the problem with:

Domain D_0:

$$D_0(x_1) = D_0(x_2) = D_0(x_3) = D_0(x_4) = D_0(x_5) = [\,1\,..\,4\,]$$

F propagators for:

$$x_2 \leq x_5,\ \texttt{alldifferent}([x_1, x_2, x_3, x_4]),\ x_1 + x_2 + x_3 + x_4 \leq 9.$$

	$\mathbf{x_1 = 1}$	*alldiff*	$x_2 \leq x_5$	$\mathbf{x_5 \leq 2}$	$x_2 \leq x_5$
x_1	1	1	1	1	1
x_2	$[\,1\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$	2
x_3	$[\,1\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$
x_4	$[\,1\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$	$[\,2\,..\,4\,]$
x_5	$[\,1\,..\,4\,]$	$[\,1\,..\,4\,]$	$[\,2\,..\,4\,]$	2	2
			D_1		

Finite Domain Propagation Example

Consider the problem with:

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$$D_0(x_1) = D_0(x_2) = D_0(x_3) = D_0(x_4) = D_0(x_5) = [\,1 \,..\, 4\,]$$

F propagators for:

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	$\mathbf{x_1 = 1}$	*alldiff*	$x_2 \leq x_5$	$\mathbf{x_5 \leq 2}$	$x_2 \leq x_5$	*alldiff*
x_1	1	1	1	1	1	1
x_2	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$	2	2
x_3	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[3..4]$
x_4	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[3..4]$
x_5	$[1..4]$	$[1..4]$	$[2..4]$	2	2	2
			D_1			

Finite Domain Propagation Example

Consider the problem with:

Domain D_0:

$$D_0(x_1) = D_0(x_2) = D_0(x_3) = D_0(x_4) = D_0(x_5) = [\,1\,..\,4\,]$$

F propagators for:

$$x_2 \le x_5,\ \texttt{alldifferent}([x_1, x_2, x_3, x_4]),\ x_1 + x_2 + x_3 + x_4 \le 9.$$

	$\mathbf{x_1 = 1}$	*alldiff*	$x_2 \le x_5$	$\mathbf{x_5 \le 2}$	$x_2 \le x_5$	*alldiff*	$\sum \le 9$
x_1	1	1	1	1	1	1	1
x_2	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$	2	2	2
x_3	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[3..4]$	3
x_4	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[3..4]$	3
x_5	$[1..4]$	$[1..4]$	$[2..4]$	2	2	2	2
			D_1				

Finite Domain Propagation Example

Consider the problem with:

Domain D_0:

$$D_0(x_1) = D_0(x_2) = D_0(x_3) = D_0(x_4) = D_0(x_5) = [\,1 \mathinner{..} 4\,]$$

F propagators for:

$$x_2 \leq x_5,\ \texttt{alldifferent}([x_1, x_2, x_3, x_4]),\ x_1 + x_2 + x_3 + x_4 \leq 9.$$

	$\mathbf{x_1 = 1}$	*alldiff*	$x_2 \leq x_5$	$\mathbf{x_5 \leq 2}$	$x_2 \leq x_5$	*alldiff*	$\sum \leq 9$	*alldiff*
x_1	1	1	1	1	1	1	1	1
x_2	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$	2	2	2	2
x_3	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[3..4]$	3	$\emptyset$
x_4	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[3..4]$	3	$\emptyset$
x_5	$[1..4]$	$[1..4]$	$[2..4]$	2	2	2	2	2
			D_1					*fail*

Backtrack

Finite Domain Propagation Example

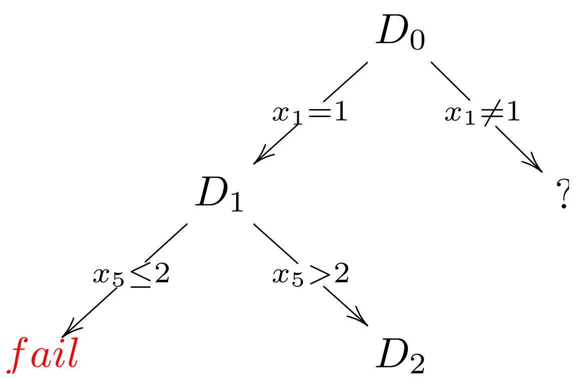
	$\mathbf{x_1 = 1}$	*alldiff*	$x_2 \leq x_5$
x_1	1	1	1
x_2	$[1..4]$	$[2..4]$	$[2..4]$
x_3	$[1..4]$	$[2..4]$	$[2..4]$
x_4	$[1..4]$	$[2..4]$	$[2..4]$
x_5	$[1..4]$	$[1..4]$	$[2..4]$
			D_1

Finite Domain Propagation Example

	$\mathbf{x_1 = 1}$	*alldiff*	$x_2 \leq x_5$	$\mathbf{x_5 > 2}$
x_1	1	1	1	1
x_2	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$
x_3	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$
x_4	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$
x_5	$[1..4]$	$[1..4]$	$[2..4]$	$[3..4]$
			D_1	D_2

Finite Domain Propagation Example

	$\mathbf{x_1 = 1}$	*alldiff*	$x_2 \leq x_5$	$\mathbf{x_5 > 2}$
x_1	1	1	1	1
x_2	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$
x_3	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$
x_4	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$
x_5	$[1..4]$	$[1..4]$	$[2..4]$	$[3..4]$
			D_1	D_2



Strengths and Weaknesses of FD solving

- Strengths
  - high level modelling
  - specialized global propagators
  - programmable search
- Weaknesses
  - Search often needs programming (weak autonomous search)
  - Optimization by repeated satisfaction search

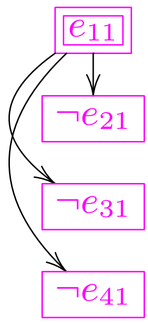
Outline

Terminology

- **literal** $l = b$ or $l = \neg b$ where b is a Boolean
- **clause** $l_1 \vee \cdots \vee l_n$ (or set of literals $\{l_1, \ldots, l_n\}$) also $\neg l_1 \wedge \cdots \wedge \neg l_{n-1} \rightarrow l_n$
- **CNF** set of clauses C
- **assignment** A is a set of literals $\{b, \neg b\} \not\subseteq A$
- **unit propagation** $up(C, A) = A'$
  - foreach clause $l_1 \vee \cdots \vee l_{n-1} \vee l_n$ where $\{\neg l_1, \ldots, \neg l_{n-1}\} \subseteq A$ add l_n to A.
  - continue to fixpoint
- **SAT solving**
  - Choose a literal l: $A' := up(C, A \cup \{l\})$
  - On failure determine a nogood $c \subseteq A$ and add it to C, backjump
  - Repeat until all variables fixed

Outline

SAT Implication Graph

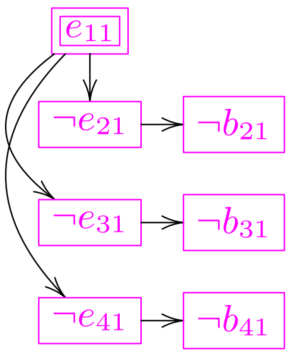


Decision e_{11}

Resolving clauses: $\neg e_{11} \vee \neg e_{21}$, $\neg e_{11} \vee \neg e_{31}$, $\neg e_{11} \vee \neg e_{41}$.

.

SAT Implication Graph

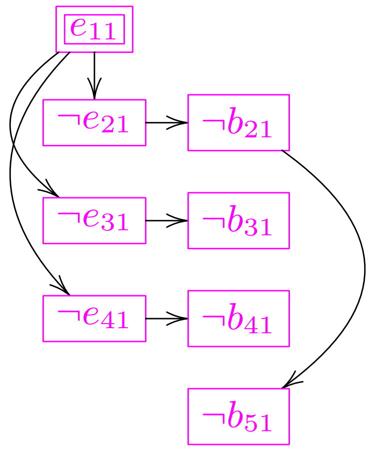


Decision e_{11}

Resolving clauses: $e_{21} \vee \neg b_{21}$, $e_{31} \vee \neg b_{31}$, $e_{41} \vee \neg b_{41}$.

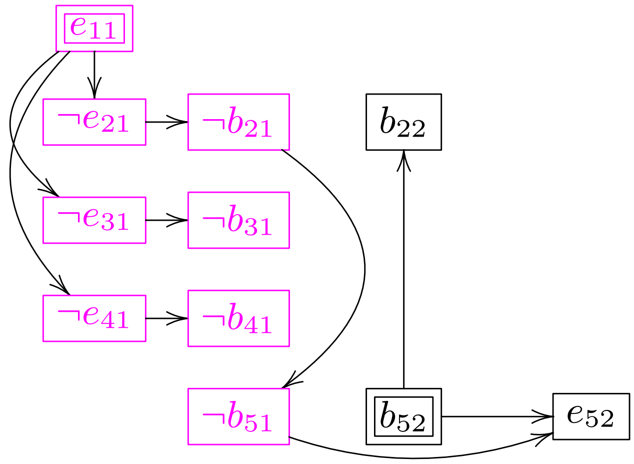
.

SAT Implication Graph



Decision e_{11}
Resolving clause: $b_{21} \vee \neg b_{51}$
Unit fixpoint

SAT Implication Graph

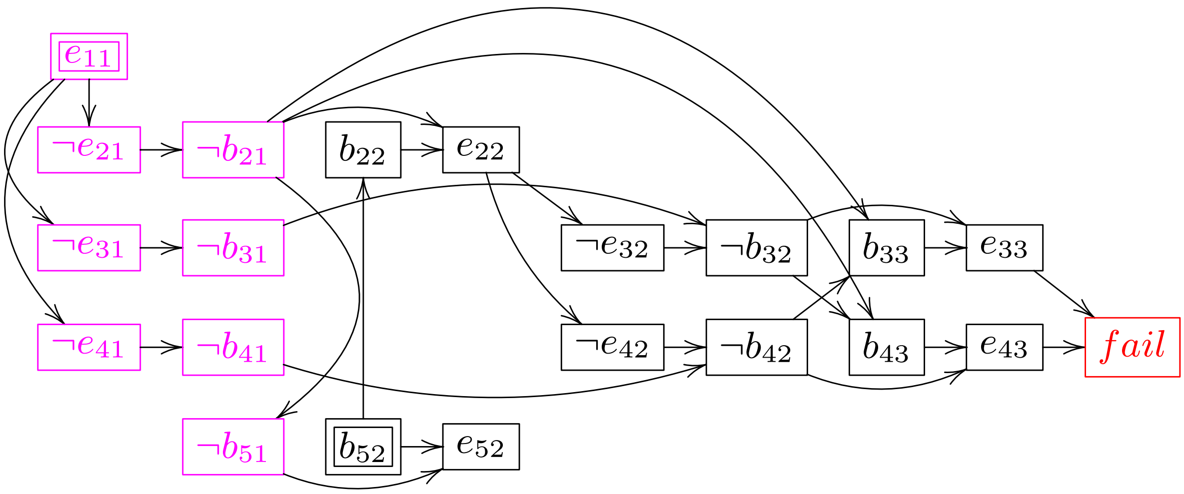


New Decision b_{52}

Resolving clauses: $b_{51} \vee \neg b_{52} \vee e_{52}$, $\neg b_{52} \vee b_{22}$

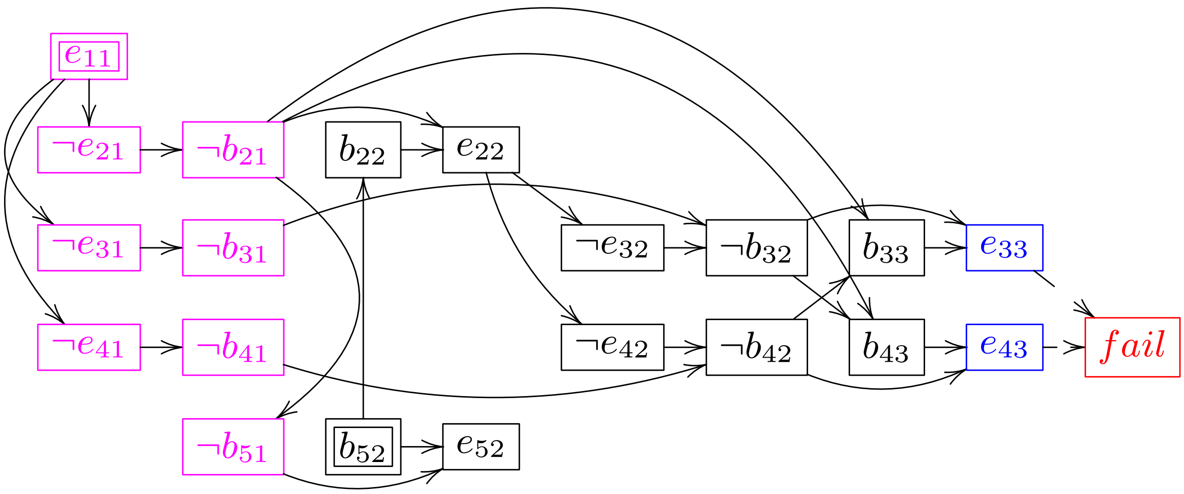
.

