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Stratified Flow over Finite Obstacles with Weak Stratification

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The time-dependent problem of inviscid stratified flow over finite obstacles in fluid of infinite depth is considered in the limit of vanishing stratification ($Na/U \to 0$), as a perturbation about the state of potential flow. The object of this study is to shed light on the "upstream influence" question, namely, under what circumstances does the motion of the obstacle generate steady motions which will eventually be felt at an arbitrarily large distance upstream? In the present study the flow develops on both short and long length and time scales, and a matched asymptotic expansion procedure is required. The results from this expansion are consistent with those from the corresponding "classical" expansion in small obstacle height [McIntyre (1972)]. In particular, no permanent effects far upstream are obtained by this procedure, and this implies that "Long's model" is applicable for the case of infinite depth if Nh/U is sufficiently small.

INTRODUCTION

The problem of two-dimensional flow of inviscid stratified fluids over obstacles is of considerable importance in meteorological and other contexts. A major difficulty, which has aroused much controversy and is as yet unresolved, concerns the "upstream influence" question, namely: under what circumstances (if any) does the flow over the obstacle generate motion which will eventually propagate unattenuated to an infinite distance upstream? To resolve this theoretically requires consideration of an initial value problem, and the most common way of tackling this problem for inviscid stratified flow over obstacles is to assume that the obstacle height is small. The most complete exposition of this procedure as a formal expansion with height as small parameter is due to McIntyre (1972).

However, the results from this procedure are not entirely in accord with experimental observations in fluid of finite depth with obstacles with "moderately small" height (Baines, 1977, 1979). In particular, in many cases, transient and steady-state flow components are observed upstream which are not predicted theoretically, suggesting that the expansion in powers of obstacle height is inadequate in practice. Hence, it seems desirable to avoid the small-height assumption if possible, and this is accomplished in the present paper by an expansion with the stratification (or more exactly, stratification/fluid velocity) as small parameter. It transpires that this expansion necessitates the use of the technique of matched inner and outer asymptotic expansions, and in order to minimize the number of length scales involved we will consider only the case of infinite depth.

Although the nature of this expansion is quite different from that of the small-height expansion, the results obtained are consistent with it and are qualitatively the same. Consequently, they provide further formal justification for the validity of Long's finite amplitude model (Long, 1955) for the infinite-depth case, when Nh/U is sufficiently small. The procedure is also of interest in the way it describes the character and evolution of the flow field.

2. BASIC EQUATIONS

The basic variables and parameters are indicated in Figure 1. We take x and z as horizontal and vertical coordinates respectively in a frame fixed relative to the topography; $\mathbf{u}=(u,w)$ denotes the fluid velocity, and ψ is the stream function defined by

$$u=-\psi_z, \quad w=\psi_x. \tag{2.1}$$

$\rho_0+\rho$ denotes the total fluid density, where $\rho_0(z)$ is the equilibrium value, and g is the acceleration due to gravity. The inviscid equations for the vorticity and density are then

$$\left.\begin{aligned}(\partial/\partial t+\mathbf{u}\cdot\nabla)\nabla^2\psi&=-(g/\bar{\rho}_0)\rho_x,\\ (\partial/\partial t+\mathbf{u}\cdot\nabla)\rho&=-(d\rho_0/dz)\psi_x,\end{aligned}\right\} \tag{2.2}$$

with boundary conditions

$$\psi\to -Uz \quad \text{as} \quad x^2+z^2\to\infty,$$

for all finite t, and

$$\psi = 0, \quad \text{on} \quad z = hf(x).$$

Here the Boussinesq approximation has been made, and $\bar{\rho}_0$ denotes the mean value of $\rho_0(z)$ over a sufficiently large depth of fluid (conceptually this presents no difficulty, particularly since we are concerned with the weak-stratification limit). If we write

$$\tilde{\psi} = \psi + Uz, \quad u = U - \tilde{\psi}_z, \quad w = \tilde{\psi}_x, \tag{2.3}$$

the boundary conditions on $\tilde{\psi}$ take the more convenient form

$$\left.\begin{aligned} &\tilde{\psi} \to 0 \quad \text{as} \quad x^2 + z^2 \to \infty, \\ &\tilde{\psi} = Uz \quad \text{on} \quad z = hf(x), \end{aligned}\right\} \tag{2.4}$$

and the equations (2.2) still hold with $\nabla^2\tilde{\psi}$ replacing $\nabla^2\psi$. For initial conditions, we assume that at $t=0$ the fluid commences motion in the positive x direction with a uniform mean velocity U, or, equivalently, that the obstacle abruptly commences motion at a uniform speed U through the undisturbed fluid in the negative x-direction.

