Chapter 8: Modelling with Finite Domain Constraints

Where we examine how modelling and controlling search interact with finite domain constraints

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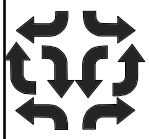

Modelling wth Finite Domains

- Domains and Labelling
- Complex Constraints
- Labelling
- Different Problem Modellings
- Example: Scheduling

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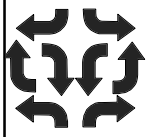

Domains and Labelling

- New requirement: need to define the domains of variables
  - *arithmetic* `X :: [1,2,3,4]` or `X :: [1..4]`
  - *non-arithmetic* `X :: [red, blue, yellow]`
  - *multiple variables* `[X,Y,Z] :: [0..10]`
- If no domain is given a default is used (say [-10000..10000]

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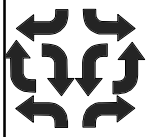

Domains Example

Goal for the smugglers knapsack problem:

```
[W,P,C] :: [0..9], 4*W + 3*P + 2*C <= 9,
15*W + 10*P + 7*C >= 30.
```

$$D(W) = [0..2], D(P) = [0..3], D(C) = [0..4]$$

The CLP(FD) system returns *unknown*.

But how do we find a solution?

Invoke a **complete** (backtracking) solver

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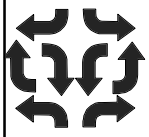

Labelling

- Built-in predicate `labeling` invokes the complete constraint solver
- `labeling(Vs)` takes a list of finite domain variables *Vs* and finds a solution

```
[W,P,C] :: [0..9], 4*W + 3*P + 2*C <= 9,
15*W + 10*P + 7*C >= 30, labeling([W,P,C]).
```

has solutions:

$$W = 0 \wedge P = 1 \wedge C = 3 \quad W = 0 \wedge P = 3 \wedge C = 0$$
$$W = 1 \wedge P = 1 \wedge C = 1 \quad W = 2 \wedge P = 0 \wedge C = 0$$

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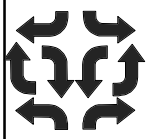

Constrain and Generate

- Typical form of a finite domain program
  - variables and domains are defined
  - constraints modelling problem are given
  - labelling is added to invoke complete solver
- Minimization can be applied on labelling

```
[W,P,C] :: [0..9], 4*W + 3*P + 2*C <= 9,
15*W + 10*P + 7*C >= 30,
minimize(labeling([W,P,C]), -15*W-10*P-7*C).
```

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Send+More=Money Example

Cryptarithmetic problem, each digit is different and the equation holds

$$\begin{array}{cccccc} & & S & E & N & D \\ + & & M & O & R & E \\ \hline = & M & O & N & E & Y \end{array}$$

```
smm(S,E,N,D,M,O,R,Y) :-
     [S,E,N,D,M,O,R,Y] :: [0..9],
     constrain(S,E,N,D,M,O,R,Y),
     labeling([S,E,N,D,M,O,R,Y]).
constrain(S,E,N,D,M,O,R,Y) :-
     S != 0, M != 0,
     alldifferent_neq([S,E,N,D,M,O,R,Y]),
                 1000*S + 100*E + 10*N + D
     +           1000*M + 100*O + 10*R + E
     = 10000*M + 1000*O + 100*R + 10*E + Y.
```

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Send+More=Money Example

After domain declarations:

$$D(S) = D(E) = D(N) = D(D) = D(M) = D(O) = D(R) = D(Y) = [0..9]$$

After $S \neq 0 \wedge M \neq 0$ then $D(S) = D(M) = [1..9]$

`alldifferent_neq` adds disequations (no chng)

final equation: (one propagation rule)

$$M \leq \frac{1}{9}\max(D,S) + \frac{91}{9000}\max(D,E) + \frac{1}{9000}\max(D,D) + \frac{1}{900}\max(D,R) - \frac{1}{10}\min(D,O) - \frac{1}{100}\min(D,N) - \frac{1}{9000}\min(D,Y)$$

With the current domain:

$$M \leq \frac{9}{9} + \frac{91\times 9}{9000} + \frac{9}{9000} + \frac{9}{900} - \frac{0}{10} - \frac{0}{100} - \frac{0}{9000} = 1.102$$

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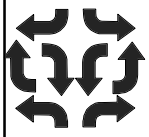

