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ROSSBY WAVE HYDRAULICS

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The work described here was begun in conjunction with Peter Rhines and Peter Haynes at the University of Washington, in late 1984, and was presented at the AGU meeting in San Francisco in 1985. The study is incomplete, but it is presented here because of the current interest in the topic and its relevance to mid-ocean circulation studies. It is due in large measure to numerous discussions with Peter Rhines on the subject.

INTRODUCTION

The dynamics of mid-ocean flows are strongly influenced by baroclinic Rossby waves. The speeds of mid-ocean currents and of these Rossby waves are of comparable magnitude, namely about 1 cm/sec, so that advective effects are very important. For such flows, the relative vorticity ζ is normally much less than the planetary vorticity f. Furthermore, large topographic ridges with heights reaching to a substantial fraction of the total depth are prominent features of oceanic topography, and these must have a dominant effect on the dynamics.

The hydraulic approach (i.e., long horizontal length scales) has proved to be very informative for the study of non-linear stratified flow over topography of substantial height (Baines, 1988; Baines and Guest, 1988). The equations for flow on a beta-plane have some similarities to those for stratified flows which suggest that the same approach may be useful. The effects of non-linear advection may be studied with the assumption that the topography and flow properties vary on length scales which are large compared to the (relevant) Rossby radius of deformation.

In order to investigate the effects of topography of finite height on beta-plane flows, we consider the simple prototype system of a $1\frac{1}{2}$-layer model.

2. The Hydraulic $1\frac{1}{2}$-layer Model

We consider a three-dimensional system as shown in Figure 1: a lower layer with thickness d_1, density ρ_1, and velocity $\underset{\sim}{u}_1 = (u_1, v_1)$ is surmounted by an infinitely deep upper layer which has density ρ_2 and constant velocity $u_b(y)$ in the x direction. The flow in both layers is assumed to be zonal (i.e., in the x direction), apart from the effects caused by three-dimensional topography whose height above the level surface $z = 0$ is given by $z = h(x,y)$. For this beta-plane model we also take $f = f_\circ + \beta y$ where $f_\circ$ and β are constants. For steady-state flow, at each point (x,y) where the local lower-layer streamline originates from upstream, we may define the variable $y_\circ(x,y)$ which is the upstream value of the y-coordinate for that streamline. This variable cannot be defined for closed recirculating regions, which are discussed below.

We also assume that the length scale L of the topographic variations is long compared with the Rossby deformation radius $(g'\overline{d}_1)^{\frac{1}{2}}/f_\circ$, where $g' = g(1 - \rho_2/\rho_1)$ with g the acceleration due to gravity, and $\overline{d}_1$ is a mean value of d_1. With this assumption, and provided that no abrupt variations occur in the flow, we may assume that the flow is geostrophic. Hence, for the lower layer we have

$$(f_\circ + \beta y)\,(u_1 - u_b) = -g'\,\frac{\partial}{\partial y}\,(d_1 + h) \qquad (2.1)$$

$$(f_\circ + \beta y)\,v_1 = g'\,\frac{\partial}{\partial x}\,(d_1 + h) \qquad (2.2)$$

when the upper layer or barotropic velocity $u_b(y)$ is independent of x. Upstream of the topography we write $u_1 = u_{1\circ}(y)$, $d_1 = d_{1\circ}(y)$ so that

$$(f_\circ + \beta y)\,.\,(u_{1\circ} - u_b) = -g'\,\frac{d}{dy}\,d_{1\circ} \qquad (2.3)$$

To be consistent with these equations we also assume that potential vorticity is conserved in the lower layer. Hence, in the usual notation

$$\frac{D}{Dt}\left(\frac{f+\zeta_1}{d_1}\right) = 0, \qquad (2.4)$$

where $\zeta_1 = \partial v_1/\partial x - \partial u_1/\partial y$. For vorticity variations forced by topography we will have $\zeta_1 \sim \Delta u/L$ where Δu is representative of the variation of velocity in the flow. We have already assumed that $L >> (g'\overline{d}_1)^{\frac{1}{2}}/f_\circ$, and hence $\Delta u/L << f_\circ\,\Delta u/\sqrt{g'\overline{d}_1} << f_\circ$, so that $|\zeta_1| << f_\circ$. Accordingly, we may neglect ζ in equation 2.4, which may be integrated to give

$$\frac{f_\circ + \beta y}{d_1} = \frac{f_\circ + \beta y_\circ}{d_{1\circ}}\,. \qquad (2.5)$$