SAT Implication Graph



Decision b_{52}
Resolving clauses many
Conflict detected!

SAT 1UIP Conflict Resolution

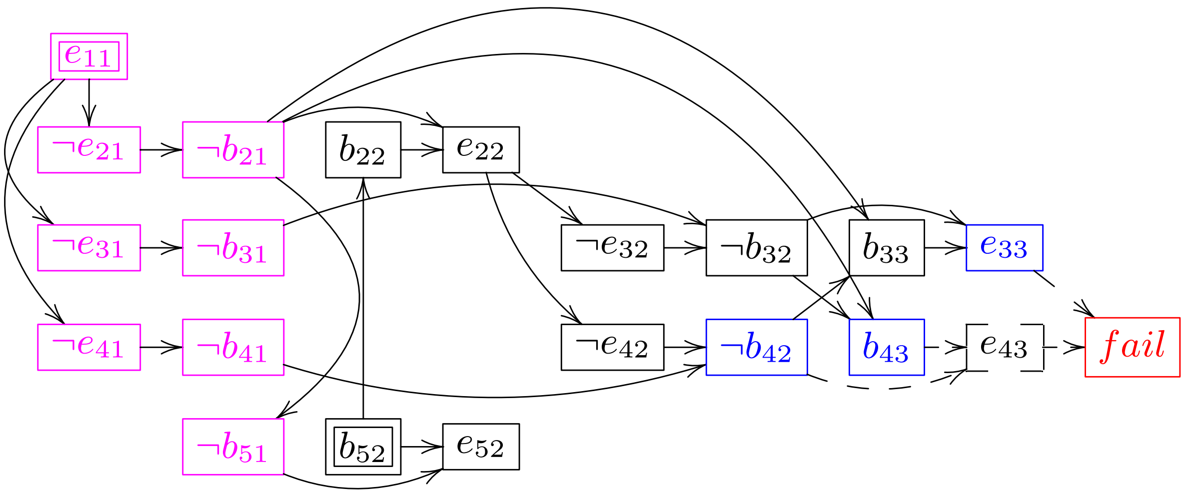


Initial nogood $(\neg e_{33} \vee \neg e_{43})$

$$e_{33} \wedge e_{43} \rightarrow false$$

.

SAT 1UIP Conflict Resolution

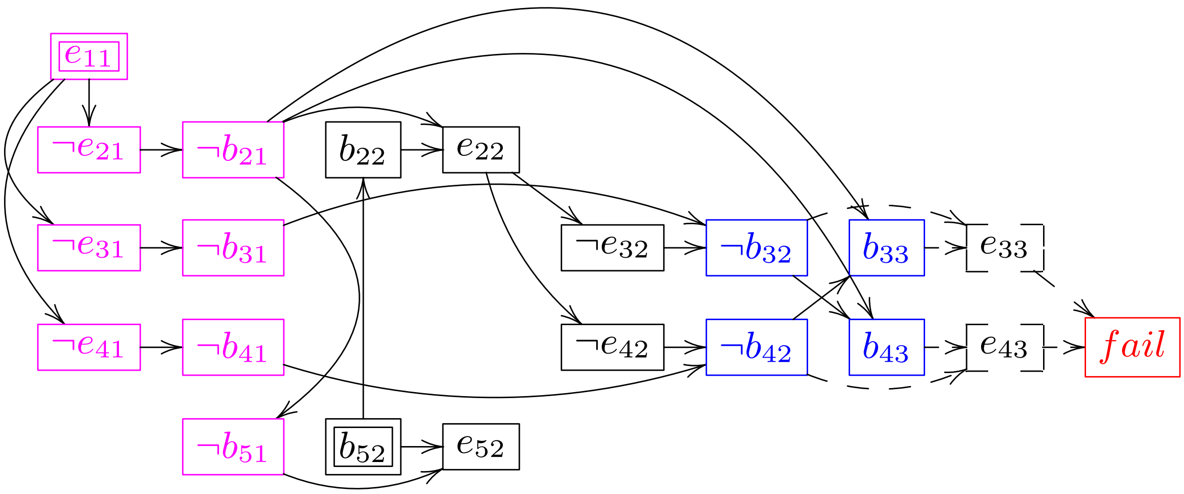


Resolving $b_{42} \vee \neg b_{43} \vee e_{43}$ gives

$$\neg b_{42} \wedge b_{43} \wedge e_{33} \rightarrow \mathit{false}$$

.

SAT 1UIP Conflict Resolution

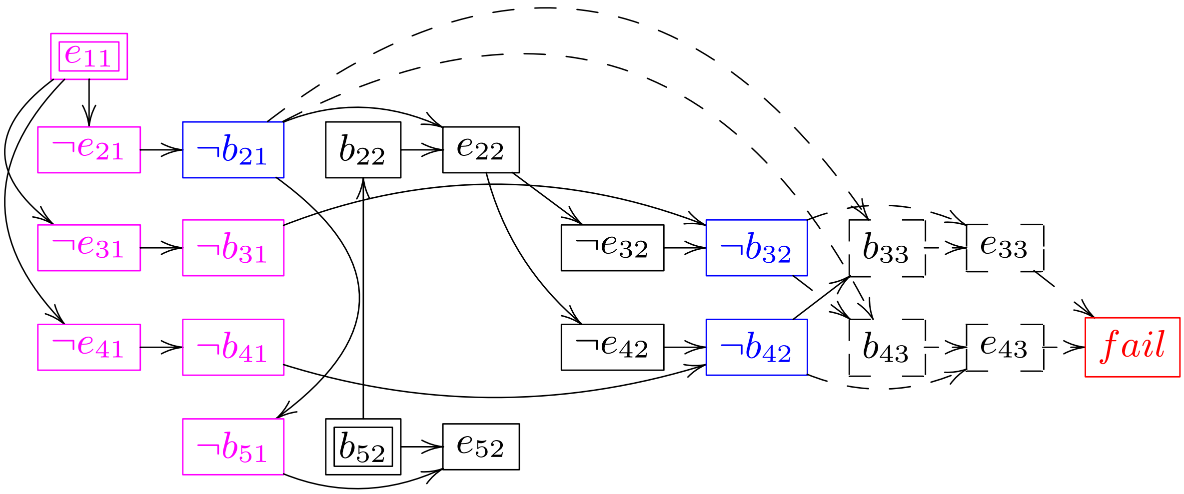


Resolving $b_{32} \vee \neg b_{33} \vee e_{33}$ gives

$$\neg b_{32} \wedge \neg b_{42} \wedge b_{33} \wedge b_{43} \rightarrow \mathit{false}$$

.

SAT 1UIP Conflict Resolution

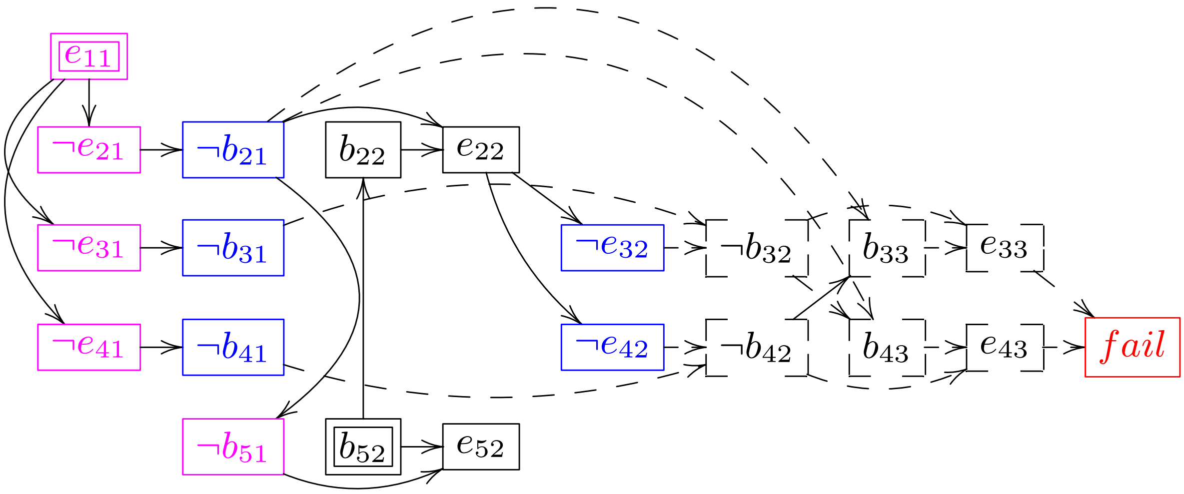


Resolving $b_{21} \vee b_{42} \vee b_{33}$ and $b_{21} \vee b_{32} \vee b_{43}$ gives

$$\neg b_{21} \wedge \neg b_{32} \wedge \neg b_{42} \rightarrow \mathit{false}$$

.

SAT 1UIP Conflict Resolution

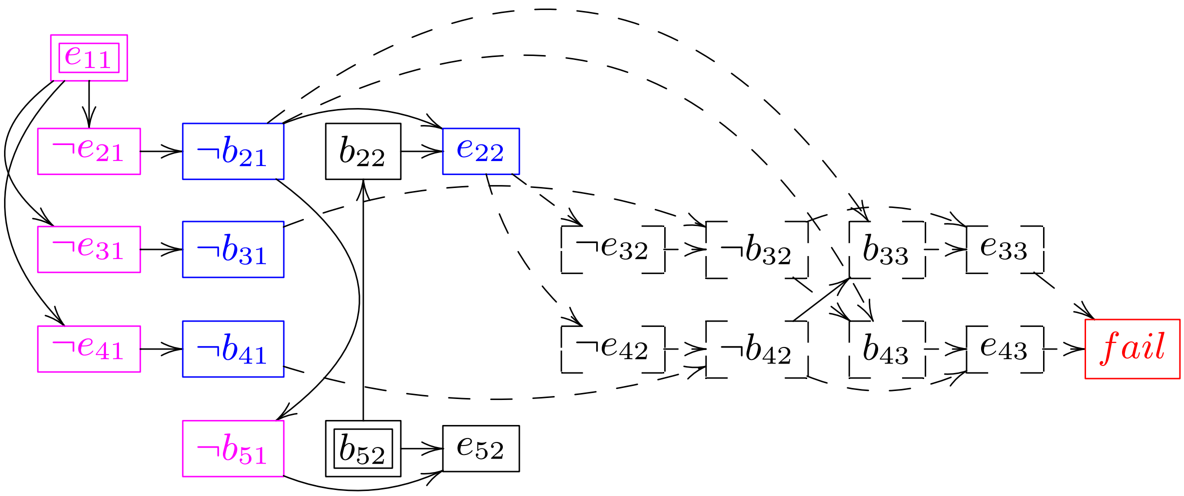


Resolving $b_{31} \vee e_{32} \vee \neg b_{32}$ and $b_{41} \vee e_{42} \vee \neg b_{42}$ gives

$$\neg b_{21} \wedge \neg b_{31} \wedge \neg b_{41} \wedge \neg e_{32} \wedge \neg e_{42} \rightarrow \mathit{false}$$

.