The Brunt–Väisälä frequency N is defined by

$$N^2 = -(g/\bar{\rho}_0)d\rho_0/dz,$$

and is assumed to be constant here. If $N=0$, the complete solution to the above problem is given by potential flow where, for $t>0$, $\tilde{\psi}$ satisfies $\nabla^2\tilde{\psi} = 0$. We suppose that for this potential flow we have

$$(\tilde{\psi}_z, \tilde{\psi}_x)_{\max} = \varepsilon U, \tag{2.5}$$

where ε will be small if the topographic slope is everywhere small. In general, ε will not be large unless the obstacle has fairly sharp convex corners (as seen from the fluid side), and we will assume for present purposes that $\varepsilon = O(1)$. This restriction is not a serious one, and may be relaxed later if necessary. If h is the height of the obstacle we may define a such that $\varepsilon = h/a$. For simple obstacles such as that shown in Figure 1, $2a$ and h/a are representative of the width and maximum slope of the obstacle respectively.

The dimensional quantitites governing the flow are therefore h, a, U and N, from which we may construct the following dimensionless quantities:

$$\varepsilon = h/a, \quad \delta = Na/U, \quad R = \varepsilon\delta = Nh/U, \tag{2.6}$$

only two of which are independent.

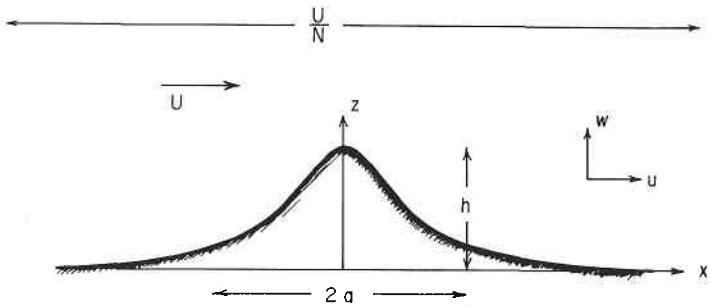

FIGURE 1 Definition sketch.

We now scale the equations in a manner which is consistent with N small, viz we take a as unit of length, U as unit of velocity and a/U as unit of time, so that

$$x = a\bar{x}, \quad t = (a/U)\bar{t}, \quad \mathbf{u} = U\mathbf{i} + \varepsilon U\bar{\mathbf{u}}, \tag{2.7}$$

where $\mathbf{i}$ is the unit vector in the positive x direction, and

$$\bar{\psi} = Uh\bar{\psi}, \quad \sigma = g\rho/\bar{\rho}_0 = N^2 h\bar{\sigma}. \tag{2.8}$$

The dimensionless equations then are

$$\left.\begin{aligned} \left(\frac{\partial}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}\right)\bar{\nabla}^2\bar{\psi} &= -\delta^2\bar{\sigma}_{\bar{x}} - \varepsilon\bar{\mathbf{u}}\cdot\bar{\nabla}\bar{\nabla}^2\bar{\psi}, \\ \left(\frac{\partial}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}\right)\bar{\sigma} &= \bar{\psi}_{\bar{x}} - \varepsilon\bar{\mathbf{u}}\cdot\bar{\nabla}\bar{\sigma}, \end{aligned}\right\} \tag{2.9}$$

with

$$\bar{\psi} = f(a\bar{x}) \quad \text{on} \quad \bar{z} = \varepsilon f(a\bar{x}).$$