For steady flow we may also obtain an equation of conservation of mass by considering a streamtube of fluid: for two neighbouring streamlines which have a north-south separation $\delta y_\circ$ upstream and a separation δy further downstream, we may write

$$u_{1\circ}\,d_{1\circ}\,\delta y_\circ = u_1\,d_1\,\delta y\,, \qquad (2.6)$$

so that

$$u_1\,d_1 = u_{1\circ}(y_\circ)\,d_{1\circ}(y_\circ)\frac{\partial y_\circ(x,y)}{\partial y}. \qquad (2.7)$$

Equations 2.1-2.3, 2.5 and 2.7 form the basic set of equations for our system. Note that we have not assumed that the topographic height h and resultant streamline displacements are small. Eliminating u_1 from equation 2.1

via equation 2.7, and using equation 2.5 and integrating with respect to y gives the relationship

$$d_\circ - d_{1\circ}(y_\circ) + \frac{u_b}{2\beta g'}\left\{(f_\circ + \beta y_\circ)^2 - (f_\circ + \beta y)^2\right\} = -h(x,y) \quad . \qquad (2.8)$$

Differentiating with respect to x then gives

$$\frac{\partial y_\circ}{\partial x}\left[u_b - \frac{\beta g' d_1}{(f_\circ + \beta y_\circ)^2} - \left(\frac{d_1}{d_{1\circ}} - 1\right)(u_{1\circ} - u_b)\right] = -\frac{g'}{f_\circ + \beta y_\circ}\frac{\partial h}{\partial x} , \qquad (2.9)$$

so that

$$\frac{\partial h}{\partial x} = 0 \text{ implies either } \frac{\partial y_\circ}{\partial x} = 0 = \frac{\partial d_1}{\partial x} , \qquad (2.10)$$

or the square-bracketed term in equation 2.9 must vanish. This latter term may be manipulated to give

$$[\ ] = \frac{u_{1\circ}}{u_1}\left\{u_b - \frac{\beta g' d_1}{(f_\circ + \beta y)^2} - \frac{g'\partial h/\partial y}{f_\circ + \beta y}\right\} = 0, \quad (2.11)$$

so that this equation implies that either $u_{1\circ}/u_1$ vanishes, or that the local propagation speed of long linear Rossby waves is zero (see next section). In the latter case, the flow is said to be critical, in the same sense that systems with gravity waves are critical. A flow which permits Rossby waves to propagate upstream is subcritical, whereas if the barotropic flow speed u_b is too large for this to be possible the flow is said to be super-critical. Hence, as for the gravity wave case, at a point where $\partial h/\partial x$ vanishes and $u_{1\circ}$ is not zero, the flow either has zero gradients in the east-west direction or is locally critical. Equation 2.11 may be used in conjunction with equation 2.8 to give the obstacle height at which the flow becomes critical for given upstream conditions $u_{1\circ}, d_{1\circ}$.

If we define

$$c_\circ = \frac{\beta g' d_{1\circ}(y_\circ)}{(f_\circ + \beta y_\circ)^2}, \quad F_\circ = \frac{u_b}{c_\circ}, \quad F_i = \frac{u_{1\circ} - u_b}{c_\circ}, \quad H = \frac{h}{d_{1\circ}}, \qquad (2.12)$$

then the critical height h_c is given by

$$H_c = \frac{h_c}{d_{1\circ}} = \frac{F_\circ - 1}{1+F_i}\left\{\frac{F_\circ}{2}\,\frac{1 + F_\circ + 2F_i}{1 + F_i} - 1\right\} \quad . \qquad (2.13)$$

Note that $c_\circ$, $F_\circ$, F_i and consequently H_c are all functions of $y_\circ$. For the simplest case, where $F_i = 0$ so that the upper and lower layer velocities are equal, the critical height is given in terms of $F_\circ$ in Figure 2. If $H < H_c$, $F_\circ < 1$ everywhere, the flow pattern over the obstacle will have the form shown in Figure 3a; if on the other hand $H < H_c$ but $F_\circ > 1$ everywhere, the flow will have the form of Figure 3b. These have been drawn for the northern hemisphere; for the southern hemisphere the changes in d_1 will have the same sign but the changes in y will be reversed. Hence, if the obstacle height is everywhere less than its critical value the overall flow pattern is determined by equations 2.8 and 2.5, and the downstream state is the same as the upstream state. But what happens if $H > H_c$ for some region of the topography? The same solutions cannot be applicable in this case because, for example, they would imply a critical state at some point where $\partial h/\partial x \neq 0$, which violates equation 2.9. By analogy with the gravity wave case we might expect that, after the initiation of motion, a disturbance is sent upstream from the obstacle which alters the oncoming flow. In order to address this phenomenon we need to consider the time-dependent equations.