SAT 1UIP Conflict Resolution

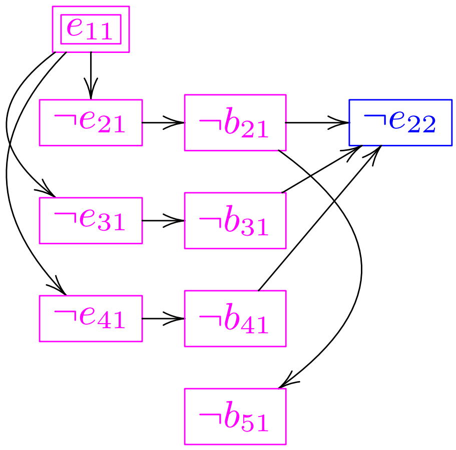


Resolving $\neg e_{22} \vee \neg e_{32}$ and $\neg e_{22} \vee \neg e_{42}$ gives

$$\neg b_{21} \wedge \neg b_{31} \wedge \neg b_{41} \wedge e_{22} \rightarrow \mathit{false}$$

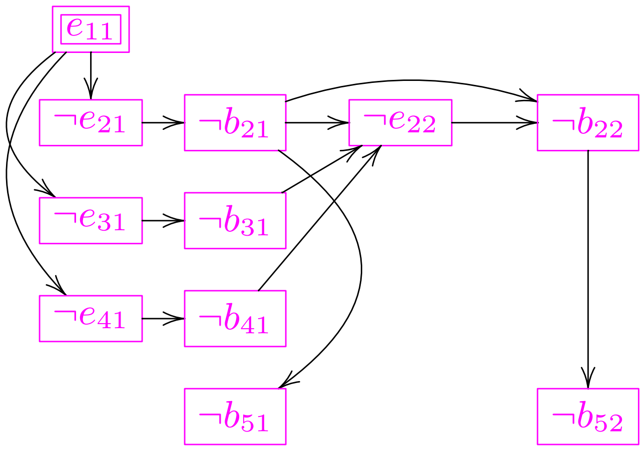
The 1UIP nogood! $b_{21} \vee b_{31} \vee b_{41} \vee \neg e_{22}$

SAT Backjumping



Backjump
Apply nogood: $b_{21} \vee b_{31} \vee b_{41} \vee \neg e_{22}$
Continue to unit fixpoint

SAT Implication Graph



Continue to unit fixpoint

Resolving clauses $b_{21} \vee \neg b_{22} \vee e_{22}$, $\neg b_{52} \vee b_{22}$

Unit fixpoint

SAT engineering

- Cornerstones of modern SAT solvers
  - Watched literals: efficient implementation of unit propagation
  - 1UIP nogoods: record effective nogoods (efficiently)
  - Activity-based search: concentrate on variables involved in recent failures
  - Restarts
- Other features
  - Deep backjumping
  - Activity based forgetting of nogoods
  - Retry last used value for a variable

Strengths and Weaknesses of SAT solving

- Strengths
  - Learning avoids repeating the same subsearch
  - Can deal with (low) millions of variables and clauses
  - Strong autonomous search
- Weaknesses
  - Optimization by repeated satisfaction search
  - Have to model entirely in clauses/Booleans (can definitely blow the limits above)

Outline

Outline

Representing Integer and Set Variables

- Integer variable x: represented using Booleans
  - $[\![x = d]\!], d \in [\, l \,..\, u\,] = D_0(x)$,
  - $[\![x \leq d]\!], l \leq d < u$.
- Clauses to maintain consistency: DOM

$$\begin{aligned} [\![x \leq d]\!] &\rightarrow [\![x \leq d+1]\!] \quad l \leq d < u-1 \\ [\![x = d]\!] \leftrightarrow [\![x \leq d]\!] &\wedge \neg[\![x \leq d-1]\!] \quad l < d \leq u \end{aligned}$$

- Unary arithmetic representation (linear in size)
- One to one correspondence domains D and assignment A unit fixpoints of DOM $A = up(DOM, A)$

Atomic Constraints

- **atomic constraints** define changes in domains
  - Fixing variable: $x_i = d$
  - Removing value: $x_i \neq d$
  - Bounding variable: $x_i \leq d$, $x_i \geq d$
- Atomic constraints are just Boolean literals!

$$
\begin{aligned}
x_i = d &\equiv [\![x_i = d]\!] \\
x_i \neq d &\equiv \neg[\![x_i = d]\!] \\
x_i \leq d &\equiv [\![x_i \leq d]\!] \\
x_i \geq d &\equiv \neg[\![x_i \leq d - 1]\!]
\end{aligned}
$$

Lazy Clause Generation Propagators

- When $f(D) \neq D$ (new information)
- Propagator explains each atomic constraint change
- What part of the current domain D created the new inference!
  - $D(x_1) = \{1\}$, $D(x_2) = D(x_3) = D(x_4) = [1..4]$,
    $\texttt{alldifferent}([x_1, x_2, x_3, x_4])$
  - $f_{alldiff}(D)$ implies $x_2 \neq 1$, $x_3 \neq 1$, $x_4 \neq 1$
  - explanations $x_1 = 1 \rightarrow x_2 \neq 1$, $x_1 = 1 \rightarrow x_3 \neq 1$, $x_1 = 1 \rightarrow x_4 \neq 1$,
- Adds explanation as clauses, unit propagate on Booleans
- Propagator similarly explains failure.
  - $D(x_3) = \{3\}$, $D(x_4) = \{3\}$, $\texttt{alldifferent}([x_1, x_2, x_3, x_4])$
  - $f_{alldiff}(D)$ gives a false domain
  - explanation $x_3 = 3 \wedge x_4 = 3 \rightarrow \mathit{fail}$

Finite Domain Propagation Example Redux

Consider the problem with:

Domain D_0:

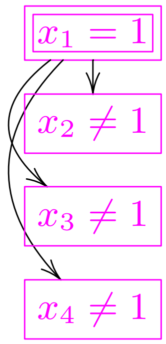
$$D_0(x_1) = D_0(x_2) = D_0(x_3) = D_0(x_4) = D_0(x_5) = [\,1\,..\,4\,]$$

F propagators for:

$$x_2 \leq x_5,\ \texttt{alldifferent}([x_1, x_2, x_3, x_4]),\ x_1 + x_2 + x_3 + x_4 \leq 9.$$

Lazy Clause Generation example

alldiff



Search: $x_1 = 1$

$D(x_1) = \{1\}$, $D(x_2) = D(x_3) = D(x_4) = D(x_5) = [1\,..\,4]$,

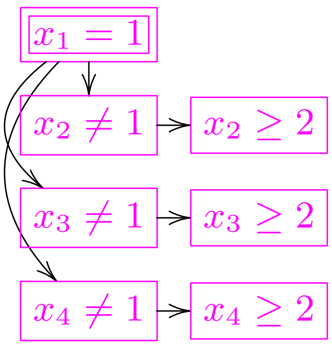
Propagate `alldifferent`$([x_1, x_2, x_3, x_4])$ on D

Determines $x_2 \neq 1$, $x_3 \neq 1$, $x_4 \neq 1$

Explanations $x_1 = 1 \rightarrow x_2 \neq 1$, $x_1 = 1 \rightarrow x_3 \neq 1$, $x_1 = 1 \rightarrow x_4 \neq 1$

Lazy Clause Generation example

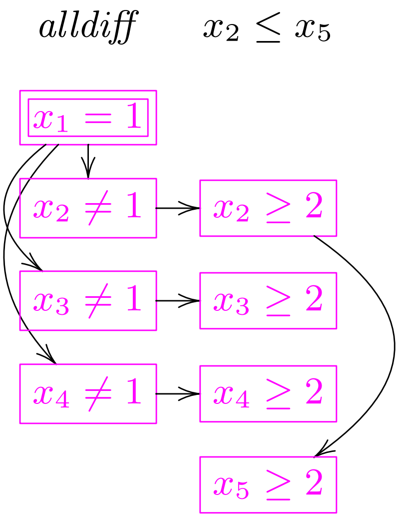
alldiff



Propagate DOM clauses: $x_2 \neq 1 \rightarrow x_2 \geq 2$, ...
Ignoring DOM clauses: $x_1 = 1 \rightarrow x_1 \neq 2$, $x_1 = 1 \rightarrow x_1 \leq 3$, ...
Domain

$$D(x_1) = \{1\},\ D(x_2) = D(x_3) = D(x_4) = [\,2\,..\,4\,],\ D(x_5) = [\,1\,..\,4\,]$$

Lazy Clause Generation example



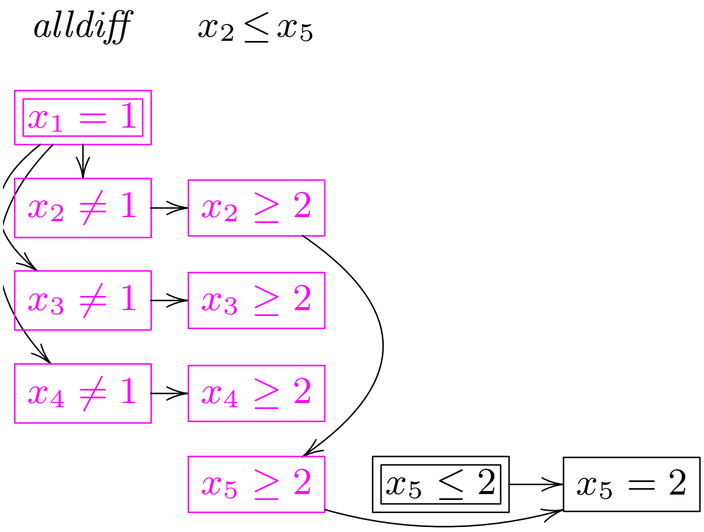
Propagate $x_2 \leq x_5$

Determines $x_5 \geq 2$ with explanation $x_2 \geq 2 \rightarrow x_5 \geq 2$

FIXPOINT:

$D_1(x_1) = \{1\}, \ D_1(x_2) = D_1(x_3) = D_1(x_4) = D_1(x_5) = [2 \,..\, 4]$

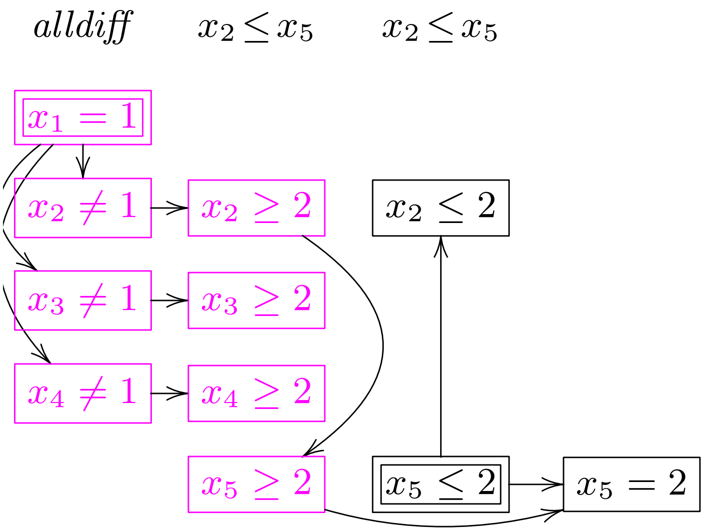
Lazy Clause Generation example



Search $x_5 \leq 2$

Domain constraints determine $x_5 = 2$ with explanation $x_5 \geq 2 \wedge x_5 \leq 2 \rightarrow x_5 = 2$

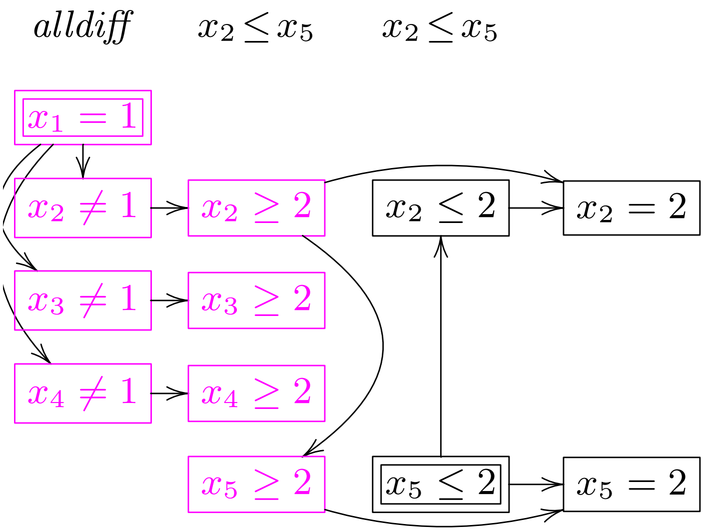
Lazy Clause Generation example



Propagate $x_2 \leq x_5$

Determine $x_2 \leq 2$ with explanation $x_5 \leq 2 \rightarrow x_2 \leq 2$

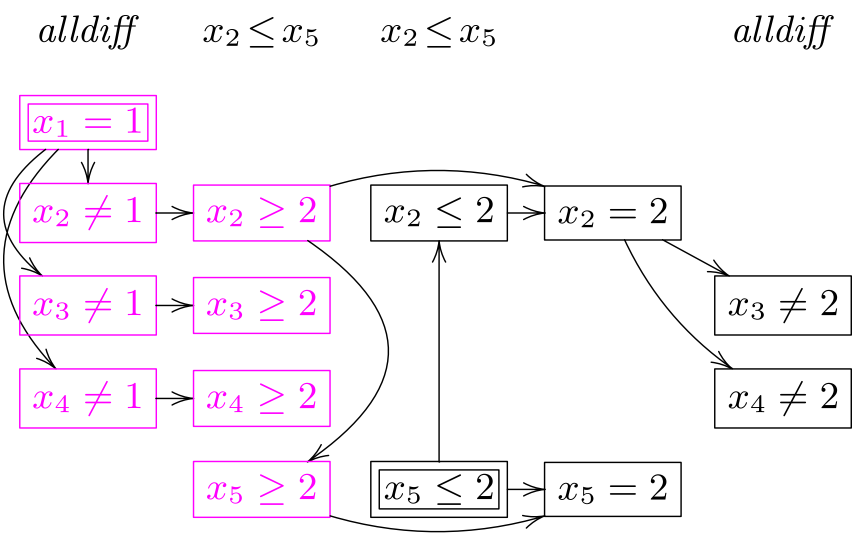
Lazy Clause Generation example



Domain constraints determine $x_2 = 2$ with explanation
$x_2 \geq 2 \wedge x_2 \leq 2 \rightarrow x_2 = 2$
Domain:

$D(x_1) = \{1\}$, $D(x_2) = \{2\}$, $D(x_3) = D(x_4) = [2..4]$, $D(x_5) = \{2\}$

Lazy Clause Generation example

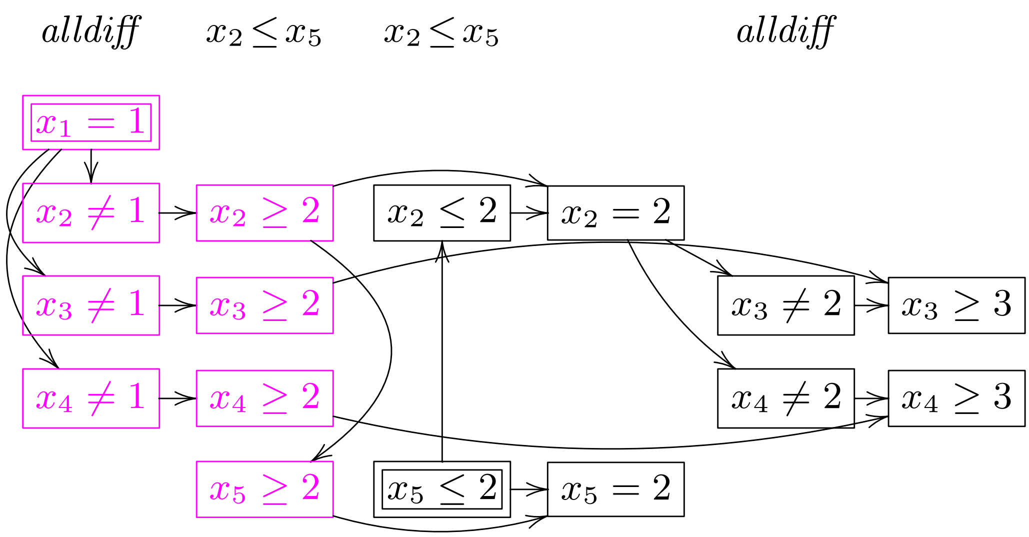


Propagate `alldifferent`$([x_1, x_2, x_3, x_4])$

Determines $x_3 \neq 2$ and $x_4 \neq 2$

with explanations $x_2 = 2 \rightarrow x_3 \neq 2$, $x_2 = 2 \rightarrow x_4 \neq 2$,

Lazy Clause Generation example

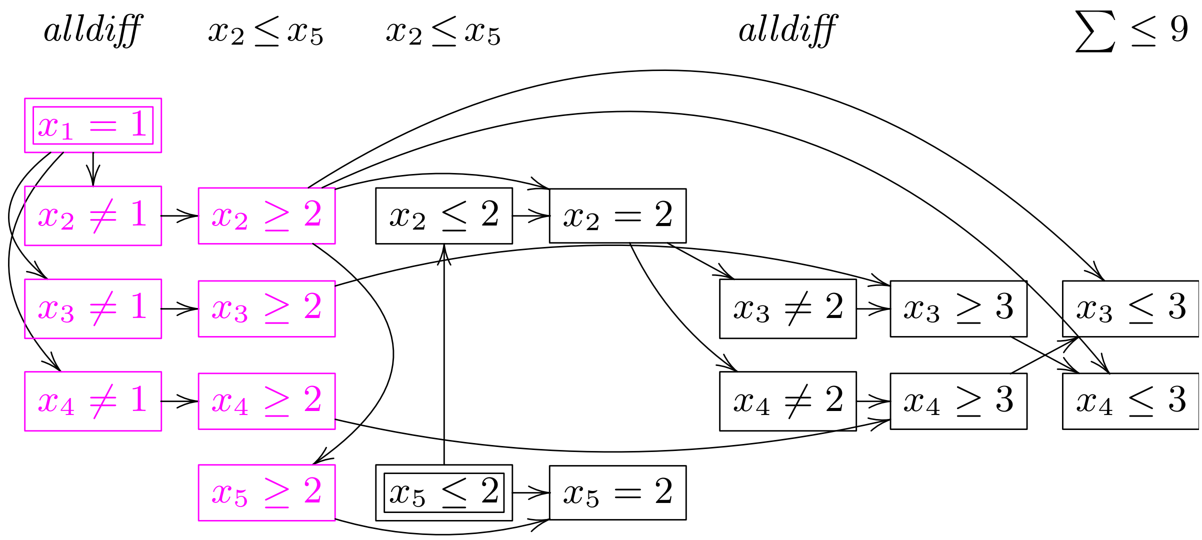


Domain constraints determine $x_3 \geq 3$ and $x_4 \geq 3$

Domain

$$D(x_1) = \{1\},\ D(x_2) = \{2\},\ D(x_3) = D(x_4) = [3\,..\,4],\ D(x_5) = \{2\}$$

Lazy Clause Generation example



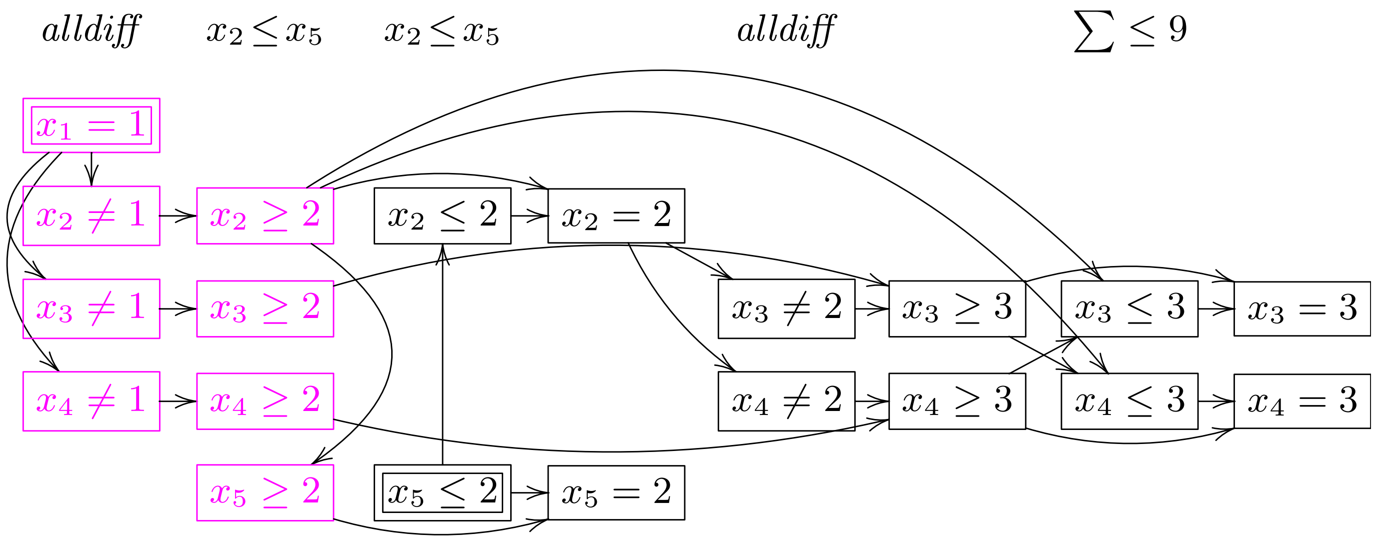
Propagate $x_1 + x_2 + x_3 + x_4 \leq 9$

Determines $x_3 \leq 3$ and $x_4 \leq 3$

with explanations $x_2 \geq 2 \wedge x_4 \geq 3 \rightarrow x_3 \leq 3$ and similar

.