Send+More=Money Example

Hence *D(M) = [1..1]*

Propagation continues arriving at

$D(S) = [9..9]$ $D(E) = [4..7]$ $D(N) = [5..8]$ $D(D) = [2..8]$

$D(M) = [1..1]$ $D(O) = [0..0]$ $D(R) = [2..8]$ $D(Y) = [2..9]$

Note: 3 variables fixed, all domains reduced

Labelling tries S=0, S=1, ..., S=8 (which fail) and then S=9. Then E=0, E=1, E=2, E=3 (which fail) and E=4 (this fails eventually), then E=5 gives

$D(S) = [9..9]$ $D(E) = [5..5]$ $D(N) = [6..6]$ $D(D) = [7..7]$

$D(M) = [1..1]$ $D(O) = [0..0]$ $D(R) = [8..8]$ $D(Y) = [2..2]$

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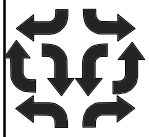

Generate and Test

Methodology without constraints:

first generate a valuation and

then test if it is a solution

```
smm(S,E,N,D,M,O,R,Y) :-
     [S,E,N,D,M,O,R,Y] :: [0..9],
     labeling([S,E,N,D,M,O,R,Y]),
     constrain(S,E,N,D,M,O,R,Y).
```

This program requires 95671082 choices before finding the solution, the previous required 35 (or 2 that did not immediately fail)

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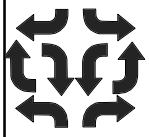

Complex Constraints

- Complex constraints such as *alldifferent, cumulative* and *element*
- More succinct problem modelling
- Better propagation = more efficiency
- Example replace
  - `alldifferent_neq([S,E,N,D,M,O,R,Y])`
  - `alldifferent([S,E,N,D,M,O,R,Y])`

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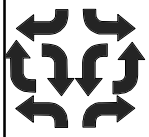
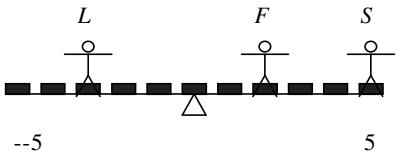


Complex Constraints

- There use may not always be obvious

10 foot seesaw with seats, Liz, Fi and Sarah want to sit 3 feet apart and balance. They weight 9,8,4 stone (resp).




```
apart(X,Y,N) :- X >= Y + N.
apart(X,Y,N) :- Y >= X + N.

[L,F,S] :: [-5..5], 9*L + 8*F + 4*S = 0,
apart(L,F,3), apart(L,S,3), apart(F,S,3),
labeling([L,F,S)]
```

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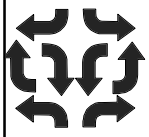
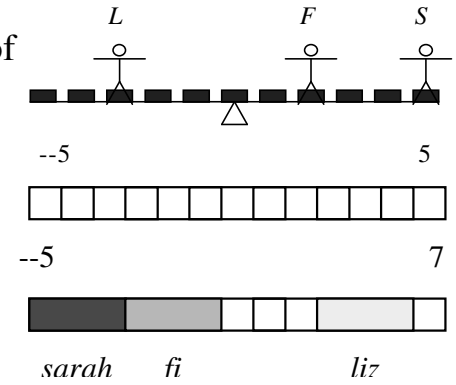


Complex Constraints

Think of each girl as a box of width 3 fitting into a box of length 13

This is an example of a *cumulative* constraint




```
[L,F,S] :: [-5..5],
9*L + 8*F + 4*S = 0,
cumulative([L,F,S],[3,3,3],[1,1,1],1)
labeling([L,F,S)]
```

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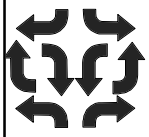

Labelling

- Labelling can be programmed in CLP(FD) using built-in predicates
  - `dom(V,D)` list of values *D* in current domain *V*
  - `maxdomain(V,M)` current max value *M* of *V*
  - `mindomain(V,M)` current min value *M* of *V*

```
labeling([]).
labeling([V|Vs]) :- indomain(V), labeling(Vs).
indomain(V) :- dom(V,D), member(V,D).
```

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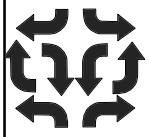

Labelling

- Makes use of current domains of variables
- Example (Send-More-Money)

$D(S) = [9..9]$ $D(E) = [4..7]$ $D(N) = [5..8]$ $D(D) = [2..8]$
$D(M) = [1..1]$ $D(O) = [0..0]$ $D(R) = [2..8]$ $D(Y) = [2..9]$

