Stratified Flow over Finite Obstacles with Weak Stratification

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The time-dependent problem of inviscid stratified flow over finite obstacles in fluid of infinite depth is considered in the limit of vanishing stratification ($N h U \to 0$), as a perturbation about the state of potential flow. The object of this study is to shed light on the “upstream influence” question, namely, under what circumstances does the motion of the obstacle generate steady motions which will eventually be felt at an arbitrarily large distance upstream? In the present study the flow develops on both short and long length and time scales, and a matched asymptotic expansion procedure is required. The results from this expansion are consistent with those from the corresponding “classical” expansion in small obstacle height [McIntyre (1972)]. In particular, no permanent effects far upstream are obtained by this procedure, and this implies that “Long’s model” is applicable for the case of infinite depth if $N h U$ is sufficiently small.

INTRODUCTION

The problem of two-dimensional flow of inviscid stratified fluids over obstacles is of considerable importance in meteorological and other contexts. A major difficulty, which has aroused much controversy and is as yet unresolved, concerns the “upstream influence” question, namely: under what circumstances (if any) does the flow over the obstacle generate motion which will eventually propagate unattenuated to an infinite distance upstream? To resolve this theoretically requires consideration of an initial value problem, and the most common way of tackling this problem for inviscid stratified flow over obstacles is to assume that the obstacle height is small. The most complete exposition of this procedure as a formal expansion with height as small parameter is due to McIntyre (1972).
However, the results from this procedure are not entirely in accord with experimental observations in fluid of finite depth with obstacles with “moderately small” height (Baines, 1977, 1979). In particular, in many cases, transient and steady-state flow components are observed upstream which are not predicted theoretically, suggesting that the expansion in powers of obstacle height is inadequate in practice. Hence, it seems desirable to avoid the small-height assumption if possible, and this is accomplished in the present paper by an expansion with the stratification (or more exactly, stratification/fluid velocity) as small parameter. It transpires that this expansion necessitates the use of the technique of matched inner and outer asymptotic expansions, and in order to minimize the number of length scales involved we will consider only the case of infinite depth.

Although the nature of this expansion is quite different from that of the small-height expansion, the results obtained are consistent with it and are qualitatively the same. Consequently, they provide further formal justification for the validity of Long’s finite amplitude model (Long, 1955) for the infinite-depth case, when $NhlU$ is sufficiently small. The procedure is also of interest in the way it describes the character and evolution of the flow field.

2. BASIC EQUATIONS

The basic variables and parameters are indicated in Figure 1. We take $x$ and $z$ as horizontal and vertical coordinates respectively in a frame fixed relative to the topography; $\mathbf{u} = (u, w)$ denotes the fluid velocity, and $\psi$ is the stream function defined by

$$u = -\psi_x, \quad w = \psi_z.$$  

(2.1)

$\rho_0 + \rho$ denotes the total fluid density, where $\rho_0(z)$ is the equilibrium value, and $g$ is the acceleration due to gravity. The inviscid equations for the vorticity and density are then

$$\begin{aligned}
\left\{ \begin{array}{l}
(\partial/\partial t + \mathbf{u} \cdot \nabla)\nabla^2\psi = - (g/\rho_0)\rho_x, \\
(\partial/\partial t + \mathbf{u} \cdot \nabla)\rho = - (d\rho_0/dz)\psi_x,
\end{array} \right.
\end{aligned}$$

(2.2)

with boundary conditions

$$\psi \to -Uz \quad \text{as} \quad x^2 + z^2 \to \infty,$$
for all finite \( t \), and

\[
\psi = 0, \quad \text{on} \quad z = h f(x).
\]

Here the Boussinesq approximation has been made, and \( \bar{\rho}_0 \) denotes the mean value of \( \rho_0(z) \) over a sufficiently large depth of fluid (conceptually this presents no difficulty, particularly since we are concerned with the weak-stratification limit). If we write

\[
\tilde{\psi} = \psi + U z, \quad u = U - \tilde{\psi}_z, \quad w = \tilde{\psi}_x,
\]  

(2.3)

the boundary conditions on \( \tilde{\psi} \) take the more convenient form

\[
\begin{align*}
\tilde{\psi} & \to 0 \quad \text{as} \quad x^2 + z^2 \to \infty, \\
\tilde{\psi} & = U z \quad \text{on} \quad z = h f(x),
\end{align*}
\]

(2.4)

and the equations (2.2) still hold with \( \nabla^2 \tilde{\psi} \) replacing \( \nabla^2 \psi \). For initial conditions, we assume that at \( t=0 \) the fluid commences motion in the positive \( x \) direction with a uniform mean velocity \( U \), or, equivalently, that the obstacle abruptly commences motion at a uniform speed \( U \) through the undisturbed fluid in the negative \( x \)-direction.

The Brunt–Väisälä frequency \( N \) is defined by

\[
N^2 = - \frac{(g/\bar{\rho}_0)}{d\rho_0/dz},
\]

and is assumed to be constant here. If \( N=0 \), the complete solution to the above problem is given by potential flow where, for \( t>0 \), \( \tilde{\psi} \) satisfies \( \nabla^2 \tilde{\psi} = 0 \). We suppose that for this potential flow we have

\[
(\tilde{\psi}_x, \tilde{\psi}_x)_{\text{max}} = \varepsilon U,
\]

(2.5)

where \( \varepsilon \) will be small if the topographic slope is everywhere small. In general, \( \varepsilon \) will not be large unless the obstacle has fairly sharp convex corners (as seen from the fluid side), and we will assume for present purposes that \( \varepsilon = O(1) \). This restriction is not a serious one, and may be relaxed later if necessary. If \( h \) is the height of the obstacle we may define \( a \) such that \( \varepsilon = h/a \). For simple obstacles such as that shown in Figure 1, \( 2a \) and \( h/a \) are representative of the width and maximum slope of the obstacle respectively.