Lazy Clause Generation example



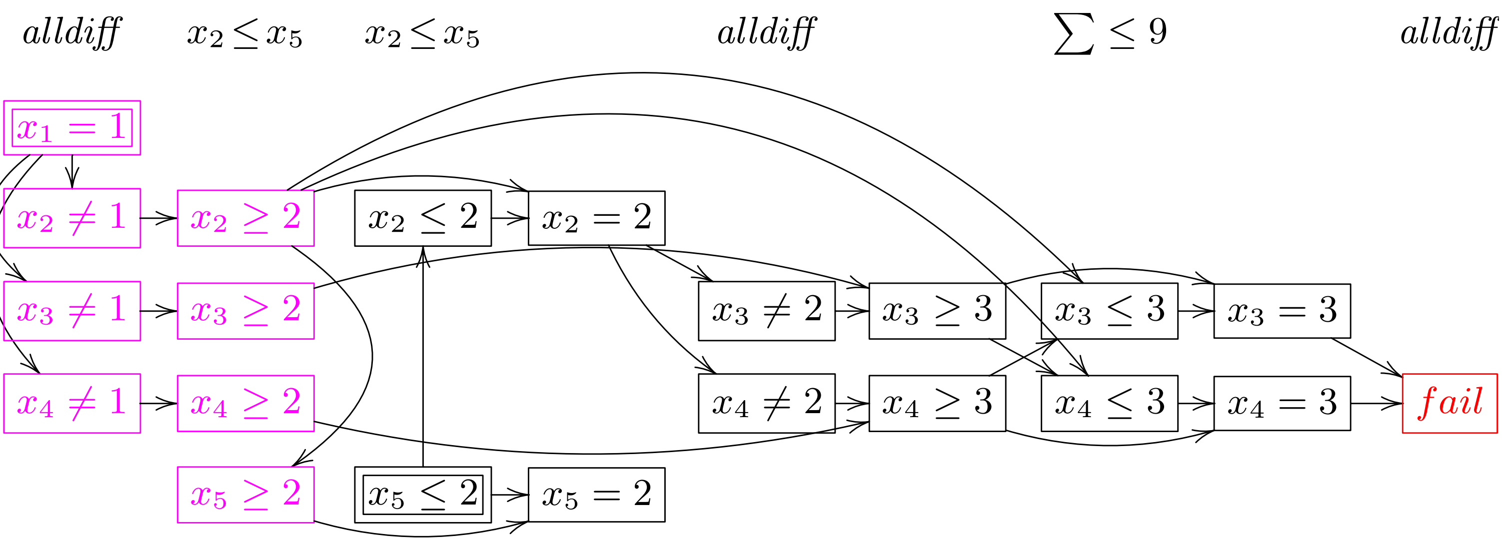
Domain constraints determine $x_3 = 3$ and $x_4 = 3$

With explanations $x_3 \geq 3 \wedge x_3 \leq 3 \rightarrow x_3 = 3$, $x_4 \geq 3 \wedge x_4 \leq 3 \rightarrow x_4 = 3$

Domain $D(x_1) = \{1\}$, $D(x_2) = \{2\}$, $D(x_3) = D(x_4) = \{3\}$, $D(x_5) = \{2\}$

.

Lazy Clause Generation example



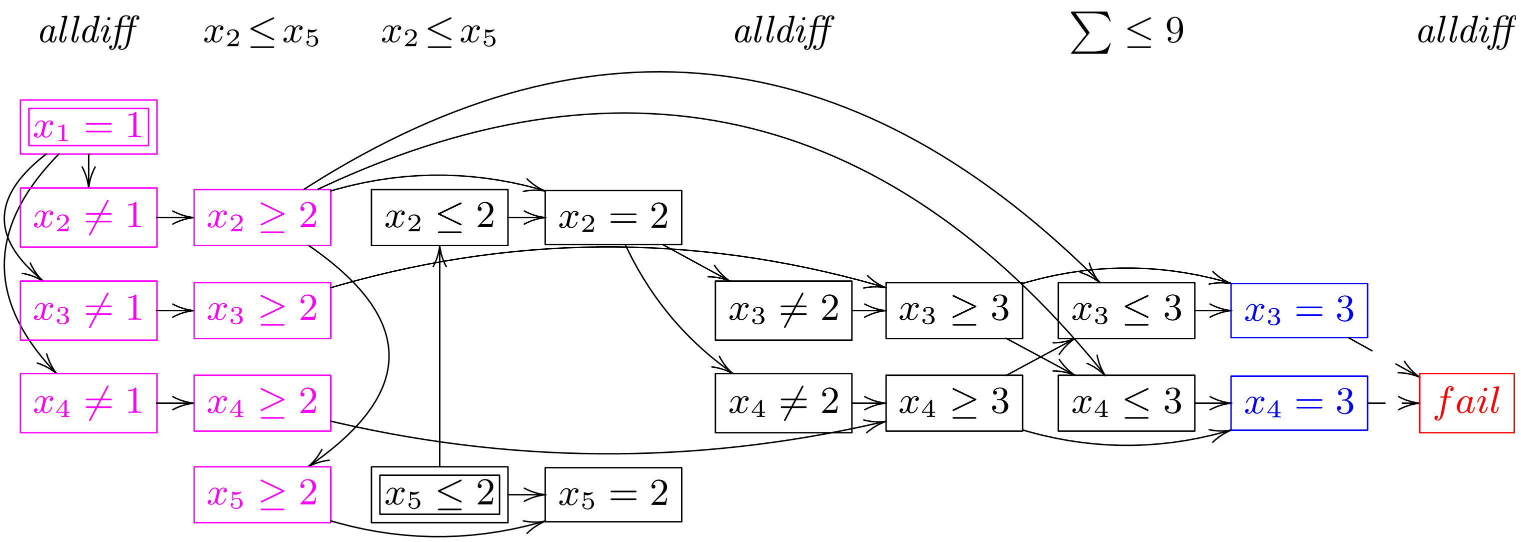
Propagate `alldifferent`$([x_1, x_2, x_3, x_4])$

Failure detected: explanation $x_3 = 3 \wedge x_4 = 3 \rightarrow \mathit{false}$

Lazy Clause Generation Example

	$\mathbf{x_1 = 1}$	*alldiff*	$x_2 \leq x_5$	$\mathbf{x_5 \leq 2}$	$x_2 \leq x_5$	*alldiff*	$\sum \leq 9$	*alldiff*
x_1	1	1	1	1	1	1	1	1
x_2	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$	2	2	2	2
x_3	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[3..4]$	3	$\emptyset$
x_4	$[1..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[2..4]$	$[3..4]$	3	$\emptyset$
x_5	$[1..4]$	$[1..4]$	$[2..4]$	2	2	2	2	2
			D_1					*fail*

Lazy Clause Generation Explanation

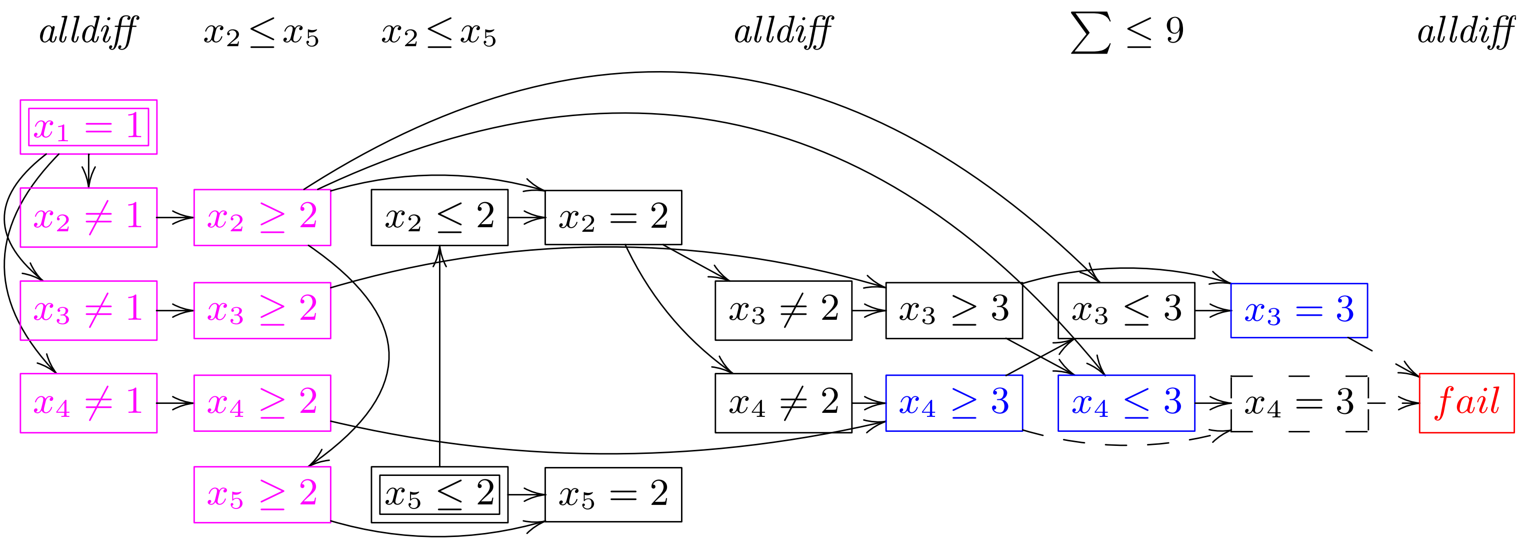


The initial nogood

$$x_3 = 3 \wedge x_4 = 3 \rightarrow \mathit{false}$$

.

Lazy Clause Generation Explanation

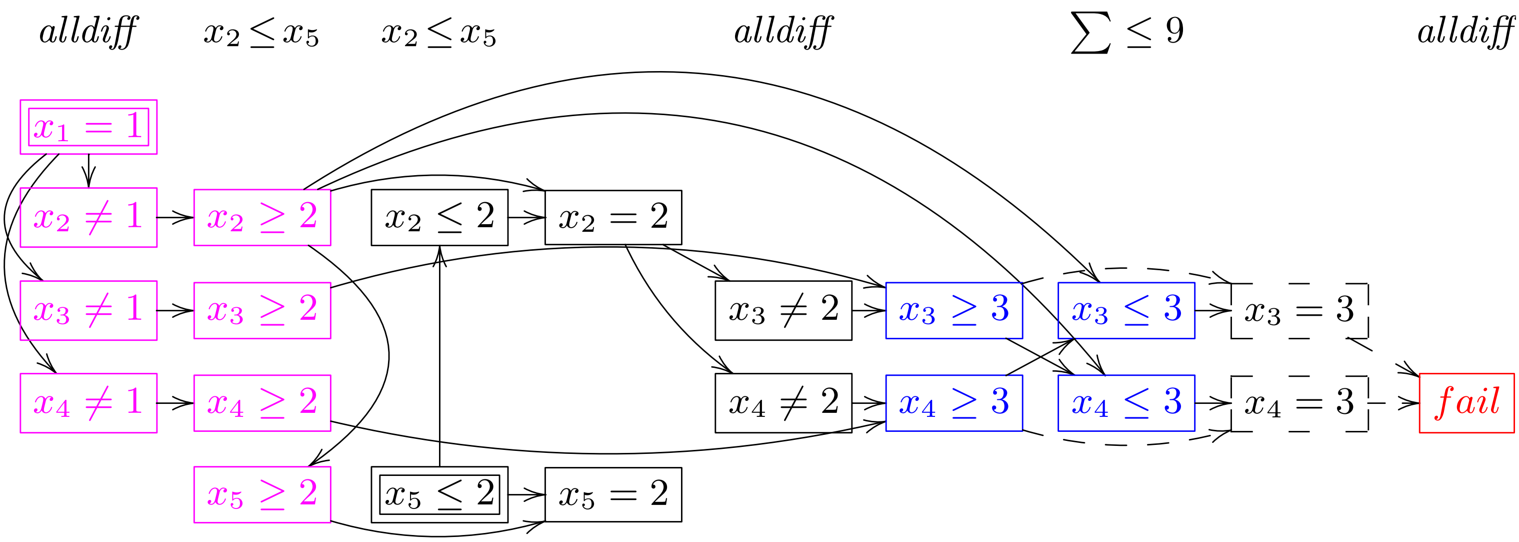


Resolving

$$x_4 \geq 3 \wedge x_4` \leq 3 \wedge x_3 = 3 \rightarrow \mathit{false}$$

.