Henceforth we omit the overbars and use ψ, σ to denote $\bar{\psi}$, $\bar{\sigma}$ etc, with $u = -\psi_z$, $w = \psi_x$, so that the above equations are

$$\left.\begin{aligned} \left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right]\nabla^2\psi &= -\delta^2\sigma_x - \varepsilon[\psi_x(\nabla^2\psi)_z - \psi_z(\nabla^2\psi)_x], \\ \left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right]\sigma &= \psi_x - \varepsilon(\psi_x\sigma_z - \psi_z\sigma_x), \end{aligned}\right\} \tag{2.10}$$

with

$$\psi = f(ax) \quad \text{on} \quad z = \varepsilon f(ax).$$

The initial conditions give

$$\psi = \psi_p(x, z), \quad \sigma = 0 \quad \text{at} \quad t = 0+, \tag{2.11}$$

where ψ_p denotes the stream function for potential flow, and is the complete solution in the limit $\delta = 0$ (i.e. zero stratification). We shall assume that the obstacle is such that

$$\psi_p = \frac{C_1 z}{x^2 + z^2} + O\left(\frac{1}{(x^2 + z^2)}\right), \quad \text{as} \quad (x^2 + z^2) \to \infty. \tag{2.12}$$

Here the dipole moment C_1 is a constant of order unity, and for small ε is approximately equal to (area of the obstacle)/π. The assumption (2.12) is satisfied by obstacles of finite width, such as that shown in Figure 1.

It is convenient to change co-ordinates from (x, z) by using a conformal transformation to the set (α, β) consisting of the streamlines and potential lines of the total potential flow where β is defined by

$$\beta = z - \varepsilon \psi_p(x, z), \tag{2.13}$$

and α is obtained from the Cauchy–Reimann equations $\alpha_x = \beta_z$ and $\alpha_z = -\beta_x$, (where the suffixes denote derivatives), and the requirement that $\alpha \to x$ as $x^2 + z^2 \to \infty$. The Jacobian of the transformation is

$$V(\alpha, \beta) = \frac{\partial(x, z)}{\partial(\alpha, \beta)} = z_\alpha^2 + z_\beta^2, \tag{2.14}$$

which is always positive. The vorticity is

$$\psi_{xx} + \psi_{zz} = V^{-1}(\psi_{\alpha\alpha} + \psi_{\beta\beta}), \tag{2.15}$$

and for any variables P and Q we have

$$J(P, Q) = V\frac{\partial(P, Q)}{\partial(x, z)} = \frac{\partial(P, Q)}{\partial(\alpha, \beta)}. \tag{2.16}$$

It is also convenient to write

$$\psi = \psi_p + \psi', \tag{2.17}$$

and for the remainder of this paper we define $\nabla^2\psi' = \psi'_{\alpha\alpha} + \psi'_{\beta\beta}$. Equations (2.10) then become

$$\left.\begin{aligned} \left(V\frac{\partial}{\partial t}+\frac{\partial}{\partial \alpha}\right)(V^{-1}\nabla^2\psi') &= -\delta^2[\sigma_\alpha + \varepsilon J(\sigma, \psi_p)] - \varepsilon J(\psi', V^{-1}\nabla^2\psi'), \\ \left(V\frac{\partial}{\partial t}+\frac{\partial}{\partial \alpha}\right)\sigma &= \psi_{p\alpha} + \psi'_\alpha + \varepsilon J(\psi, \psi_p) - \varepsilon J(\psi', \sigma). \end{aligned}\right\} \quad (2.18)$$

The flow domain is now $-\infty < \alpha < \infty$, $\beta > 0$, the boundary condition is $\psi' = 0$ at $\beta = 0$, and the initial conditions are that ψ', $\sigma = 0$ at $t = 0+$.