The dimensional quantities governing the flow are therefore \( h, a, U \) and \( N \), from which we may construct the following dimensionless quantities:

\[
\varepsilon = h/a, \quad \delta = N a/U, \quad R = \varepsilon \delta = Nh/U,
\]

(2.6)

only two of which are independent.
We now scale the equations in a manner which is consistent with \( N \) small, viz we take \( a \) as unit of length, \( U \) as unit of velocity and \( a/U \) as unit of time, so that

\[
x = a \bar{x}, \quad t = (a/U) \bar{t}, \quad \mathbf{u} = U \mathbf{i} + \varepsilon U \bar{u},
\]

where \( \mathbf{i} \) is the unit vector in the positive \( x \) direction, and

\[
\ddot{\psi} = U \dot{\psi}, \quad \sigma = g \rho/\bar{\rho}_0 = N^2 h \bar{\sigma}.
\]

The dimensionless equations then are

\[
\begin{align*}
\left( \frac{\partial}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}} \right) \nabla^2 \psi &= - \delta^2 \dot{\sigma}_x - \varepsilon \bar{u} \cdot \nabla^2 \psi, \\
\left( \frac{\partial}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}} \right) \ddot{\sigma} &= \ddot{\psi}_x - \varepsilon \bar{u} \cdot \nabla \ddot{\sigma},
\end{align*}
\]

with

\[
\ddot{\psi} = f(a \bar{x}) \quad \text{on} \quad \ddot{z} = \xi f(a \bar{x}).
\]

Henceforth we omit the overbars and use \( \psi, \sigma \) to denote \( \ddot{\psi}, \ddot{\sigma} \) etc, with \( u = -\psi_x, w = \psi_x \), so that the above equations are

\[
\begin{align*}
\left[ \frac{\partial}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}} \right] \nabla^2 \psi &= - \delta^2 \sigma_x - \varepsilon [\psi_x (\nabla^2 \psi)_x - \psi_x (\nabla^2 \psi)_z], \\
\left[ \frac{\partial}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}} \right] \sigma &= \psi_x - \varepsilon (\psi_x \sigma_x - \psi_x \sigma_z),
\end{align*}
\]
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\[ \psi = f(ax) \quad \text{on} \quad z = ef(ax). \]

The initial conditions give

\[ \psi = \psi_p(x, z), \quad \sigma = 0 \quad \text{at} \quad t = 0^+, \quad (2.11) \]

where \( \psi_p \) denotes the stream function for potential flow, and is the complete solution in the limit \( \delta = 0 \) (i.e. zero stratification). We shall assume that the obstacle is such that

\[ \psi_p = \frac{C_1 z}{x^2 + z^2} + O\left(\frac{1}{(x^2 + z^2)}\right), \quad \text{as} \quad (x^2 + z^2) \to \infty. \quad (2.12) \]

Here the dipole moment \( C_1 \) is a constant of order unity, and for small \( \varepsilon \) is approximately equal to \( (\text{area of the obstacle})/\pi \). The assumption (2.12) is satisfied by obstacles of finite width, such as that shown in Figure 1.

It is convenient to change co-ordinates from \((x, z)\) by using a conformal transformation to the set \((\alpha, \beta)\) consisting of the streamlines and potential lines of the total potential flow where \( \beta \) is defined by

\[ \beta = z - \varepsilon \psi_p(x, z), \quad (2.13) \]

and \( \alpha \) is obtained from the Cauchy–Reimann equations \( \alpha_x = \beta_z \) and \( \alpha_z = -\beta_x \) (where the suffixes denote derivatives), and the requirement that \( \alpha \to x \) as \( x^2 + z^2 \to \infty \). The Jacobian of the transformation is

\[ V(\alpha, \beta) = \frac{\partial(x, z)}{\partial(\alpha, \beta)} = z_x^2 + z_y^2, \quad (2.14) \]

which is always positive. The vorticity is

\[ \psi_{xx} + \psi_{zz} = V^{-1}(\psi_{\alpha x} + \psi_{\beta \beta}), \quad (2.15) \]

and for any variables \( P \) and \( Q \) we have

\[ J(P, Q) = V \frac{\partial(P, Q)}{\partial(x, z)} = \frac{\partial(P, Q)}{\partial(\alpha, \beta)}. \quad (2.16) \]

It is also convenient to write

\[ \psi = \psi_p + \psi', \quad (2.17) \]
and for the remainder of this paper we define $V^2\psi' = \psi_{\alpha\alpha} + \psi_{\beta\beta}$. Equations (2.10) then become

$$
\begin{align*}
\left( V \frac{\partial}{\partial t} + \frac{\partial}{\partial \alpha} \right) (V^{-1} V^2 \psi') &= -\delta^2 [\sigma_\alpha + \varepsilon J(\sigma, \psi_\beta)] - \varepsilon J(\psi', V^{-1} V^2 \psi'), \\
\left( V \frac{\partial}{\partial t} + \frac{\partial}{\partial \alpha} \right) \sigma &= \psi_{\alpha\alpha} + \psi_{\beta\beta} + \varepsilon J(\psi, \psi_\beta) - \varepsilon J(\psi', \sigma).
\end{align*}
$$

(2.18)

The flow domain is now $-\infty < \alpha < \infty$, $\beta > 0$, the boundary condition is $\psi' = 0$ at $\beta = 0$, and the initial conditions are that $\psi', \sigma = 0$ at $t = 0^+$. Now it follows from (2.12) that we may write

$$
\psi_\alpha = \sum_{n=1}^{\infty} \frac{C_n \sin n\phi}{\rho^n} = \sum_{n=1}^{\infty} \frac{C_n (-1)^{n-1} \delta^{n-1} \sin \phi}{(n-1)! \frac{\partial \alpha^{n-1}}{\partial \alpha}} \left( \frac{\sin \phi}{\rho} \right),
$$