Labelling tries S=9. Then E=4 (this fails eventually), then E=5 which gives

$D(S) = [9..9]$ $D(E) = [5..5]$ $D(N) = [6..6]$ $D(D) = [7..7]$
$D(M) = [1..1]$ $D(O) = [0..0]$ $D(R) = [8..8]$ $D(Y) = [2..2]$

Labelling then add N=6, D=7, M=1, O=0, R=9, and Y=2, and succeeds

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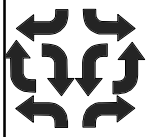

Labelling

- There are two choices made in labelling
  - choice of which variable to label first
  - choice of which value to try first
- Default labelling
  - try variables in order of the given list
  - try value in order min to max (returned by dom)
- We can program different strategies

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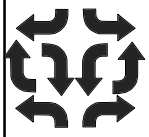

Choice of Variable

- Variable choice effects size of deriv. tree
- choose variable with smallest current domain (hence smallest possible answers)
- Example

$D(S) = [9..9]$ $D(E) = [4..7]$ $D(N) = [5..8]$ $D(D) = [2..8]$
$D(M) = [1..1]$ $D(O) = [0..0]$ $D(R) = [2..8]$ $D(Y) = [2..9]$

Labelling first tries S, M, O (need do nothing), then either E or N. Much better than Y first

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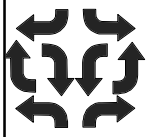

First-Fail Labelling

```
labelingff([]).
labelingff(Vs) :- deleteff(Vs,V,R),
     indomain(V), labelingff(R).
deleteff([V0|Vs],V,R) :-
     getsize(V0,S), minsize(Vs,S,U,V,R)).
minsize([],_,V,V,[]).
minsize([V1|Vs],S0,V0,V,[V0|R]) :-
     getsize(V1,S1), S1 < S0,
     minsize(Vs,S1,V1,V,R).
minsize([V1|Vs],S0,V0,V,[V1|R]) :-
     getsize(V1,S1), S1 >= S0,
     minsize(Vs,S0,V0,V,R).
```

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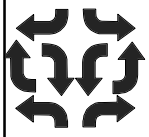

First-Fail Labelling

- `labellingff` selects variable with least domain to label first
- `minsize(Vs,S0,V0,V,R)` searchs through variables *Vs* to find var *V* with minimal size with current min var *V0* and size *S0*, *R* is the other variables.
- Application of **first-fail principle**
- *"To succeed, try first where you are most likely to fail"*

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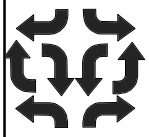

Choice of Domain Value

- Value choice only effects order in which branches are explored
- Problem specific knowledge may lead to finding a solution faster
- Also important for optimization
  - good solutions found earlier reduce search later

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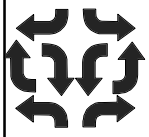
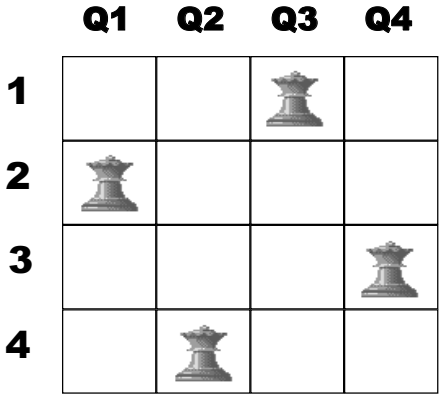


N-Queens Example

Problem of placing *N* queens on an *N*x*N* chessboard so that none can take another.

For large *N* solutions are more likely with values in the middle of variable domains



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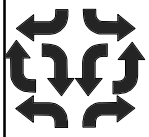

Middle-Out Ordering

```
indomain_mid(V) :-
     ordervalues(V,L), member(V,L).
ordervalues(V,L) :-
     dom(V,D), length(D,N), H = N//2,
     halvelist(H,D,[],F,S), merge(S,F,L).
halvelist(0,L2,L1,L1,L2).
halvelist(N,[E|R],L0,L1,L2) :-
     N >= 1, N1 = N - 1,
     halvelist(N1,R,[E|L0],L1,L2).
merge([],L,L).
merge([X|L1],L2,[X|L3]) :- merge(L2,L1,L3).
```

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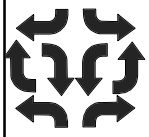