Now it follows from (2.12) that we may write

$$\psi_p = \sum_{n=1}^{\infty} \frac{C_n \sin n\phi}{\rho^n} = \sum_{n=1}^{\infty} \frac{C_n(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial \alpha^{n-1}}\left(\frac{\sin \phi}{\rho}\right), \quad (2.19)$$

where $\alpha = \rho\cos\phi$, $\beta = \rho \sin\phi$ and the C_n are constants. Then it follows from (2.13) and the Cauchy–Riemann equations that

$$x = \alpha - \varepsilon \sum_{n=1}^{\infty} \frac{C_n \cos n\phi}{\rho^n}, \quad z = \beta + \varepsilon \sum_{n=1}^{\infty} \frac{C_n \sin n\phi}{\rho^n}. \quad (2.20)$$

Also from (2.14) we can show that

$$V = 1 + \frac{2\varepsilon C_1 \cos 2\phi}{\rho^2} + \frac{4\varepsilon C_2 \cos 3\phi}{\rho^3} + O\left(\frac{1}{\rho^4}\right). \quad (2.21)$$

3. THE INNER EXPANSION

We wish to solve equations (2.18) with δ small and $\varepsilon = O(1)$. To do this we might first look for a solution in which ψ', σ are expanded as power series in δ. Such an expansion is valid only when ρ is O(1), and fails as $\rho \to \infty$, where it must be matched with an outer expansion (e.g. Van Dyke, 1964, or Cole 1968). It transpires that the matching procedure necessitates the presence of terms proportional to $\delta^n \log \delta$ in the inner expansion. Anticipating this, we shall seek an inner expansion of the form

$$\begin{aligned} \psi' &= \psi_0 + \delta^2 \log\delta\, \psi_2^+ + \delta^2\psi_2 + \delta^3\psi_3 + \delta^4 (\log\delta)^2 \psi_4^{++} \\ &\quad + \delta^4 \log\delta\, \psi_4^+ + \delta^4\psi_4 + \ldots, \\ \sigma &= \sigma_0 + \delta^2 \log\delta\, \sigma_2^+ + \delta^2\sigma_2 + \delta^3\sigma_3 + \delta^4 (\log\delta)^2\sigma_4^{++} + \delta^4 \log\delta\, \sigma_4^+ + \delta^4\sigma_4 + \ldots. \end{aligned} \quad (3.1)$$

At $t=0$, ψ_0, ψ_2^+, ψ_2 ... and σ_0, σ_2^+, σ_2 ... and zero and on $\beta=0$, ψ_0, ψ_2^+, ψ_2 ... are zero.

Substituting (3.1) into (2.18) it is readily seen that $\psi_0=0$, and that σ_0 satisfies the equation

$$\left(V\frac{\partial}{\partial t}+\frac{\partial}{\partial\alpha}\right)\sigma_0=\psi_{p\alpha}. \tag{3.2}$$

Let $t^*=t^*(\alpha,\beta,t)$ be defined by

$$t=\int_{\alpha-t^*}^{\alpha}V(\alpha',\beta)d\alpha'. \tag{3.3}$$

Then using α, β, t^* as co-ordinates in place of α, β, t, it may be shown that for any variable P

$$\left(V\frac{\partial}{\partial t}+\frac{\partial}{\partial\alpha}\right)P=\left(\frac{\partial}{\partial t^*}+\frac{\partial}{\partial\alpha}\right)P. \tag{3.4}$$

Hence σ_0 is given by

$$\sigma_0=\psi_p(\alpha,\beta)-\psi_p(\alpha-t^*,\beta), \tag{3.5}$$

since $\sigma_0=0$ at $t=0$. Now as $t\to\infty$, $t^*\sim t+t_0(\alpha,\beta)$ where $t_0(\alpha,\beta)=O(1)$ uniformly in α, β. σ_0 therefore consists of a region of elevated density surfaces over the obstacle coinciding with the streamlines of potential flow, and an equal and opposite (apart from the geometric distortion near the obstacle) region of depressed density surfaces, which is advected downstream with the mean flow. We know on simple physical grounds that such a depressed density region will ultimately return to its equilibrium configuration, which does not constitute a correction of $O(\delta^2)$; hence the transient part of this expansion (at least) cannot be uniformly valid as $t\to\infty$ for *all* α, β. However, our interest is primarily with the steady-state solution, and for *fixed* α, β of O(1), $\sigma_0\to\psi_p$ as $t\to\infty$. It is interesting to note that if the initial conditions are such that the obstacle is suddenly introduced into a uniform stream (as considered by McIntyre, 1972) we have $\sigma_0=\psi_p$ and the transient part of σ_0 vanishes.