(2.19)

where $\alpha = \rho \cos \phi$, $\beta = \rho \sin \phi$ and the $C_n$ are constants. Then it follows from (2.13) and the Cauchy–Riemann equations that

$$
x = \alpha - \varepsilon \sum_{n=1}^{\infty} \frac{C_n \cos n\phi}{\rho^n}, \quad z = \beta + \varepsilon \sum_{n=1}^{\infty} \frac{C_n \sin n\phi}{\rho^n}.
$$

(2.20)

Also from (2.14) we can show that

$$
V = 1 + \frac{2\varepsilon C_1 \cos 2\phi}{\rho^2} + \frac{4\varepsilon C_2 \cos 3\phi}{\rho^3} + O\left( \frac{1}{\rho^4} \right).
$$

(2.21)

3. THE INNER EXPANSION

We wish to solve equations (2.18) with $\delta$ small and $\varepsilon = O(1)$. To do this we might first look for a solution in which $\psi', \sigma$ are expanded as power series in $\delta$. Such an expansion is valid only when $\rho$ is $O(1)$, and fails as $\rho \to \infty$, where it must be matched with an outer expansion (e.g. Van Dyke, 1964, or Cole 1968). It transpires that the matching procedure necessitates the presence of terms proportional to $\delta^4 \log \delta$ in the inner expansion. Anticipating this, we shall seek an inner expansion of the form

$$
\psi' = \psi_0 + \delta^2 \log \delta \psi_2^* + \delta^2 \psi_2 + \delta^3 \psi_3 + \delta^4 (\log \delta)^2 \psi_4^* + \delta^4 \log \delta \psi_4 + \ldots,
$$

$$
\sigma = \sigma_0 + \delta^2 \log \delta \sigma_2^* + \delta^2 \sigma_2 + \delta^3 \sigma_3 + \delta^4 (\log \delta)^2 \sigma_4^* + \delta^4 \log \delta \sigma_4 + \delta^4 \sigma_4 + \ldots.
$$

(3.1)
At $t=0$, $\psi_0$, $\psi_2^+$, $\psi_2^-$ and $\sigma_0$, $\sigma_2^+$, $\sigma_2^-$ and zero and on $\beta=0$, $\psi_0$, $\psi_2^+$, $\psi_2^-$ are zero.

Substituting (3.1) into (2.18) it is readily seen that $\psi_0=0$, and that $\sigma_0$ satisfies the equation

$$
\left( V \frac{\partial}{\partial t} + \frac{\partial}{\partial \alpha} \right) \sigma_0 = \psi_{px}.
$$

(3.2)

Let $t^*=t^*(\alpha, \beta, t)$ be defined by

$$
t^* = \int_{\alpha-t^*}^{\alpha} V(\alpha', \beta) d\alpha'.
$$

(3.3)

Then using $\alpha$, $\beta$, $t^*$ as co-ordinates in place of $\alpha$, $\beta$, $t$, it may be shown that for any variable $P$

$$
\left( V \frac{\partial}{\partial t} + \frac{\partial}{\partial \alpha} \right) P = \left( \frac{\partial}{\partial t^*} + \frac{\partial}{\partial \alpha} \right) P.
$$

(3.4)

Hence $\sigma_0$ is given by

$$
\sigma_0 = \psi_p(\alpha, \beta) - \psi_p(\alpha - t^*, \beta),
$$

(3.5)

since $\sigma_0=0$ at $t=0$. Now as $t \to \infty$, $t^* \to t + t_0(\alpha, \beta)$ where $t_0(\alpha, \beta) = O(1)$ uniformly in $\alpha, \beta$. $\sigma_0$ therefore consists of a region of elevated density surfaces over the obstacle coinciding with the streamlines of potential flow, and an equal and opposite (apart from the geometric distortion near the obstacle) region of depressed density surfaces, which is advected downstream with the mean flow. We know on simple physical grounds that such a depressed density region will ultimately return to its equilibrium configuration, which does not constitute a correction of $O(\delta^2)$; hence the transient part of this expansion (at least) cannot be uniformly valid as $t \to \infty$ for all $\alpha, \beta$. However, our interest is primarily with the steady-state solution, and for fixed $\alpha, \beta$ of $O(1)$, $\sigma_0 \to \psi_p$ as $t \to \infty$. It is interesting to note that if the initial conditions are such that the obstacle is suddenly introduced into a uniform stream (as considered by McIntyre, 1972) we have $\sigma_0 = \psi_p$ and the transient part of $\sigma_0$ vanishes.

Turning next to the terms in $\delta^2 \log \delta$ it follows from (2.18) and (3.5) that

$$
\begin{align*}
\left( V \frac{\partial}{\partial t} + \frac{\partial}{\partial \alpha} \right) (V^{-1} V^2 \psi_2^+) &= 0, \\
\left( V \frac{\partial}{\partial t} + \frac{\partial}{\partial \alpha} \right) \sigma_2^+ &= g_2^+, 
\end{align*}
$$

(3.6)
where
\[ g_2^+ = \psi_{2a}^+ + \epsilon J(\psi_2^+, \psi_p(\alpha - t^*, \beta)). \]

Then using \( \alpha, \beta, t^* \) as co-ordinates [vide (3.3)], it follows from (3.4) and the initial conditions that
\[
\begin{align*}
\nabla^2 \psi_2^+ &= 0, \\
\sigma_2^+ &= \int_{\alpha-t^*}^{\alpha} g_2^+(\alpha', \beta, \alpha' - \alpha + t^*) d\alpha'.
\end{align*}
\] (3.7)