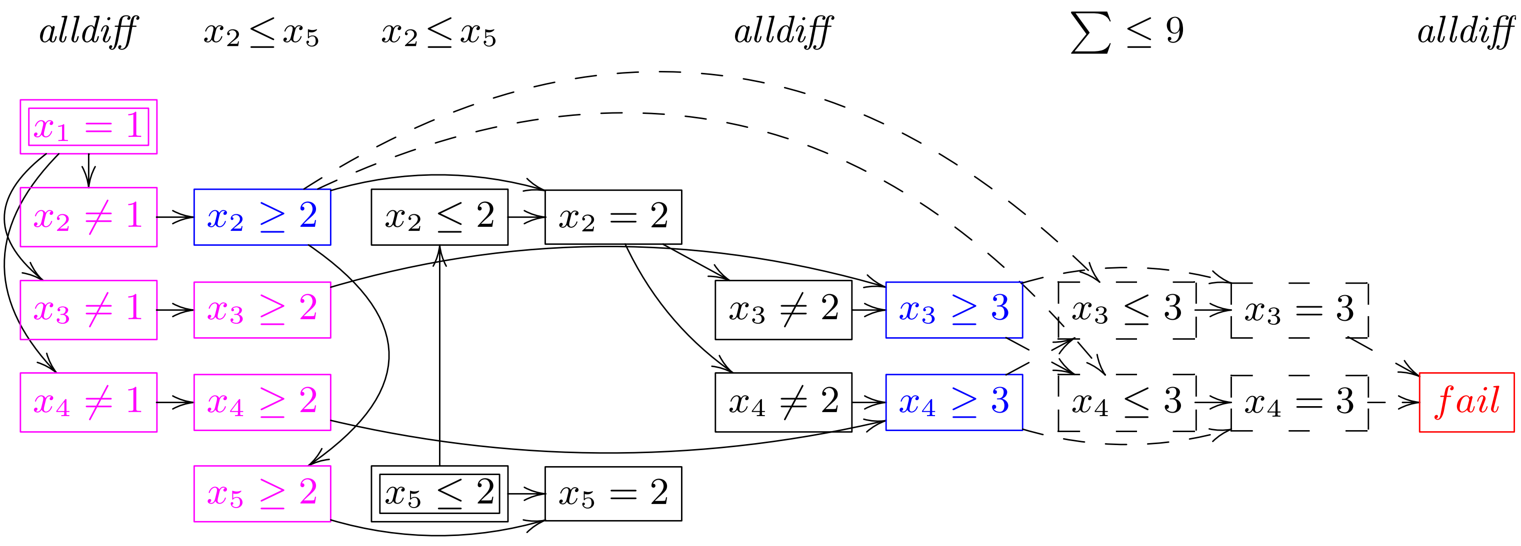
Lazy Clause Generation Explanation



Resolving

$$x_3 \geq 3 \wedge x_4 \geq 3 \wedge x_3 \leq 3 \wedge x_4 \leq 3 \rightarrow false$$

Lazy Clause Generation Explanation

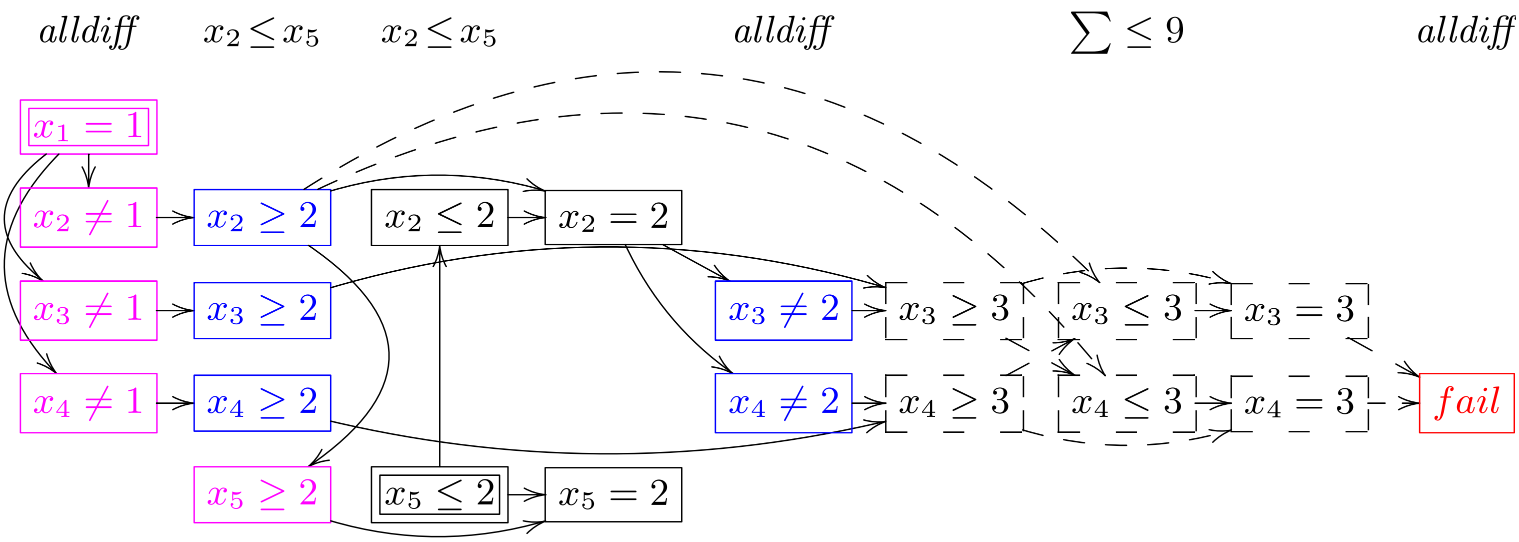


Resolving

$$x_2 \geq 2 \wedge x_3 \geq 3 \wedge x_4 \geq 3 \rightarrow \mathit{false}$$

.

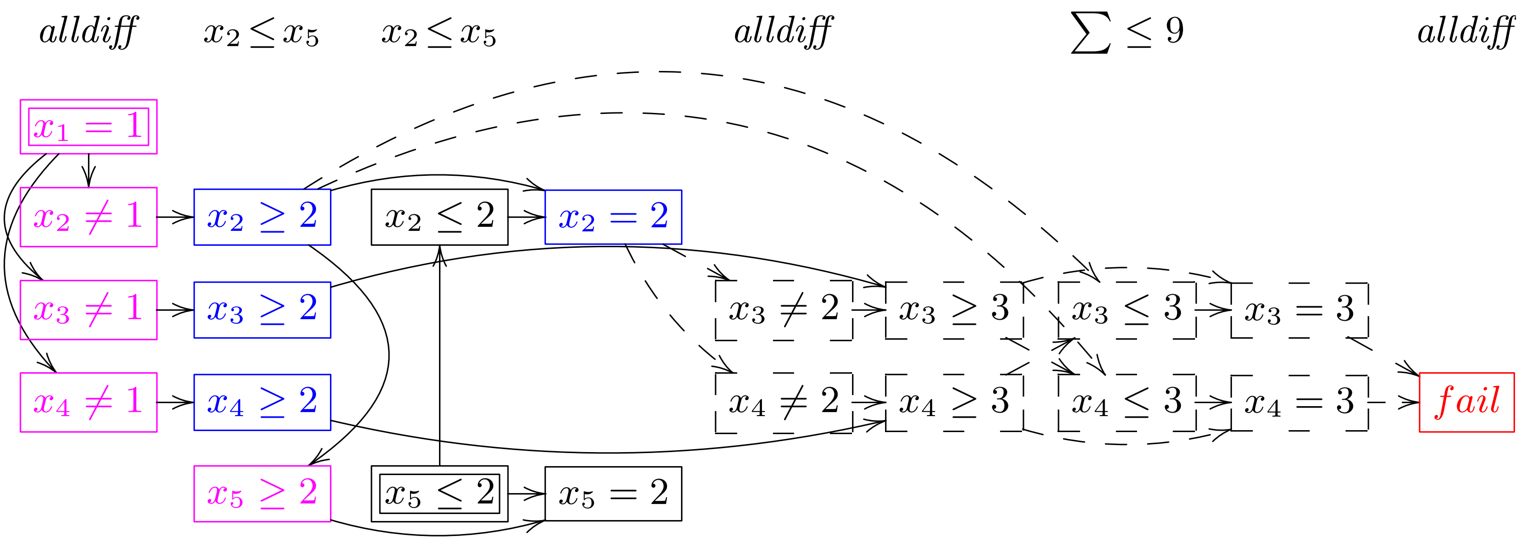
Lazy Clause Generation Explanation



Resolving

$$x_2 \geq 2 \land x_3 \geq 2 \land x_4 \geq 2 \land x_3 \neq 2 \land x_4 \neq 2 \rightarrow false$$

Lazy Clause Generation Explanation



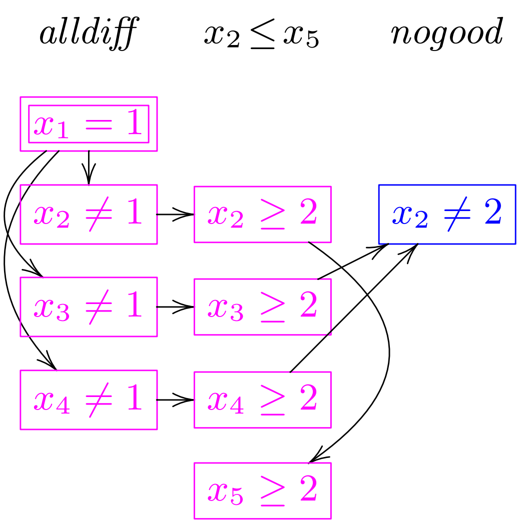
Resolving

$$x_2 \geq 2 \wedge x_3 \geq 2 \wedge x_4 \geq 2 \wedge x_2 = 2 \rightarrow false$$

Simplify! $\quad x_3 \geq 2 \wedge x_4 \geq 2 \wedge x_2 = 2 \rightarrow false$

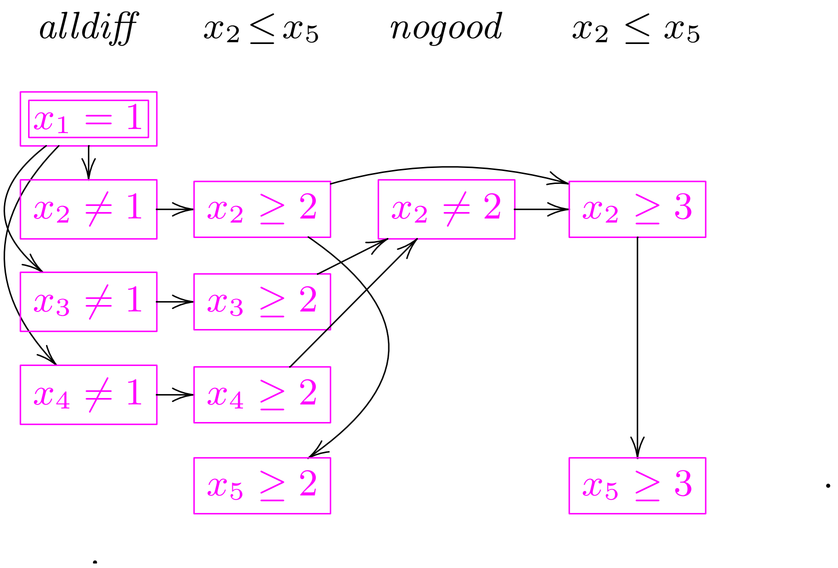
.