Middle-Out Ordering

- `indomain_mid(V)` trys the values in the domain of *V* in middle out order
- `ordervalues` creates the ordered list of values
- `halvelist` breaks a list into two halves, the first half reversed
- `merge(L1,L2,L3)` interleaves two lists *L1, L2* to get *L3*

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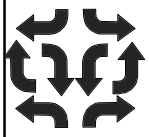
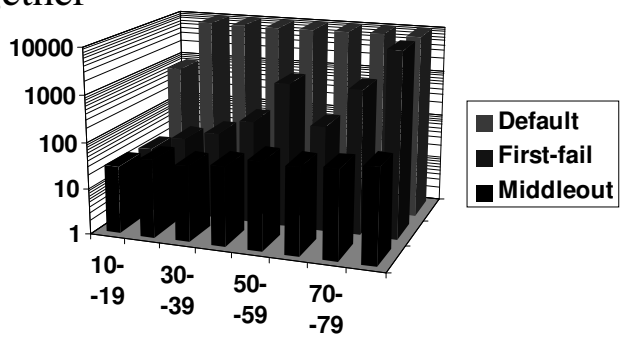


Labelling Efficiency

- We can use both variable and value ordering together

Different number of choices for N-queens in different ranges



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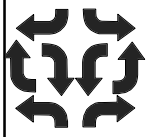

Domain Splitting

- Labelling doesnt have to be
  - setting a variable to a value
- Simply has to reduce the domain of a variable
- **Domain splitting** splits the current domain of chosen variable in half

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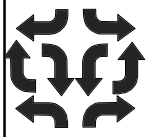

Domain Splitting

```
labelingsplt([]).
labelingsplt([V|Vs]) :-
     mindomain(V,Min), maxdomain(V,Max),
     (Min = Max -> NVs = Vs
     ;    Mid = (Min+Max)//2,
          labelsplt(V,Mid)
          append(Vs,[V],NVs)
     ),
     labelingsplt(NVs).
labelsplt(V,M) :- V <= M,.
labelsplt(V,M) :- V >= M+1.
```

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Domain Splitting

- `labelingsplt` recursively splits the domains of each variable in half until all domains are singletons. If *Min*=*Max* the variable is eliminated from list, otherwise it is split and added on the end of list of vars to be split.
- `labelsplt(V,M)` choose *V* to be either <= or > midpoint *M*

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Different Problem Modellings

- Different views of the problem lead to different models
- Different model = different set of variables
- Depending on solver capabilities one model may require less search to find answer
- Empirical comparison may be worthwile

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Different Problem Modellings

Simple assignment problem: four workers *w1,w2,w3,w4* and four products *p1,p2,p3,p4*. Assign workers to products to make profit >= 19

Profit matrix is:

	*p*1	*p*2	*p*3	*p*4
*w*1	7	1	3	4
*w*2	8	2	5	1
*w*3	4	3	7	2
*w*4	3	1	6	3

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Operations Research Model

16 Boolean variables *Bij* meaning worker *i* is assigned product *j*

$$\bigwedge_{i=1}^{4} \sum_{j=1}^{4} B_{ij} = 1$$

$$\bigwedge_{j=1}^{4} \sum_{j=1}^{4} B_{ij} = 1$$

$$P = \begin{array}{llll} 7B_{11} & +B_{12} & +3B_{13} & +4B_{14} \\ +8B_{21} & +2B_{22} & +5B_{23} & +B_{24} \\ +4B_{31} & +3B_{32} & +7B_{33} & +2B_{34} \\ +3B_{41} & +B_{42} & +6B_{43} & +3B_{44} \end{array}$$

$$P \geq 19$$

	*p*1	*p*2	*p*3	*p*4
*w*1	7	1	3	4
*w*2	8	2	5	1
*w*3	4	3	7	2
*w*4	3	1	6	3

B23

11 prim. constraints

28 choices to find all four solutions

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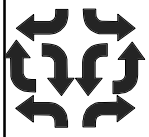

Better Model

Make use of disequality and complex constraints. Four variables *W1,W2,W3,W4* correponding to workers

$$alldifferent([W1,W2,W3,W4]) \wedge$$
$$element(W1,[7,1,3,4],WP1) \wedge$$
$$element(W2,[8,2,5,1],WP2) \wedge$$
$$element(W3,[4,3,7,2],WP3) \wedge$$
$$element(W4,[3,1,6,3],WP4) \wedge$$
$$P = WP1 + WP2 + WP3 + WP4 \wedge$$
$$P \geq 19$$