Thus \( \psi_2^+ \) is a potential which satisfies the boundary condition \( \psi_2^+ = 0 \) on \( \beta = 0 \). A complete determination of \( \psi_2^+ \) requires a knowledge of the behaviour of \( \psi_2^+ \) as \( \rho \to \infty \); this, in turn, is provided by the inner limit of the outer expansion, and must await the calculation of the outer expansion, which is described in §4. Once \( \psi_2^+ \) has been determined, \( g_2^+ \) is known and hence \( \sigma_2^+ \) is given by (3.7). However, our main concern here is with the steady-state terms, which we shall denote by an additional suffix "s". It is easily seen from (3.6) and (3.7) that
\[
V^2 \psi_{2s}^+ = 0, \quad \sigma_{2s}^+ = \psi_{2s}^+.
\] (3.8)

The complete determination of \( \psi_{2s}^+ \) is described in §5 where the behaviour of \( \psi_{2s}^+ \) as \( \rho \to \infty \) is determined by the inner limit of the steady part of the outer expansion.

For the terms proportional to \( \delta^2 \) it follows from (2.18) and (3.5) that
\[
\left( V \frac{\partial}{\partial t} + \frac{\partial}{\partial \alpha} \right) (V^{-1} \nabla^2 \psi_2) = f_2,
\]
\[
\left( V \frac{\partial}{\partial t} + \frac{\partial}{\partial \alpha} \right) \sigma_2 = g_2,
\]
where
\[
\begin{align*}
\sigma_2 &= - \sigma_{0s} - \epsilon J(\psi_p(\alpha, \beta), \psi_p(\alpha - t^*, \beta)), \\
g_2 &= \psi_{2a} + \epsilon J(\psi_2, \psi_p(\alpha - t^*, \beta)).
\end{align*}
\] (3.9)

Again, using \( \alpha, \beta, t^* \) as co-ordinates [vide (3.3)] it follows from (3.4) and the initial conditions that
\[
\begin{align*}
V^{-1} \nabla^2 \psi_2 &= \int_{\alpha-t^*}^{\alpha} f_2(\alpha', \beta, \alpha' - \alpha + t^*) d\alpha', \\
\sigma_2 &= \int_{\alpha-t^*}^{\alpha} g_2(\alpha', \beta, \alpha' - \alpha + t^*) d\alpha'.
\end{align*}
\] (3.10)
Here \( f_2 \) is known, and (3.10) with the boundary condition \( \psi_2 = 0 \) on \( \beta = 0 \) is a Poisson equation for \( \psi_2 \). As for \( \psi_3^+ \), a complete determination of \( \psi_2 \) requires a knowledge of the behaviour of \( \psi_2 \) as \( \rho \to \infty \), which must await the calculation of the inner limit of the outer expansion. Once \( \psi_2 \) is known, \( g_2 \) is known and hence \( \sigma_2 \) is known. Considering only the steady-state terms, it follows easily from (3.5), (3.9) and (3.10) that

\[
V^{-1} \nabla^2 \psi_2 = -\psi_p, \quad \sigma_2 = \psi_2.
\]

The transient parts are centred on \( \alpha = t^* \) and are \( O(t^{-1}) \). Equation (3.11) shows that \( \psi_{2s} \) has vorticity \( -\psi_p \); since \( \psi_p \) is generally positive, this flow has negative vorticity.

The calculation of the terms \( \psi_3, \psi_4^+, \psi_4^+, \psi_4 \ldots \) and \( \sigma_3, \sigma_4^+, \sigma_4^+, \sigma_4 \ldots \) proceeds in a similar way. At each stage a pair of equations analogous to (3.9) is obtained, whose solution is analogous to (3.10); at each stage the forcing terms \( f_3, f_4^+, f_4^+, f_4 \ldots \) are known, and hence each of \( \psi_3, \psi_4^+, \psi_4^+, \psi_4 \ldots \) is governed by a Poisson equation. The complete determination of \( \psi_3, \psi_4^+, \psi_4^+, \psi_4 \ldots \) requires a knowledge of the behaviour of \( \psi_3, \psi_4^+, \psi_4^+, \psi_4 \ldots \) as \( \rho \to \infty \), which is obtained from the outer expansion. Once \( \psi_3, \psi_4^+, \psi_4^+, \psi_4 \ldots \) are known, \( g_3, g_4^+, g_4^+, g_4 \ldots \) are known, and hence \( \sigma_3, \sigma_4^+, \sigma_4^+, \sigma_4 \ldots \) are known. Considering only the steady-state terms it may be shown that

\[
\begin{align*}
\nabla^2 \psi_{3s} &= 0, & \sigma_{3s} &= \psi_{3s}, \\
V^{-1} \nabla^2 \psi_{4s}^+ &= -\psi_{2s}^+, & \sigma_{4s}^+ &= \psi_{4s}^+, \\
V^{-1} \nabla^2 \psi_{4s}^- &= -\psi_{2s}^-, & \sigma_{4s}^- &= \psi_{4s}^-.
\end{align*}
\]

while \( \psi_{4s}^+ \) and \( \sigma_{4s}^+ \) are both zero. The steady-state terms thus satisfy the sequence of equations (3.5), (3.8), (3.11) and (3.12). Consequently the steady-state parts \( \psi'_s \) and \( \sigma_s \) of the expansions (3.1) satisfy the equations

\[
V^{-1} \nabla^2 \psi'_s + \delta^2 \psi'_s = -\delta^2 \psi_p, \quad \sigma_s = \psi_p + \psi'_s.
\]
4. THE OUTER EXPANSION