Lazy Clause Generation Example



Backjump
Propagate $x_3 \geq 2 \wedge x_4 \geq 2 \rightarrow x_2 \neq 2$

Lazy Clause Generation Example



Domain constraints determine $x_2 \geq 3$

Propagate $x_2 \leq x_5$ determines $x_5 \geq 3$

Different Domain

$D'_2(x_1) = \{1\}$, $D'_2(x_2) = D'_2(x_5) = [3..4]$, $D'_2(x_3) = D'_2(x_4) = [2..4]$

What's Really Happening

- A high level "Boolean" model of the problem
- Clausal representation of the Boolean model is generated "as we go"
- All generated clauses are redundant and can be removed at any time
- We can control the size of the active "Boolean" model

Comparing with SAT on Tai open shop scheduling: (averages)
SAT generates the full Boolean model before starting solving

	Time	*solve only*	*Fails*	Max Clauses Generated
SAT	318	(89)	**3597**	13.17
LCG	**62**		6611	**1.00**

Strengths and Weaknesses of Lazy Clause Generation

- Strengths
  - High level modelling
  - Learning avoids repeating the same subsearch
  - Strong autonomous search
  - Programmable search
  - Specialized global propagators (but requires work)
- Weaknesses
  - Optimization by repeated satisfaction search
  - Overhead compared to FD when nogoods are useless

Outline

Lazy Boolean Variable Creation

- Many Boolean variables are never used
- Create them on demand
- **Array encoding**
  - Create bounds variables initially $x \leq d$
  - Only create equality variables $x = d$ on demand
    Add $x \geq d \wedge x \leq d \rightarrow x = d$
- **List encoding**
  - Create bounds variables on demand $x \leq d$
    Add $x \leq d' \rightarrow x \leq d$, $x \leq d \rightarrow x \leq d''$ where d' (d'') is next lowest (highest) existing bound
  - At most $2\times$ bounds clauses
  - Create equality variables on demand as before

Lazy Boolean Variable Creation Tradeoffs

- List versus array
- List always works! Array may require too many variables
- Implementation complexity
- List hampers learning

Tai open shop scheduling: 15x15 (average of 10 problems)

	AverageTime
array	**13.38**
list	56.66

Views (Schulte + Tack, 2005)

- **View** is a pseudo variable defined by a "bijective" function to another variable
  - $x = \alpha y + \beta$
  - $x = \texttt{bool2int}(y)$
  - $x = \neg y$
- The view variable x, does not exist, operations on it are mapped to y
- More important for lazy clause generation
  - Reduce Boolean variable representation
  - Improve nogoods (reduce search)

Constrained path covering problems: Average of 5 problems

	Time	*Fails*
views	**0.71**	**950**
no views	1.12	1231

Explanation Deletion

- Explanations only really needed for nogood learning
  - **Forward** add explanations as they are generated
  - **Backward** delete explanations as we backtrack past them
- Smaller set of clauses
- Can hamper search "Reprioritization"

Tai open shop scheduling:

	15x15	20x20
deletion	**13.38**	**39.96**
no deletion	20.58	95.88

But RCPSP worse with deletion!

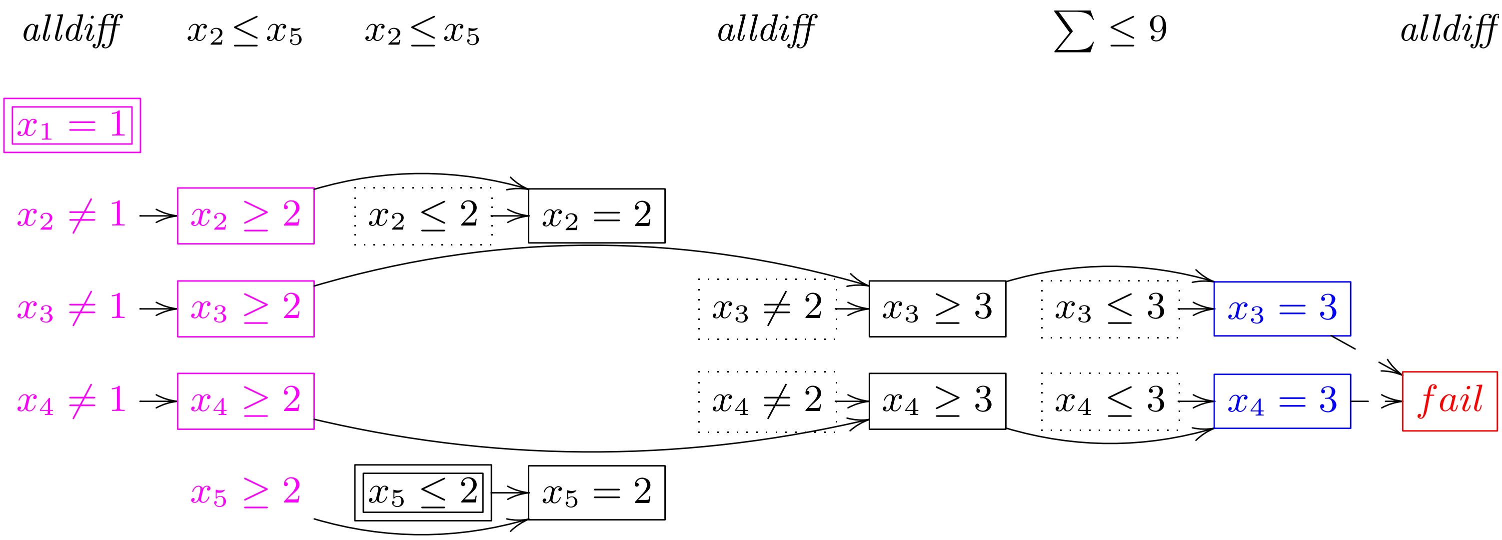
Lazy Explanation

- Explanations only needed for nogood learning
  - **Forward** record propagator causing each atomic constraint
  - **Backward** ask propagator to explain atomic constraint (if required)
- Standard for SAT extensions (MiniSAT 1.14) [See Gent et al PADL2010]
- Only create needed explanations!
- Harder implementation

Social Golfers Problems: using an MDD propagator
(each explanation as expensive as running entire propagator)

	Time	*Reasons*	*Fails*
lazy explanation	**2.38**	**14347**	**2751**
eager explanation	4.92	78177	5126

Lazy Clause Generation Explanation

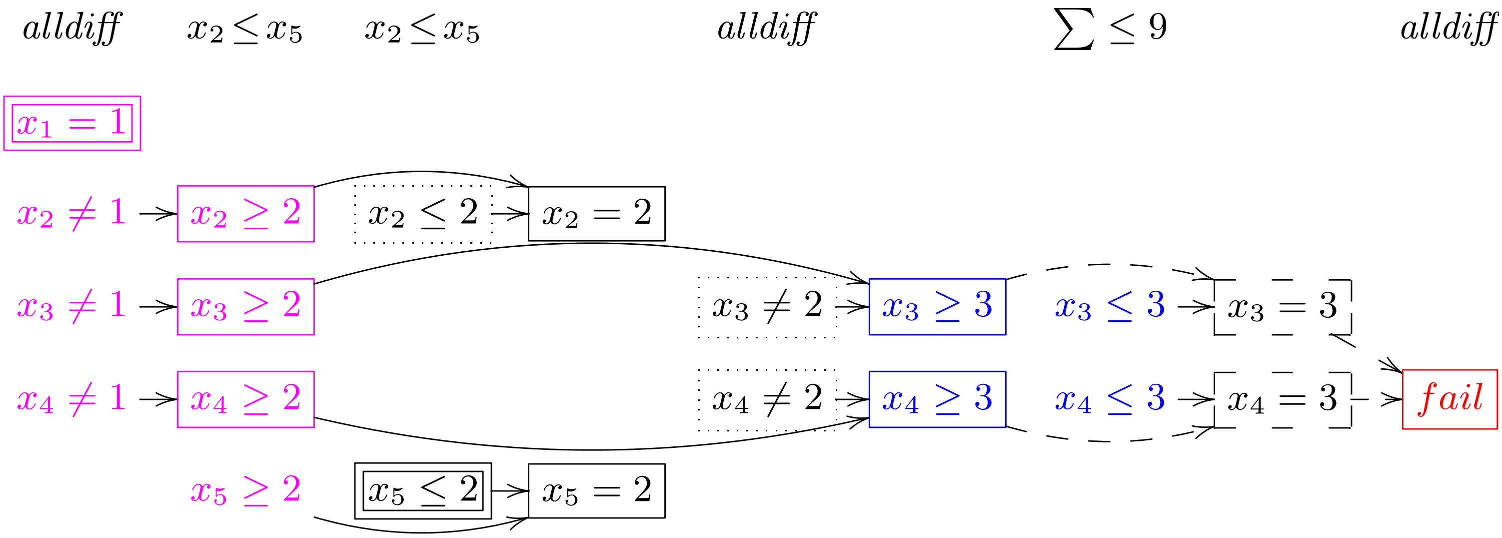


Dotted boxes explained by above propagator.
Initial nogood

$$x_3 = 3 \wedge x_4 = 3 \rightarrow fail$$

.