Turning next to the terms in $\delta^2\log\delta$ it follows from (2.18) and (3.5) that

$$\left.\begin{aligned}\left(V\frac{\partial}{\partial t}+\frac{\partial}{\partial\alpha}\right)(V^{-1}\nabla^2\psi_2^+)&=0,\\ \left(V\frac{\partial}{\partial t}+\frac{\partial}{\partial\alpha}\right)\sigma_2^+&=g_2^+,\end{aligned}\right\} \tag{3.6}$$

where

$$g_2^+ = \psi_{2\alpha}^+ + \varepsilon J(\psi_2^+, \psi_p(\alpha - t^*, \beta)).$$

Then using α, β, t^* as co-ordinates [*vide* (3.3)], it follows from (3.4) and the initial conditions that

$$\left.\begin{aligned} \nabla^2\psi_2^+ &= 0, \\ \sigma_2^+ &= \int_{\alpha - t^*}^{\alpha} g_2^+(\alpha', \beta, \alpha' - \alpha + t^*)d\alpha'. \end{aligned}\right\} \tag{3.7}$$

Thus ψ_2^+ is a potential which satisfies the boundary condition $\psi_2^+ = 0$ on $\beta = 0$. A complete determination of ψ_2^+ requires a knowledge of the behaviour of ψ_2^+ as $\rho \to \infty$; this, in turn, is provided by the inner limit of the outer expansion, and must await the calculation of the outer expansion, which is described in §4. Once ψ_2^+ has been determined, g_2^+ is known and hence σ_2^+ is given by (3.7). However, our main concern here is with the steady-state terms, which we shall denote by an additional suffix "*s*". It is easily seen from (3.6) and (3.7) that

$$\nabla^2\psi_{2s}^+ = 0, \qquad \sigma_{2s}^+ = \psi_{2s}^+. \tag{3.8}$$

The complete determination of ψ_{2s}^+ is described in §5 where the behaviour of ψ_{2s}^+ as $\rho \to \infty$ is determined by the inner limit of the steady part of the outer expansion.

For the terms proportional to δ^2 it follows from (2.18) and (3.5) that

$$\left.\begin{aligned} \left(V\frac{\partial}{\partial t} + \frac{\partial}{\partial \alpha}\right)(V^{-1}\nabla^2\psi_2) &= f_2, \\ \left(V\frac{\partial}{\partial t} + \frac{\partial}{\partial \alpha}\right)\sigma_2 &= g_2, \\ \text{where} \qquad f_2 &= -\sigma_{0\alpha} - \varepsilon J(\psi_p(\alpha, \beta), \psi_p(\alpha - t^*, \beta)), \\ g_2 &= \psi_{2\alpha} + \varepsilon J(\psi_2, \psi_p(\alpha - t^*, \beta)). \end{aligned}\right\} \tag{3.9}$$

Again, using α, β, t^* as co-ordinates [*vide* (3.3)] it follows from (3.4) and the initial conditions that

$$\left.\begin{aligned} V^{-1}\nabla^2\psi_2 &= \int_{\alpha - t^*}^{\alpha} f_2(\alpha', \beta, \alpha' - \alpha + t^*)d\alpha', \\ \sigma_2 &= \int_{\alpha - t^*}^{\alpha} g_2(\alpha', \beta, \alpha' - \alpha + t^*)d\alpha'. \end{aligned}\right\} \tag{3.10}$$

Here f_2 is known, and (3.10) with the boundary condition $\psi_2=0$ on $\beta=0$ is a Poisson equation for ψ_2. As for ψ_2^+, a complete determination of ψ_2 requires a knowledge of the behaviour of ψ_2 as $\rho\to\infty$, which must await the calculation of the inner limit of the outer expansion. Once ψ_2 is known, g_2 is known and hence σ_2 is known. Considering only the steady-state terms, it follows easily from (3.5), (3.9) and (3.10) that

$$V^{-1}\nabla^2\psi_2=-\psi_p, \quad \sigma_{2s}=\psi_{2s}. \tag{3.11}$$

The transient parts are centred on $\alpha=t^*$ and are $O(t^{-1})$. Equation (3.11) shows that ψ_{2s} has vorticity $-\psi_p$; since ψ_p is generally positive, this flow has negative vorticity.