The inner expansion fails as \( \rho \to \infty \), and it is necessary to introduce an outer expansion to describe the flow on length and time scales of \( \delta^{-1} \) (which in dimensional terms are \( UN^{-1} \) and \( N^{-1} \) respectively). Hence we put

\[
A = \delta x, \quad B = \delta \beta, \quad T = \delta t, \quad R = \delta \rho.
\]

From (2.19) it follows that

\[
\psi_p = \delta \Psi_p, \quad \Psi_p = \sum_{n=1}^{\infty} \delta^{n-1} \Psi_{pn}
\]

where

\[
\Psi_{pn} = C_n \sin n\phi / R^n.
\]

Here \( A = R \cos \phi, B = R \sin \phi \). Then from (2.14) and (2.21) we may also put

\[
V = 1 + \sum_{n=2}^{\infty} \delta^n V_n,
\]

where

\[
V_2 = 2\varepsilon C_1 \cos 2\phi / R^2, \quad \text{and} \quad V_3 = 4\varepsilon C_2 \cos 3\phi / R^3,
\]

while from (3.3) and (3.5) we may write

\[
\sigma_0 = \delta S_0, \quad S_0 = \sum_{n=1}^{\infty} \delta^{n-1} S_{0n},
\]

where

\[
S_{01} = \Psi_{p1}(A, B) - \Psi_{p1}(A - T, B),
\]

\[
S_{02} = \Psi_{p2}(A, B) - \Psi_{p2}(A - T, B), \text{etc.}
\]

Next we put

\[
\psi = \delta (\Psi_p + \Psi), \quad \sigma = \delta (S_0 + S),
\]

where \( \Psi, S \) are regarded as functions of \( A, B, T \). Then equations (2.18) become

\[
\left( \frac{\partial}{\partial T} + \frac{\partial}{\partial A} \right) \nabla^2 \Psi + S_A = F,
\]

\[
\left( \frac{\partial}{\partial T} + \frac{\partial}{\partial A} \right) S - \Psi_A = G,
\]
where the “forcing” terms on the right-hand side are given by
\[ F = -S_{o,A} - \frac{\partial}{\partial A}\left\{(V^{-1} - 1)V^2\Psi\right\} - \epsilon \delta^2 J(S_o + S, \Psi_p) - \epsilon \delta^2 J(\Psi, V^{-1}V^2\Psi), \]
and
\[ G = \Psi_{p,A} - \left(\frac{\partial}{\partial T} + \frac{\partial}{\partial A}\right)S_o - (V - 1)\frac{\partial}{\partial T}(S_o + S) + \epsilon \delta^2 J(\Psi, \Psi_p - S_o - S). \]

We then look for an asymptotic solution of (4.6) of the form
\[ \Psi = \sum_{n=1}^{\infty} \delta^{n-1} \Psi_n, \quad S = \sum_{n=1}^{\infty} \delta^{n-1} S_n. \tag{4.8} \]

At each stage we obtain a pair of equations of the form
\[ \left\{ \begin{array}{l} \frac{\partial}{\partial T} + \frac{\partial}{\partial A}\right\}V^2\Psi_n + S_{n,A} = F_n \\frac{\partial}{\partial T} + \frac{\partial}{\partial A}\right\}S_n - \Psi_{n,A} = G_n \end{array} \right\} \tag{4.9} \]

where it is apparent from (4.7) that each \( F_n, G_n \) depends only on \( \Psi_1, \ldots, \Psi_{n-1} \) and \( S_1, \ldots, S_{n-1} \), and so can be regarded as known forcing terms. Equations (4.9) constitute a pair of “lee-wave” equations, which may be combined into the single equation for \( \Psi_n \):
\[ \left\{ \begin{array}{l} \frac{\partial}{\partial T} + \frac{\partial}{\partial A}\right\}^2V^2\Psi_n + \Psi_{n,A,A} = H_n \end{array} \right\} \tag{4.10} \]

where
\[ H_n = \left(\frac{\partial}{\partial T} + \frac{\partial}{\partial A}\right)F_n - \frac{\partial}{\partial A}G_n. \]

The initial conditions are that \( \Psi_n, S_n = 0 \) at \( T=0 \), and the boundary conditions \( \Psi_n = 0 \) on \( B=0 \). Examination of (4.2), (4.3) and (4.4) indicates that the forcing terms \( F_n, G_n \), and hence \( H_n \), are singular as \( R \rightarrow 0 \). Hence equation (4.10) describes a “lee-wave” problem which is being forced by known singular forcing terms. These singular forcing terms are a “far-field” representation of the obstacle, and one of the main advantages in using the \( \alpha, \beta \) co-ordinate system, or the \( A, B \) coordinate system, is the succinct way in which the obstacle is represented by appropriate singular forcing terms.
Now putting $n=1$, we see from (4.2), (4.3), (4.4) and (4.7) that $F_1 = -S_{0,1A}$, $G_1 = 0$, and hence from (4.4) and (4.9)

$$H_1 = -\partial^2 \Psi_{p1}/\partial A^2.$$  

(4.11)

Since $\Psi_{p1}$ is $C_1 \sin \phi/R$ by (4.2), the $n=1$ term in the outer expansion is the response to a dipole placed at the origin. This problem (4.9) with $n=1$ may be solved in a conventional manner using Laplace and Fourier transforms, and has been considered by Miles (1969) for the analogous problem for a dipole in a rotating fluid. It may be shown that the steady-state limit ($T \to \infty$) is given by

$$\Psi_{1s} = -\Psi_{p1} + \Psi_L, \quad S_{1s} = \Psi_{1s},$$

(4.12)

where

$$\Psi_L = C_1 \int_0^1 \cos \left[ kA + (1 - k^2)^{1/2}B \right] dk + C_1 \int_1^\infty \exp\left[ -(k^2 - 1)^{1/2}B \right] \cos kA \, dk.$$  