		1	2	3	4
		$p1$	$p2$	$p3$	$p4$
W1	$w1$	7	1	3	4
W2	$w2$	8	2	5	1
W3	$w3$	4	3	7	2
W4	$w4$	3	1	6	3

7 prim. constraints

14 choices to find all four solutions

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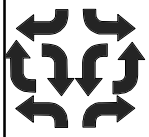

Different Model

Four variables *T1,T2,T3,T4* correponding to products.

$$alldifferent([T1,T2,T3,T4]) \wedge$$
$$element(T1,[7,8,4,3],TP1) \wedge$$
$$element(T2,[1,2,3,1],TP2) \wedge$$
$$element(T3,[3,5,7,6],TP3) \wedge$$
$$element(T4,[4,1,2,3],TP4) \wedge$$
$$P = TP1 + TP2 + TP3 + TP4 \wedge$$
$$P \geq 19$$

		T1	T2	T3	T4
		$p1$	$p2$	$p3$	$p4$
1	$w1$	7	1	3	4
2	$w2$	8	2	5	1
3	$w3$	4	3	7	2
4	$w4$	3	1	6	3

7 prim. constraints

7 choices to find all four solutions

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Comparing Models

- Relative efficiency comes from
  - more direct mapping to prim. constraints
  - fewer variables
  - usually requires empirical evaluation
- Other criteria
  - flexibility (adding new constraints)
  - e.g. worker 3 works on product > worker 2

$B_{31} = 0 \wedge B_{24} = 0 \wedge$
$B_{32} < B_{21} \wedge$
$B_{33} \leq B_{21} + B_{22} \wedge$
$B_{34} \leq B_{21} + B_{22} + B_{23} \wedge$

$W3 > W2$?????

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Combining Models

- Combine models by relating the variables and there values in each model
- eg. B13 = 1 means W1 = 3 means T3 = 1
- Combined models can gain more information through propagation
- Suppose prim. constraint *iff(V1,D1,V2,D2)* which holds if *V1=D1* iff *V2=D2*

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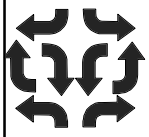
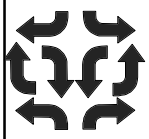


Combined Model

$$
\begin{array}{ll}
alldifferent([W1,W2,W3,W4]) \wedge & alldifferent([T1,T2,T3,T4]) \wedge \\
element(W1,[7,1,3,4],WP1) \wedge & element(T1,[7,8,4,3],TP1) \wedge \\
element(W2,[8,2,5,1],WP2) \wedge & element(T2,[1,2,3,1],TP2) \wedge \\
element(W3,[4,3,7,2],WP3) \wedge & element(T3,[3,5,7,6],TP3) \wedge \\
element(W4,[3,1,6,3],WP4) \wedge & element(T4,[4,1,2,3],TP4) \wedge \\
P = WP1 + WP2 + WP3 + WP4 \wedge & P = TP1 + TP2 + TP3 + TP4 \\
P \geq 19 \wedge &
\end{array}
$$

$$
\begin{array}{l}
iff\,(W1,1,T1,1) \wedge iff\,(W1,2,T2,1) \wedge iff\,(W1,3,T3,1) \wedge iff\,(W1,4,T4,1) \wedge \\
iff\,(W2,1,T1,2) \wedge iff\,(W2,2,T2,2) \wedge iff\,(W2,3,T3,2) \wedge iff\,(W2,4,T4,2) \wedge \\
iff\,(W3,1,T1,3) \wedge iff\,(W3,2,T2,3) \wedge iff\,(W3,3,T3,3) \wedge iff\,(W3,4,T4,3) \wedge \\
iff\,(W4,1,T1,4) \wedge iff\,(W4,2,T2,4) \wedge iff\,(W4,3,T3,4) \wedge iff\,(W4,4,T4,4)
\end{array}
$$

39 prim. constraints. 5 choices to find all solutions [35]



Example: Scheduling

- Given a set of tasks
  - with precedences (one finished before another)
  - and shared resources (some require same machine)
- Determine a suitable schedule, so
  - constraint are satisfied
  - overall time is minimized
- We will start with just a fixed time limit