3. Time-Dependent Equations and Jumps

The time-dependent form of the vorticity equation is

$$\frac{\partial}{\partial t}\frac{f_\circ + \beta y}{d_1} + u_1\frac{\partial}{\partial x}\frac{f_\circ + \beta y}{d_1} + v_1\frac{\partial}{\partial y}\frac{f_\circ + \beta y}{d_1} = 0. \qquad (3.1)$$

Eliminating u_1 and v_1 via the geostrophic relations yields

$$\frac{\partial d_1}{\partial t} + \left\{u_b - \frac{\beta g' d_1}{(f_\circ + \beta y)^2} - \frac{g'}{f_\circ + \beta y}\frac{\partial h}{\partial y}\right\}\frac{\partial d_1}{\partial x} + \frac{g'}{f_\circ + \beta y}\frac{\partial h}{\partial x}\frac{\partial d_1}{\partial y} = \frac{\beta g' d_1}{(f_\circ + \beta y)^2}\frac{\partial h}{\partial x} \quad . \qquad (3.2)$$

This hyperbolic equation may be written in the characteristic form

$$\frac{d}{dt} d_1 = \frac{\beta g' d_1}{(f_\circ + \beta y)^2} \frac{\partial h}{\partial x}, \qquad (3.3)$$

on the characteristic lines $x = x(t)$, $y = y(t)$ given by

$$\frac{dx}{dt} = u_b - \frac{\beta g' d_1}{(f_\circ + \beta y)^2} - \frac{g'}{f_\circ + \beta y},$$

$$\frac{dy}{dt} = \frac{g'}{f_\circ + \beta y} \frac{\partial h}{\partial x}. \qquad (3.4)$$

Note that, excepting above the obstacle, d_1 is constant on each ray and the latter are straight lines in the x - t plane.

If the obstacle height h is small so that

$$h/\overline{d}_1(y) \leq \varepsilon << 1, \qquad (3.5)$$

we may obtain the linear solution as follows. The geometry of the rays over the obstacle is only affected to $0(\varepsilon)$ so that, writing $d_1 = \overline{d}_1(y) + d_1'(x,y,t)$ we have, on each ray

$$\frac{d}{dx} d_1' = \frac{\dfrac{\beta g' \overline{d}_1}{(f_\circ + \beta y)^2}}{\left[u_b - \dfrac{\beta g' \overline{d}_1}{(f_\circ + \beta y)^2}\right]} \frac{\partial h}{\partial x} (1 + 0(\varepsilon)). \qquad (3.6)$$

If the initial conditions for a particular ray are

$$d_1' = d_1'(x_\circ, y,_\circ), \; h = h(x_\circ, y), \; t = 0, \qquad (3.7)$$

equation 3.6 integrates to

$$d_1'(x,y,t) = d'(x_\circ,y) + \frac{h(x,y) - h(x_\circ,y)}{F_\circ(y) - 1} + 0(\varepsilon^2), \qquad (3.8)$$