The calculation of the terms ψ_3, ψ_4^{++}, ψ_4^+, ψ_4 ... and σ_3, σ_4^{++}, σ_4^+, σ_4 ... proceeds in a similar way. At each stage a pair of equations analogous to (3.9) is obtained, whose solution is analogous to (3.10); at each stage the forcing terms f_3, f_4^{++}, f_4^+, f_4 ... are known, and hence each of ψ_3, ψ_4^{++}, ψ_4^+, ψ_4 ... is governed by a Poisson equation. The complete determination of ψ_3, ψ_4^{++}, ψ_4^+, ψ_4 ... requires a knowledge of the behaviour of ψ_3, ψ_4^{++}, ψ_4^+, ψ_4 ... as $\rho\to\infty$, which is obtained from the outer expansion. Once ψ_3, ψ_4^{++}, ψ_4^+, ψ_4 ... are known, g_3, g_4^{++}, g_4^+, g_4 ... are known, and hence σ_3, σ_4^{++}, σ_4^+, σ_4 ... are known. Considering only the steady-state terms it may be shown that

$$\left.\begin{aligned} &\nabla^2\psi_{3s}=0, && \sigma_{3s}=\psi_{3s},\\ &V^{-1}\nabla^2\psi_{4s}^+=-\psi_{2s}^+, && \sigma_{4s}^+=\psi_{4s}^+,\\ &V^{-1}\nabla^2\psi_{4s}=-\psi_{2s}, && \sigma_{4s}=\psi_{4s}, \end{aligned}\right\} \tag{3.12}$$

while ψ_{4s}^{++} and σ_{4s}^{++} are both zero. The steady-state terms thus satisfy the sequence of equations (3.5), (3.8), (3.11) and (3.12). Consequently the steady-state parts ψ_s' and σ_s of the expansions (3.1) satisfy the equations

$$V^{-1}\nabla^2\psi_s'+\delta^2\psi_s'=-\delta^2\psi_p, \quad \sigma_s=\psi_p+\psi_s'. \tag{3.13}$$

Thus the steady-state stream function ψ_s' satisfies the Helmholtz equation with a forcing term $-\delta^2\psi_p$; the stream function $\psi_s'+\psi_p$ satisfies the homogeneous Helmholtz equation. Thus the steady-state part of the inner expansion reduces to Long's model (1955), with the important difference that the behaviour of ψ_s' as $\rho\to\infty$ is not specified *a priori* but is to be determined by matching with the outer expansion.

4. THE OUTER EXPANSION

The inner expansion fails as $\rho \to \infty$, and it is necessary to introduce an outer expansion to describe the flow on length and time scales of δ^{-1} (which in dimensional terms are UN^{-1} and N^{-1} respectively). Hence we put

$$A=\delta\alpha, \quad B=\delta\beta, \quad T=\delta t, \quad R=\delta\rho. \tag{4.1}$$

From (2.19) it follows that

$$\left.\begin{aligned} &\psi_p=\delta\Psi_p, \quad \Psi_p=\sum_{n=1}^{\infty} \delta^{n-1}\Psi_{pn}, \\ \text{where} \qquad & \\ &\Psi_{pn}=C_n \sin n\phi/R^n. \end{aligned}\right\} \tag{4.2}$$

Here $A=R\cos\phi$, $B=R\sin\phi$. Then from (2.14) and (2.21) we may also put

$$\left.\begin{aligned} &V=1+\sum_{n=2}^{\infty} \delta^n V_n, \\ \text{where} \qquad & \\ &V_2=2\varepsilon C_1 \cos 2\phi/R^2, \quad \text{and} \quad V_3=4\varepsilon C_2 \cos 3\phi/R^3, \end{aligned}\right\} \tag{4.3}$$

while from (3.3) and (3.5) we may write

$$\left.\begin{aligned} &\sigma_0=\delta S_0, \quad S_0=\sum_{n=1}^{\infty} \delta^{n-1} S_{0n}, \\ \text{where} \qquad &S_{01}=\Psi_{p1}(A,B)-\Psi_{p1}(A-T,B), \\ &S_{02}=\Psi_{p2}(A,B)-\Psi_{p2}(A-T,B), \text{etc.} \end{aligned}\right\} \tag{4.4}$$