Lazy Clause Generation Explanation

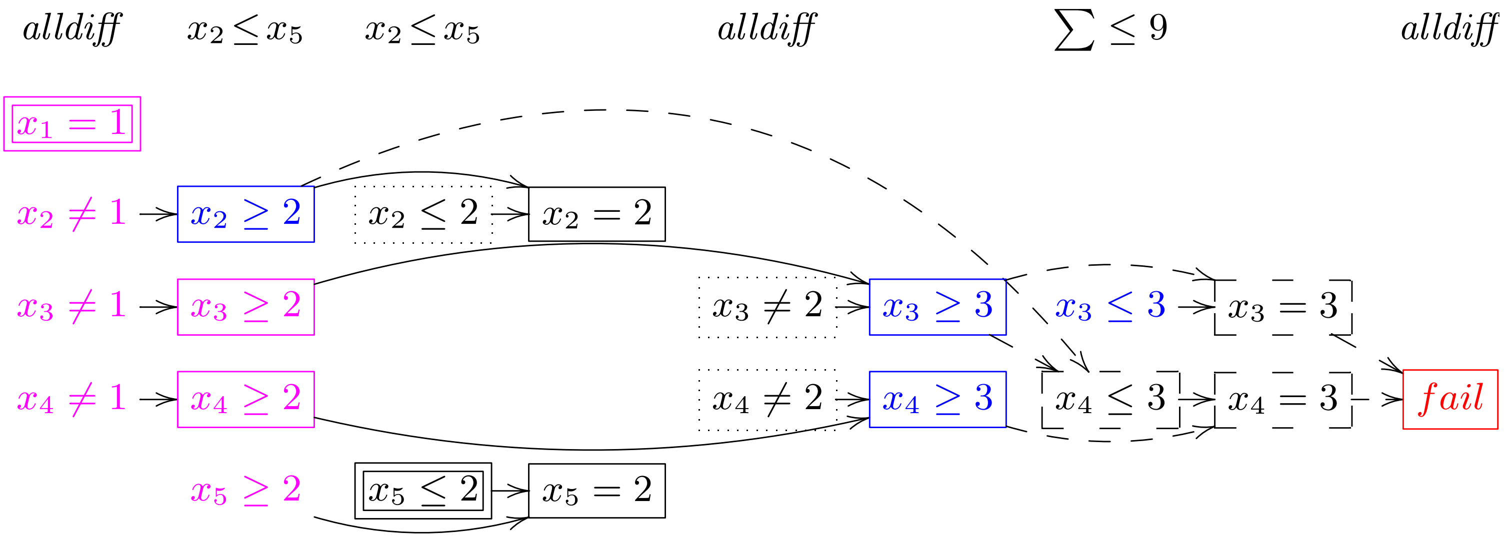


Resolving $x_3 \geq 3 \wedge x_3 \leq 3 \rightarrow x_3 = 3$ and $x_4 \geq 3 \wedge x_4 \leq 3 \rightarrow x_4 = 3$

$$x_3 \geq 3 \wedge x_4 \geq 3 \wedge x_3 \leq 3 \wedge x_4 \leq 3 \rightarrow \mathit{fail}$$

Request $x_1 + x_2 + x_3 + x_4 \leq 9$ to explain $x_4 \leq 3$

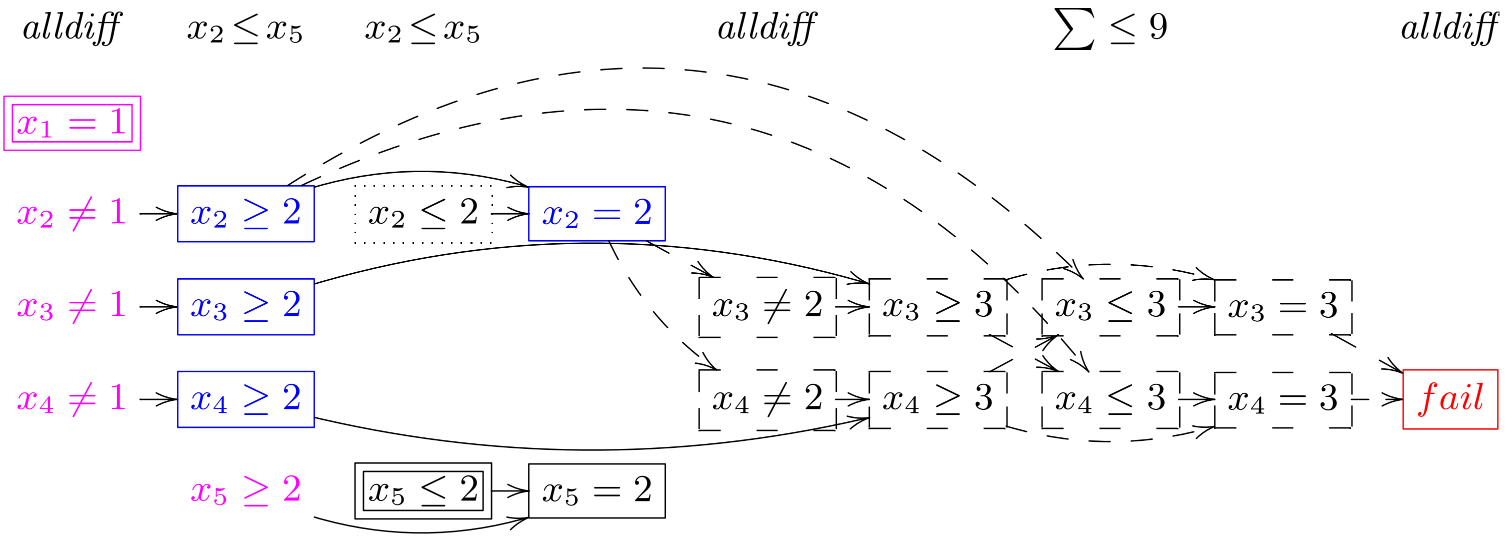
Lazy Clause Generation Explanation



Lazy Explanation $x_2 \geq 2 \wedge x_3 \geq 3 \rightarrow x_4 \leq 3$
Resolving on this gives

$$x_2 \geq 2 \wedge x_3 \geq 3 \wedge x_4 \geq 3 \wedge x_3 \leq 3 \rightarrow \mathit{fail}$$

Lazy Clause Generation Explanation



Final 1UIP nogood

$$x_2 \geq 2 \wedge x_3 \geq 2 \wedge x_4 \geq 2 \wedge x_2 = 2 \rightarrow \mathit{false}$$

Note 5 unexplained atomic constraints remain!

Outline

The Globality of Explanation

- Nogoods extract global information from the problem
- Can overcome weaknesses of local propagators
- **Example**
- $D(x_1) = D(x_2) = [\,0\,..\,100000\,]$ $x_2 \geq x_1 \wedge (b \Leftrightarrow x_1 > x_2)$
- Set $b = \mathit{true}$ and 200000 propagations later failure. YIKES
- A global difference logic propagator immediately sets $b = \mathit{false}$!
- Lazy clause generation learns $b = \mathit{false}$ after 200000 propagations
  - But never tries it again!

Globals by Decomposition

- Globals defined by decomposition
  - Don't require implementation
  - Automatically incremental
  - Allow partial state relationships to be "learned"
  - Much more attractive with lazy clause generation
- When propagation is not hampered, and size does not blowout:
  - can be good enough!

Resource constrained project scheduling problems: (`cumulative` by decomposition) closed 62 open problems % solved to optimality in time

	J60			J90			J120		
	45s	300s	1800s	45s	300s	1800s	45s	300s	1800s
Laborie	-	84.2	85.0	-	78.5	79.4	-	41.3	41.7
LCG	**85.2**	**88.1**	**89.4**	**79.8**	**81.3**	**82.5**	**42.5**	**44.8**	**45.3**

Which Decomposition?

- Different decompositions interact better or worse with lazy clause generation.
- `alldifferent`
  - **diseq**: $O(n^2)$ disequations
  - **bnd**: Bound consistent decomposition of Bessiere et al IJCAI09
  - **bnd+**: Bound consistent decomp. replacing $x \geq d \wedge x \leq d$ by $x = d$
  - **gcc**: Based on a simple global cardinality decomposition

Quasi-group completion 25x25 (average of examples solved by all)

$diseq(13)$		$bnd(11)$		$bnd+(13)$		$gcc(15)$		$CSPComp2008$ (13)	$CSPComp2008$ (12)
Time	*Fails*	*Time*	*Fails*	*Time*	*Fails*	*Time*	*Fails*	*Time*	*Time*
131	142680	757	9317	129	1144	**4.3**	**1010**	> 433	> 500

Explanations for Globals

- Globals are better than decomposition
  - More efficient
  - Stronger propagation
- Instrument global constraint to also explain its propagations
  - **mdd**: expensive each explanation as much as propagation
  - **cumulative**: choices in how to explain
- Implementation complexity, Can't learn partial state
- More efficient + stronger propagation

Resource constrained project scheduling problems:

	J60 (25% faster)			J90 (25% faster)			J120 (60% faster)		
	45s	300s	1800s	45s	300s	1800s	45s	300s	1800s
Decomp	84.8	**89.2**	89.4	79.8	81.7	82.5	42.3	45.2	45.7
Global	**85.8**	89.0	**89.6**	**80.0**	**81.9**	**82.7**	**42.7**	**45.8**	**47.0**

Outline

Nogoods and Programmed Search

- Contrary to SAT folklore
  - Activity based search can be terrible
  - Nogoods work excellently with programmed search

Constrained Path Covering Problems

	Time	*Fails*
nogoods + *VSIDS*	> 361.89	$> 30,000$
nogoods + *programmed*	**0.71**	**950**
programmed	> 240.2	$> 10,000$

Activity-based search

- An excellent default search!
- Weak at the beginning (no meaningful activities)
- Need hybrid approachs
  - Hot Restart:
    - Start with programmed search to "initialize" meaningful activities.
    - Switch to activity-based after restart
  - Use activity-based as part of a programmed search
- Much more to explore in this direction

Outline

SAT modulo theories (SMT)

- Combine a SAT solver with theory solvers to handle non Boolean constraints.
- (Original) Lazy Clause Generation is a special case
  - Each propagator is its own theory
  - Propagators do "theory propagation"
- Differences
  - LCG transmits "lower level" information
  - LCG learns "finer" nogoods
  - LCG supports programmed search
  - Global Propagators $\approx$ Theories
- Sometimes the theory view is better:
  - modulo arithmetic + Radio Link Frequency Assignment
- Sometimes finer nogoods are better
  - separation logic + Open Shop Scheduling
- Eventually the approaches will merge!

Generalized Nogoods (g-nogoods)

- Nogood learning has a long history in Constraint Programming
  - longer than in SAT?
- Traditional Nogoods: $x_1 = d_1 \wedge \cdots \wedge x_n = d_n \rightarrow \mathit{fail}$
- Generalized Nogoods: $x_1 \overset{=}{\neq} d_1 \wedge \cdots \wedge x_n \overset{=}{\neq} d_n \rightarrow \mathit{fail}$
  - Introduced by Katsirelos and Bacchus 2003
  - Used SAT technology for propagation (watched literals)
  - Equivalent to lazy clause generation without bounds constraints
  - Interesting 1UIP nogoods not effective?
  - Also defined global explanation approach for `alldifferent`
  - Didnt consider activity, forgetting and VSIDS search

Mixed Integer Programming

- Strengths
  - Can deal with 100K variables 1M linear constraints
  - Strong autonomous search
  - "Knows" where the good solutions are
- Weaknesses
  - Have to model using only linear constraints

Can we get add the optimization strength of MIP to lazy clause generation?

SCIP: Solving Constraint Integer Programs

Hybrid constraint programming and mixed integer programming (MIP)

- Linear constraints as propagators and part of global MIP
- MIP propagator explains failures (and fathoming) as nogoods

$$x_1 \begin{array}{c}\leq\\ \geq\end{array} d_1 \wedge \cdots x_n \begin{array}{c}\leq\\ \geq\end{array} d_n \rightarrow \mathit{fail}$$

- Propagates these using SAT technology
- Creates ALLUIP nogoods for MIP failures
- Very good results on some hard MIP problems

Lazy Clause Generation and MIP?

- Mixed integer programming (MIP) solvers know where the good solutions are
- Lazy clause generation and MIP are compatible
  - MIP engine explains failure and fathoming (and reduced cost bounds changes)
  - Treated like an other global propagator
  - SCIP is a lazy clause generation MIP solver!
- In order to use the MIP advantage it probably directs search
- SCIP default search:
  - pseudo costs (MIP), then activity (SAT), then impact (CP)
- Plenty more to discover on the best interaction! (see our short paper)

Outline

Conclusion

Lazy Clause Generation

- High level modelling
- Strong nogood creation
- Effective autonomous search
- Global Constraints

Defines state-of-the-art for:

- Resource constrained project scheduling (minimize makespan)
- Set constraint problems
- Nonagrams (regular constraints)

Usually 1-2 order of magnitude speedup on FD problem

Future Research

Plenty of better engineering yet to be done

Plenty of open research questions

- Best combinination with MIP solving
- Hybrid search: structured + activity based
- Parallelism
- SAT Modulo Theories and Lazy Clause Generation
- Adaptive Behaviour

Questions