(4.13)

Here $\Psi_L$ satisfies the equation

$$\nabla^2 \Psi_L + \Psi_L = 0,$$  

(4.14)

and describes the lee-wave field due to a horizontal dipole placed at the origin (Miles and Huppert, 1969). Following the method used by Miles (1969), it may be shown that the transient terms are $O(T^{-1})$. Miles and Huppert (1969) have shown that $\Psi_L$ may be expressed in the form

$$\Psi_L = C_1 \left\{ -\frac{1}{2} \pi Y_1(R) \sin \phi + \frac{1}{2} \int_0^\infty \cos \left[ R \sin(t + \phi) \right] \cos t \, dt \right\},$$  

(4.15)

where $Y_1(R)$ is the Bessel function of the second kind of order one. As $R \to \infty$, the integral in (4.15) may be evaluated by the method of stationary phase; using the known asymptotic expansions for $Y_1(R)$ it may be shown that

$$\Psi_L = C_1 (2\pi/R)^{1/2} \cos (R - \frac{1}{4}\pi) \sin \phi H(A) + C_1 \sin B/A + O(\left| A \right|^{-3/2}).$$  

as $|A| \to \infty$,  

(4.16)

where $H(A)$ is the Heaviside function (zero for $A<0$ and one for $A>0$). The first term in (4.16) describes the downstream lee-wave field. As $R \to 0$, we may use the known formulae for $Y_1(R)$, while the integral in (4.15) can be evaluated by expanding in powers of $R$. We find that,
\[ \Psi_{1s} = C_1 \sin \phi \left\{ - (\gamma + \log \frac{1}{2} R) \left[ \frac{R}{2} - \frac{1}{6} R^3 + O(R^3) \right] + \frac{1}{4} R - \frac{5}{6} R^3 + O(R^5) \right\} - \frac{1}{6} C_1 R^2 \sin 2\phi + O(R^4), \]  

(4.17)

where \( \gamma \) is Euler's constant (0.5772...).

Next we put \( n = 2 \) in (4.8), and we see from (4.2), (4.3), (4.4) and (4.7) that \( F_2 = - S_{02A}, \ G_2 = 0 \), and hence from (4.4) and (4.7)

\[ H_2 = - \delta^2 \Psi_{p2}/\partial A^2. \]  

(4.18)

But \( \Psi_{p2} = - (\partial/\partial A) (C_2 \sin \phi/R) \), and hence \( \Psi_2 = - (C_2/C_1) \partial \Psi_1/\partial A \). In particular, the steady-state limit is

\[ \Psi_{2s} = - (C_2/C_1) \partial \Psi_{1s}/\partial A, \quad S_{2s} = S_{2s}. \]  

(4.19)

For \( n = 3 \), we shall only calculate the steady-state parts \( F_{2s}, \ G_{2s} \) of the forcing terms \( F_2, \ G_2 \); it may be shown that the transient parts are \( O(T^{-2}) \) as \( T \to \infty \), and they contribute only a transient term \( O(T^{-1}) \) to \( \Psi_3 \). After some manipulation, using (4.2), (4.3), (4.4), (4.12) and (4.14) in (4.7), we find that

\[ F_{2s} = - \Psi_{p3s} - 2\epsilon (\partial/\partial A)(\Psi_{p1B} \Psi_L), \quad G_{2s} = 0, \]

and

\[ H_{2s} = F_{2s}. \]  

(4.20)

Hence, since \( \Psi_{p3} = \frac{1}{2} (\partial^2/\partial A^2)(C_3 \sin \phi/R) \), we may put

\[ \Psi_3 = (C_3/2C_1) \partial \Psi_1/\partial A^2 + \Psi_3, \]  

(4.21)

where \( \Psi_3 \) is the solution of the lee-wave equation (4.9) when the forcing term is \( -2\epsilon (\Psi_{p1B} \Psi_L)_{AA} \). The solution for \( \Psi_3 \) may be obtained in a conventional manner, using Laplace and Fourier transforms and the representation (4.13) for \( \Psi_L \). We find that the steady-state limit \( (T \to \infty) \) is given by

\[ \Psi_{3s} = (C_3/2C_1) \partial \Psi_{1s}/\partial A^2 + \Psi_{3s}, \quad S_{3s} = S_{3s}, \]

where

\[ \Psi_{3s} = \frac{1}{6} C_2^2 \int_0^1 k^2 \cos [kA + (1 - k^2)^{1/2} B] dk \]

\[ + \int_1^\infty k^2 \exp[-(k^2 - 1)^{1/2} B] \cos kAdk \]

\[ + (2/R) \int_0^1 [k \cos \phi - (1 - k^2)^{1/2} \sin \phi] \sin [kA + (1 - k^2)^{1/2} B] dk \]

\[ + (2/R) \int_1^\infty [k \cos \phi \sin kA - (k^2 - 1)^{1/2} \sin \phi \cos kA] \]

\[ \times \exp[-(k^2 - 1)^{1/2} B] dk]. \]  

(4.22)
The transient terms are $O(T^{-1})$. The method used by Miles and Huppert (1969) to transform $\Psi_L$ into the form (4.15) may be employed on $\Psi_{3s}$, and we can show that

$$\Psi_{3s} = eC_1^2 \left\{ -\frac{\pi}{8} Y_1(R) \sin 2\phi \cos \phi - \frac{\pi}{4} R^{-1} Y_0(R) \sin \phi 
+ (1/2R) \int_0^\pi \sin [R \sin(t + \phi)] \sin(t - \phi) \cos t \, dt 
+ (1/8) \int_0^\pi \cos [R \sin(t + \phi)] \sin 2t \sin t \, dt \right\}. \tag{4.23}$$