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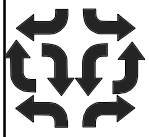
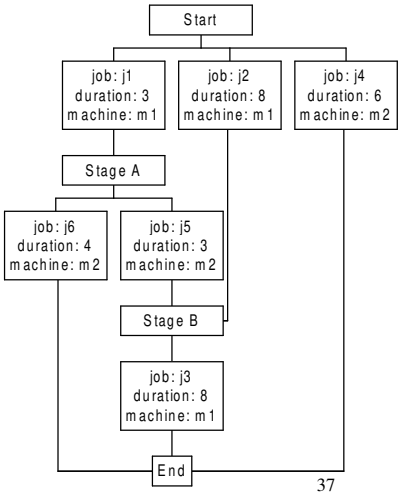


Small Schedule Data

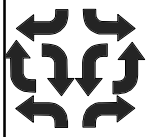
Represent data as a list of task records: (call it *problem*)

task(name,duration,[names],machine)

```
[task(j1,3,[],m1),
 task(j2,8,[],m1),
 task(j3,8,[j4,j5],m1),
 task(j4,6,[],m2),
 task(j5,3,[j1],m2),
 task(j6,4,[j1],m2)]
```



37



Scheduling Program

- Form of the program
  - Define variables: `makejobs`
    - Variables: Start time for each job
    - association list: *job(name,duration,StartVar)*
  - Precedence constraints: `precedences`
  - Machine constraints: `machines`
  - Labelling: `labeltasks`
    - get variables from job list and label

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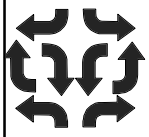

Scheduling Program

```
schedule(Data,End,Jobs) :-
     makejobs(Data, Jobs, End),
     precedences(Data, Jobs),
     machines(Data, Jobs),
     labeltasks(Jobs).
makejobs([],[],_).
makejobs([task(N,D,_,_)|Ts],
          [job(N,D,TS)|Js], End) :-
     TS :: [0..1000], TS + D <= End,
     makejobs(Ts,Js,End).
getjob(JL,N,D,TS) :- member(job(N,D,TS),JL).
```

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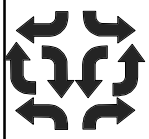

Scheduling: Precedences

```
precedences([],_).
precedences([task(N,_,P,_)|Ts],JL) :-
     getjob(JL,N,_,TS),
     prectask(P,TS,JL),
     prececedences(Ts,JL).

prectask([],_,_).
prectask([N|Ns],TS0,JL),
     getjob(JL,N,D,TS1),
     TS1 + D <= TS0,
     prectask(Ns,TS0,JL).
```

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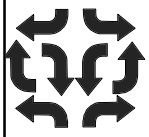

Scheduling: Machines

```
machines([],_).
machines([task(N,_,_,M)|Ts],JL) :-
     getjob(JL,N,D,TS),
     machtask(Ts,M,D,TS,JL),
     machines(Ts,JL).
machtask([],_,_,_,_).
machtask([task(N,_,_,M1)|Ts],M0,D0,TS0,JL):-
     (M = M0 -> getjob(JL,N,D1,TS1),
                exclude(D0,TS0,D1,TS1)
     ;          true),
     machtask(Ts,M0,D0,TS0,JL).
exclude(_,TS0,D1,TS1) :- D1 + TS1 <= TS0.
exclude(D0,TS0,_,TS1) :- D0 + TS0 <= TS1.
```

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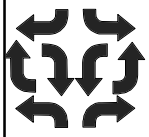

Executing Scheduling

`End=20, schedule(`*problem*`, End, JL).`

`makejobs`: Builds initial joblist and adds constraints

```
[job(j1,3,TS1),job(j2,8,TS2),job(j3,8,TS3),
job(j4,6,TS4),job(j5,3,TS5),job(j6,4,TS6)]
```

$$0 \le TS1 \quad TS1+3 \le 20 \quad 0 \le TS2 \quad TS2+8 \le 20$$
$$0 \le TS3 \quad TS3+8 \le 20 \quad 0 \le TS4 \quad TS4+6 \le 20$$
$$0 \le TS5 \quad TS5+3 \le 20 \quad 0 \le TS6 \quad TS6+4 \le 20$$

Initial variable domains:

$$D(TS1)=[0..17] \quad D(TS2)=[0..12] \quad D(TS3)=[0..12]$$
$$D(TS4)=[0..14] \quad D(TS5)=[0..17] \quad D(TS6)=[0..16]$$