where

$$F_\circ(y) = \frac{u_b(f_\circ + \beta y)^2}{\beta g' \overline{d}_1(y)}.$$

Hence, as shown in Figure 4, the linear solution consists of a transient part which appears upstream for $F_\circ(y) < 1$ and downstream for $F_\circ(y) > 1$, and a steady part over the obstacle. However, when the obstacle height becomes sufficiently large or $F_\circ$ is sufficiently close to unity, non-linear terms become important. Equation 3.3 shows that d_1 increases above the "forward face" of the topography where $\partial h/\partial x > 0$. This will tend to set up a state where d_1 increases with x. If we consider a simple state where d_1 has the form shown in Figure 5, in the absence of topography d_1 is constant on each ray, so that the latter are straight lines. These will then converge to form a shock or discontinuity as shown, in the conventional manner for a hyperbolic system (e.g., Lighthill 1978). Once such shocks or jumps form, they require special consideration to determine the relationship between the jump propagation speed and the upstream and downstream states. In parallel with the analogous cases for gravity-wave hydraulics and gas-dynamics, we seek an overall relationship which does not depend on the detailed structure of the jump. We therefore assume that the jump may be represented as a moving discontinuity with a propagations speed $c_J(y)$ which varies with y, as shown in Figure 6. If the flow upstream of the jump is represented by $u_{1\circ}$, $d_{1\circ}$ and the flow downstream by u_{11}, d_{11} relative to fixed axes, from conservation of mass we may write

$$d_{1\circ}(u_{1\circ} - c_J) - d_{11}(u_{11} - c_J) = \frac{dV}{dy}, \qquad (3.9)$$

where V is the northward transport in the jump, given by

$$V = \int_{\text{upstream}}^{\text{downstream}} d_1 v_1 \, dx = \frac{g'}{2f} \left\{ d_{11}^2 - d_{10}^2 \right\}, \qquad (3.10)$$

using equation 2.2. Substituting in equation 3.9 and rearranging then gives

$$c_J(y) = u_b - \frac{c_\circ(y)}{2} \left\{ 1 + \frac{d_{11}}{d_{10}} \right\}, \qquad (3.11)$$

so that the jump speed is simply the mean of the upstream and downstream linear wave speeds. Note that equation 3.11 follows from conservation of mass plus (north-south) geostrophy, but equation 3.2 depends only on the conservation of vorticity, plus geostrophy.

Another relation across the jump may be

obtained by integrating ∇d_1 along a streamline. At any given time we have

$$g' \int_u^d \nabla d_1 . \underset{\sim}{ds} = g' \int_u^d \frac{\partial d_1}{\partial x} dx + g' \int_u^d \frac{\partial d_1}{\partial y} dy, \tag{3.12}$$

where $\underset{\sim}{ds}$ is an element of a streamline and the terminals u and d refer to upstream and downstream of the (moving) jump respectively. Assuming geostrophic flow within the jump we obtain

$$g'(d_1(y) - d_{1\circ}(y_\circ)) = \int_u^d (f_\circ + \beta y)v_1 dx -$$

$$\int_u^d (f_\circ + \beta y)(u_1 - u_b)dy = \int_{y_\circ}^y (f_\circ + \beta y)u_b dy. \tag{3.13}$$

If u_b is independent of y we have

$$g'(d_1(y) - d_{1\circ}(y_\circ)) = u_b(y - y_\circ)(f_\circ + \tfrac{1}{2}\beta(y + y_\circ)). \tag{3.14}$$

Note that $(f_\circ + \beta y)/d_1$ is not conserved across the jump.

Equations 3.11, 3.13/14 provide relationships across the jump based solely on geostrophy and mass conservation. They are independent of the detailed internal dynamics, and, like other aspects of this work, parallel the corresponding relationship for gravity wave bores.

4. Flows with Upstream Jumps

For flows which are commenced from some initial state and for which $H > H_c$ for part of the obstacle, we may look for solutions where the flow is governed by a critical condition at the points where $\partial h/\partial x = 0$ in this region, and the fluid adjusts to the original upstream flow via an upstream-propagating jump. If a streamline in this hydraulically controlled region has depth d_c as it passes over the crest of the obstacle (where $\partial h/\partial x = 0$), the critical condition there (equation 2.11) gives

$$u_b - \frac{\beta g' d_{1c}}{(f_\circ + \beta y_1)^2} - \left(\frac{d_{1c}}{d_{11}(y_1)} - 1\right) \bullet$$

$$(u_{11}(y_1) - u_b) = 0, \tag{4.1}$$

where $d_{11}(y_1)$, $u_{11}(y_1)$ and y_1 all refer to the variables d_1, u_1 and y on the same streamline upstream of the obstacle but downstream of the jump. From equations 2.5, 2.8 we also have

$$\frac{f_\circ + \beta y_1}{d_{11}(y_1)} = \frac{f_\circ + \beta y_c}{d_{1c}}, \tag{4.2}$$

$$d_{1c}(y_c) - d_{11}(y_1) + \frac{u_b}{2\beta g'}\left[(f_\circ + \beta y_1)^2 - (f_\circ + \beta y_c)^2\right] = -h(x, y_c). \tag{4.3}$$

upstream of the obstacle and downstream of the jump we also have

$$(f_\circ + \beta y_1)(u_{11}(y_1) - u_b) = -g' \frac{d}{dy_1} d_{11}(y_1). \tag{4.4}$$

d_{1c} and y_c may be eliminated from equations 4.1-4.3 to obtain a relationship between $u_{11}(y_1)$ and $d_{11}(y_1)$; these variables may then be determined by integrating equation 4.4 over the appropriate range of y_1.