Using the representation (4.15) for $\Psi_L$ it may now be directly verified that $\Psi_{3s}$ satisfies the equation

$$\nabla^2 \Psi_{3s} + \Psi_{3s} = -2e\Psi_{p1B} \Psi_L. \tag{4.24}$$

As $R \to \infty$, the integrals in (4.22) may be evaluated by the method of stationary phase; using the known asymptotic expansions for $Y_0(R)$ and $Y_1(R)$ we can show that

$$\Psi_{3s} = \frac{1}{2} eC_1^2 (2\pi/R)^{1/2} \cos \left( R - \frac{\pi}{4} \right) \sin 2\phi \cos \phi H(A) + O(\|A\|^{-3/2}), \tag{4.25}$$

as $|A| \to \infty$.

As $R \to 0$, we may use the known formulae for $Y_1(R)$ and $Y_0(R)$ while the integrals in (4.23) can be evaluated by expanding in powers of $R$. We find that

$$\Psi_{3s} = \frac{1}{2} eC_1^2 \left\{ - (\gamma + \log 1/4R) (1/4R^{-1} \sin \phi - 1/8R \sin \phi + 1/5R \sin 2\phi \cos \phi) 
+ 1/4R^{-1} \sin 2\phi \cos \phi + 1/16R \sin 2\phi \cos \phi - 1/8R \sin \phi + O(R^2) \right\}. \tag{4.26}$$

In summary, the outer expansion describes the far-field response to the obstacle. In the steady-state limit the leading terms for $\psi$ in (4.5) are

$$\psi = \delta \Psi_L - \delta^2 \frac{C_2}{C_1} \Psi_{LA} + \delta^3 \frac{C_3}{2C_1} \Psi_{LAA} + \delta^3 \Psi_{3s} + O(\delta^4). \tag{4.27}$$

The first term is the lee-wave field due to a horizontal dipole at the origin. The next two terms are modifications of this dipole field due to a more accurate representation of the obstacle, while $\Psi_{3s}$ is a term due to the interaction of the lee-wave field $\Psi_L$ with the obstacle. As $R \to \infty$ upstream (i.e. $A \to -\infty$), the asymptotic expressions (4.16) and (4.25) show that $\psi$ is $O(\|A\|^{-1})$, and hence, at least to $O(\delta^3)$, there is no upstream influence.
5. THE MATCHING PROCEDURE

It remains to match the outer expansion with the inner expansion. In fact we shall do this only for the steady-state terms, and so match the outer expansion (4.27) to the steady-state limit of the inner expansion. The matching procedure will show that the outer expansion (4.27) is complete, and will provide the boundary conditions as \( \rho \to \infty \) which will enable the inner expansion to be completely determined. Strictly speaking, the matching should be accomplished using intermediate variables (Cole, 1968), but it will suffice here to write the outer expansion (4.27) in terms of the inner variables, and compare the result with the limit as \( \rho \to \infty \) of the inner expansion (cf. Van Dyke, 1964).

Thus we substitute \( R = \delta \rho \) in (4.27), and expand the result in \( \delta \), using (4.17) and (4.26). Recalling that \( \psi' = \delta \Psi \) by (2.17) and (4.5), we find that the inner limit of the outer expansion is

\[
\psi' = \delta^2 \log \delta \psi'_{2l} + \delta^2 \psi_{2l} + \delta^3 \psi_{3l} + \delta^4 \log \delta \psi'_{4l} + \delta^4 \psi_{4l} + O(\delta^5). \tag{5.1}
\]

where the expressions \( \psi'_{2l}, \psi_{2l} \ldots \) are given by

\[
\psi'_{2l} = -\frac{1}{2} C_1 \rho \sin \phi - \frac{1}{2} \varepsilon C_2^2 \rho^{-1} \sin \phi + O(\rho^{-2}),
\]

\[
\psi_{2l} = -\frac{1}{2} C_1 \rho \sin \phi \log \frac{\rho}{\rho_0} - \frac{1}{2} C_1 \rho \sin \phi (\gamma - \frac{1}{2})
+ \frac{1}{4} C_2 \sin 2\phi + \frac{1}{4} C_3 \rho^{-1} \cos 2\phi \sin \phi
+ \varepsilon C_1^2 \left\{ \frac{1}{4} \rho^{-1} \sin 2\phi \cos \phi - \frac{1}{2} (\gamma + \log \frac{\rho}{\rho_0}) \rho^{-1} \sin \phi \right\} + O(\rho^{-2}),
\]

\[
\psi_{3l} = -\frac{1}{6} C_1 \rho^2 \sin 2\phi + \frac{1}{2} C_2 \rho \sin \phi + O(\rho^{-1}),
\]

\[
\psi'_{4l} = \frac{1}{16} C_1 \rho^3 \sin \phi - \frac{1}{16} C_2 \rho^2 \sin 2\phi + \frac{1}{16} C_3 \rho \sin \phi
+ \varepsilon C_1^2 \left\{ \frac{1}{8} \rho \sin \phi - \frac{1}{8} \rho \sin 2\phi \cos \phi \right\} + O(\log \rho),
\]