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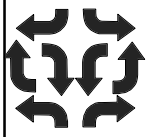

Executing Scheduling

`precedences`: adds constraints and changes domain

$$TS1 + 3 \le TS5 \wedge TS1 + 3 \le TS6 \wedge TS4 + 6 \le TS3 \wedge TS5 + 3 \le TS3$$

$$D(TS1) = [0..6] \quad D(TS2) = [0..12] \quad D(TS3) = [6..12]$$
$$D(TS4) = [0..6] \quad D(TS5) = [3..9] \quad D(TS6) = [3..16]$$

`machines`: adds choices of constrs, changes domain

$$TS2 + 8 \le TS1 \vee \underline{TS1 + 3 \le TS2} \quad TS3 + 8 \le TS1 \vee \underline{TS1 + 3 \le TS3}$$

$$D(TS1) = [0..0] \quad D(TS2) = [3..4] \quad D(TS3) = [12..12]$$
$$D(TS4) = [6..6] \quad D(TS5) = [3..3] \quad D(TS6) = [12..16]$$

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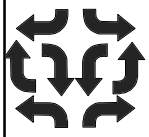

Labelling

In this case simply assigns the first possible value to every variable.

$$D(TS1) = [0..0] \quad D(TS2) = [3..3] \quad D(TS3) = [12..12]$$
$$D(TS4) = [6..6] \quad D(TS5) = [3..3] \quad D(TS6) = [12..12]$$

Solution found!

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Improving the Program

- Answer Redundant Information
  - Suppose *t1, ..., tn* are tasks on the same machine that must be finished before *t0*
  - *t0* must start the sum of the durations added to the earliest start time

    $TS0 \geq \text{minimum}(\{TS1,\ldots,TSn\}) + D1+\cdots+Dn$
  - Calculate predecessors for each task, and add these extra constraints
  - Reduces search

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Improving the Program

- *cumulative* constraints can be used to model machine exclusion constraints
  - cumulative([TS1,TS2,TS3],[3,8,8],[1,1,1],1)
  - cumulative([TS4,TS5,TS6],[6,3,4],[1,1,1],1)
- Replace `machines` with a version which builds these constraints
  - Execution produces (without choice)

    $D(TS1) = [0..1]$ $D(TS2) = [3..4]$ $D(TS3) = [11..12]$

    $D(TS4) = [0..6]$ $D(TS5) = [3..9]$ $D(TS6) = [6..16]$

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Labelling for Scheduling

- Original formulation:
  - Picking the minimum value for each value must be a solution. (default labelling is fine)
- Cumulative formulation
  - Find a task with earliest possible starting time
    - Either place task at this time,
    - Or disallow task to be at this time
  - Repeat (look for task with now earliest time)

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Labelling the Earliest

```
label_earliest([]).
label_earliest([TS0|Ts]) :-
     mindomain(TS0,M0),
     find_min(Ts,TS0,M0,TS,M,RTs),
     (TS = M, Rs=RTs ; TS != M, Rs=[TS0|Ts]),
     label_earliest(Rs).
find_min([],TS,M,TS,M,[]).
find_min([TS1|Ts],TS0,M0,TS,M,RTs) :-
     mindomain(TS1,M1),
     (M1 < M0 ->
          M2=M1, TS2=TS1, RTs=[TS0|Rs]
     ;    M2=M0, TS2=TS0, RTs=[TS1|Rs]),
     find_min(Ts,TS2,M2,TS,M,Rs).
```

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Labelling the Earliest

$D(TS1) = [0..1]$ $D(TS2) = [3..4]$ $D(TS3) = [11..12]$
$D(TS4) = [0..6]$ $D(TS5) = [3..9]$ $D(TS6) = [6..16]$

Label TS1 = 0 Label TS4 = 0 Label TS2 = 3

Label TS5 = 6 Label TS6 = 9 Label TS3 = 11

$D(TS1) = [0..0]$ $D(TS2) = [3..3]$ $\underline{D(TS3) = [11..11]}$
$D(TS4) = [0..0]$ $D(TS5) = [6..6]$ $D(TS6) = [9..9]$

Solution (and in fact a minimal solution)

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Modelling with Finite Domains Summary

- Domains must be initialized
- Labelling to invoke a complete solver
  - Many strategies are possible
    - Choice of variable
    - Choice of value
- Complex constraints can reduce search
- Problems usually have different modellings of different efficiencies

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