Unfortunately, when one applies these equations to specific examples of three-dimensional topography, it becomes apparent that the situation is really much more complex. Equations 4.1 to 4.4 are applicable for only part of the range of y where H > Hc. Regions with closed streamlines may occur. Hydraulic jumps on the downstream side of the topography seem to be necessary in order to prevent streamlines circling the topography and intersecting themselves, and upstream blocking may also be common. Rhines (1988) describes some numerical solutions which illustrate these phenomena.

In spite of these difficulties, I believe that this hydraulic approach has much to offer in improving our understanding of the effects of mid-ocean ridges on ocean circulation.

REFERENCES

Baines, P.G., 1988. A general method for determining upstream effects in stratified flow of finite depth over long two-dimensional obstacles. J. Fluid Mech., 118, 1-22.

Baines, P.G. & Guest, F., 1988. The nature of upstream blocking in uniformly stratified flow over long obstacles. J. Fluid Mech., 118, 23-45.

Lighthill, M.J., 1978. "Waves in Fluids", Cambridge University Press.

Rhines, P.B., 1988. Deep planetary circulation and topography: simple models of mid-ocean flows. J. Phys. Oceanogr., (to appear).

ON A NONLINEAR REGRESSION METHOD AS APPLIED TO "AQUA ALTA" OF VENICE

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To analyse a set of data $\{f(t)\}$ as a function of a continuous (or discrete) parameter t one can use the classical Fourier analysis

$$f(t) = \sum a_k \cos [(2\pi t/T_k) + \phi_k]$$

or

$$f(t) = \int a_k \cos [(2\pi t/T_k) + \phi_k] \, dk$$

where a_k, ϕ_k, T_k εR.

Of remarkable interest is also the more modern regression method

$$f(t) = \sum a_k f(t - \tau_k) + \delta(t) \qquad a_k, \tau_k \varepsilon R$$

for given $\{\tau_k\}$ values. Here δ is the "error". Asking that

$$\int_{ti}^{tf} \delta^2 \, dt$$

is minimum one determines the $\{a_k\}$ values. Since this linear regression technique is widely used (Draper, Smith, 1966; Toutembourg, 1982), we investigated the next step, namely a nonlinear regression technique (Massi, Parisi, Paschini, 1979)

$$f(t) = \sum \tilde{a}_k(t) \, f(t - \tau_k) + \delta(t) \equiv$$

$$\equiv \sum a_k \, f(t - \tau_k) + \sum b_{kl} \, f(t - \tau_k) \, f(t - \tau_l) + \delta(t)$$

with a_k, τ_k, C_k, b_{kl} εR.

Requiring again that $\int_{ti}^{tf} \delta^2 dt$ is minimum in some time range (t_i, t_f), we can determine $\{a_k, b_{kl}\}$. On mathematical grounds this consists of solving a linear inhomogeneous system of algebraic equations, a well posed mathematical problem that for periodic phenomena can give numerical difficulties.

Classical periodic phenomena in physical oceanography are the tides - a deeply studied problem - as well as more complex problems such as "aqua alta" at Venice (Italy), a dangerous case of intermittent lowering and rising of the sea level around this magnificent town. Its economical and social effects can be easily understood. It is well known (Accerboni, Manca, 1973; Michelato, 1975) that the "aqua alta" is a catastrophic event due to the comtemporaneous effect of tides (with periods at 12, 12.40, 24 h), inertial phenomena (17.32 h), atmospheric pressure oscillations (with main periodicities 3.7, 4.3, 4.4, 4.9, 5.8 days) as well as winds that strongly influence seiches (6.1, 7.2, 8.2, 9.5, 10.8, 21.3 h). The short-term terms make any medium-range forecast difficult since a sudden transverse wind can be taken into account only if one has information on large scale meteorological events. In applying