\[
\psi_{4l} = \frac{1}{16} C_1 \rho^3 \sin \phi \log \frac{\rho}{\rho_0} - C_1 \rho^3 \sin \phi \left( \frac{5}{16} - \frac{\gamma}{16} \right)
- \frac{1}{16} C_2 \rho^2 \sin 2\phi (\log \frac{\rho}{\rho_0} + \frac{1}{2}) + C_2 \rho^2 \sin 2\phi \left( \frac{5}{16} - \frac{\gamma}{16} \right)
+ \frac{1}{16} C_3 \rho \sin \phi \log \frac{\rho}{\rho_0} + \frac{1}{8} C_3 \rho \sin 3\phi + C_3 \rho \sin \phi \left( \frac{\gamma}{16} - \frac{1}{32} \right)
+ \varepsilon C_1^2 \left\{ -\frac{1}{8} \rho \sin 2\phi \cos \phi \log \frac{\rho}{\rho_0} + \rho \sin 2\phi \cos \phi \left( \frac{1}{16} - \frac{\gamma}{8} \right)
+ \rho \sin \phi \left( \frac{\gamma}{8} - \frac{1}{8} \right) \right\} + O(\log \rho). \tag{5.6}
\]
Here the error terms arise from higher-order terms in the outer expansion (i.e. \( \Psi_{4s} \) etc). The expressions \( \psi_{21}, \psi_{22}, \ldots \) form the boundary conditions as \( \rho \to \infty \) for \( \psi_{2s}, \psi_{2s}, \ldots \).

Consider first \( \psi_{2s}^+ \), which is a potential flow (3.7), which satisfies the boundary condition \( \psi_{2s}^+ = 0 \) on \( \beta = 0 \), and \( \psi_{2s}^+ \sim \psi_{21}^+ \) as \( \rho \to \infty \). Hence

\[
\psi_{2s}^+ = -\frac{1}{2}C_1 \rho \sin \phi - \frac{1}{2} \varepsilon C_1^2 \rho^{-1} \sin \phi + \sum_{n=2}^{\infty} D_n \rho^{-n} \sin n\phi,
\]

where the constants \( D_n^+ \) can only be determined by matching to higher-order terms in the outer expansion.

Next \( \psi_{2s}^+ \) satisfies the Poisson equation (3.11)

\[
\nabla^2 \psi_{2s} = -V \psi_p.
\]

Using the expressions (2.19) and (2.21) for \( \psi_p \) and \( V \) respectively we can show that

\[
\psi_{2s} = -\frac{1}{2}C_1 \rho \sin \phi \log \rho + \frac{1}{3} C_2 \sin 2\phi + \frac{1}{8} (C_3 + \varepsilon C_1^2) \rho^{-1} \sin 3\phi
\]

\[-\frac{1}{2} \varepsilon C_1^2 \rho^{-1} \sin \phi \log \rho + D_{-1} \rho \sin \phi + D_1 \rho^{-1} \sin \phi + O(\rho^{-2}),
\]

where the constants \( D_{-1}, D_1 \) associated with the potential terms (i.e. solutions of \( \nabla^2 \psi_{2s} = 0 \)) are to be determined. Comparing (5.9) with (5.3), we see that the matching is achieved by choosing

\[
D_{-1} = \frac{1}{2} C_1 (\gamma - \frac{1}{2}), \quad D_1 = -\frac{1}{8} C_3 + \varepsilon C_1^2 \left( \frac{1}{8} - \gamma \right).
\]

Proceeding in this manner, we see from (3.12) that \( \psi_{3s} \) is a potential flow, and its outer boundary condition \( \psi_{3l} \) is also a potential flow, so that the matching can be achieved. For \( \psi_{4s}^+ \), we can use (5.7) to determine a particular integral for \( \psi_{4s}^+ \), which we can show matches with the non-potential terms in \( \psi_{4l}^+ \); the remaining potential terms in \( \psi_{4s}^+ \) determine the potential flow part of \( \psi_{4s}^+ \). Similarly we can show that \( \psi_{4s}^+ \) matches to \( \psi_{4l}^+ \) as \( \rho \to \infty \). In fact the expressions \( \psi_{2l}, \psi_{2t}, \ldots \) are the leading terms for \( \psi_{2s}, \psi_{2s} \ldots \) as \( \rho \to \infty \).

6. CONCLUSIONS AND COMMENTS

A perturbation expansion in powers and logarithms of \( \delta = Na/U \) about the state of potential flow over the obstacle shows that, after starting from rest, the flow develops on two length and time scales. In addition to potential flow, flow on the length scale \( a \) develops in the time scale \( a/U \) in the vicinity of the obstacle, and consists of a vortex with negative vorticity situated over the obstacle, together with transients which are advected...
downstream. Physically, this motion is due to buoyancy forces acting on the deformed density field caused by potential flow. On the longer time-scale $1/N$, flow develops with the length scale $U/N$, forced by the smaller-scale motion. This larger-scale motion is mostly composed of the lee-waves and hence is mainly on the downstream side. In fact, the leading order term for this large-scale motion is just that due to a dipole situated at the origin (i.e. the position of the obstacle). The expansion has been carried out to $O(\delta^3)$ in $\delta$ [i.e. error of $O(\delta^4)$], and to this order it does not yield any terms which are independent of the horizontal coordinate, or which are found far upstream in the steady state.

These results are fully consistent with those of the corresponding expansion in small obstacle height, and the regions of applicability of the two theories are shown in Figure 2. It seems probable that, in the steady state, both expansions are converging to Long's finite amplitude model (Long, 1955, Miles and Huppert, 1969, etc.) when $Nh/U$ is sufficiently small, and consequently they provide support for the assumptions required by this model. However, experimental observations (Baines, 1977, 1979) in fluid of finite depth with values of $\delta$ and/or $\varepsilon$ which are substantially less than unity show transient and steady-state effects in the flow (particularly upstream) which are not described by the small $\delta$ or small $\varepsilon$ expansions, and it appears in view of the results of the present study that these effects are primarily a consequence of the finite depth.
References