Next we put

$$\psi=\delta(\Psi_p+\Psi), \quad \sigma=\delta(S_0+S), \tag{4.5}$$

where Ψ, S are regarded as functions of A, B, T. Then equations (2.18) become

$$\left.\begin{aligned} &\left(\frac{\partial}{\partial T}+\frac{\partial}{\partial A}\right)\nabla^2\Psi+S_A=F, \\ &\left(\frac{\partial}{\partial T}+\frac{\partial}{\partial A}\right)S-\Psi_A=G, \end{aligned}\right\} \tag{4.6}$$

where the "forcing" terms on the right-hand side are given by

$$\left.\begin{aligned} F &= -S_{0_A} - \frac{\partial}{\partial A}\{(V^{-1}-1)\nabla^2\Psi\} - \varepsilon\delta^2 J(S_0+S, \Psi_p) - \varepsilon\delta^2 J(\Psi, V^{-1}\nabla^2\Psi), \\ &\text{and} \\ G &= \Psi_{pA} - \left(\frac{\partial}{\partial T}+\frac{\partial}{\partial A}\right)S_0 - (V-1)\frac{\partial}{\partial T}(S_0+S) + \varepsilon\delta^2 J(\Psi, \Psi_p - S_0 - S). \end{aligned}\right\} \quad (4.7)$$

We then look for an asymptotic solution of (4.6) of the form

$$\Psi = \sum_{n=1}^{\infty} \delta^{n-1}\Psi_n, \quad S = \sum_{n=1}^{\infty} \delta^{n-1} S_n. \quad (4.8)$$

At each stage we obtain a pair of equations of the form

$$\left.\begin{aligned} \left(\frac{\partial}{\partial T}+\frac{\partial}{\partial A}\right)\nabla^2\Psi_n + S_{nA} &= F_n, \\ \left(\frac{\partial}{\partial T}+\frac{\partial}{\partial A}\right)S_n - \Psi_{nA} &= G_n, \end{aligned}\right\} \quad (4.9)$$

where it is apparent from (4.7) that each F_n, G_n depends only on $\Psi_1, \ldots, \Psi_{n-1}$ and $S_1, \ldots, S_{n-1}$, and so can be regarded as known forcing terms. Equations (4.9) constitute a pair of "lee-wave" equations, which may be combined into the single equation for Ψ_n:

$$\left.\begin{aligned} \left(\frac{\partial}{\partial T}+\frac{\partial}{\partial A}\right)^2\nabla^2\Psi_n + \Psi_{nAA} &= H_n, \\ &\text{where} \\ H_n = \left(\frac{\partial}{\partial T}+\frac{\partial}{\partial A}\right)F_n - \frac{\partial}{\partial A}G_n&. \end{aligned}\right\} \quad (4.10)$$

The initial conditions are that Ψ_n, $S_n = 0$ at $T=0$, and the boundary conditions $\Psi_n = 0$ on $B=0$. Examination of (4.2), (4.3) and (4.4) indicates that the forcing terms F_n, G_n, and hence H_n, are singular as $R \to 0$. Hence equation (4.10) describes a "lee-wave" problem which is being forced by known singular forcing terms. These singular forcing terms are a "far-field" representation of the obstacle, and one of the main advantages in using the α, β co-ordinate system, or the A, B coordinate system, is the succinct way in which the obstacle is represented by appropriate singular forcing terms.

Now putting $n=1$, we see from (4.2), (4.3), (4.4) and (4.7) that $F_1 = -S_{01A}$, $G_1=0$, and hence from (4.4) and (4.9)

$$H_1 = -\partial^2\Psi_{p1}/\partial A^2. \tag{4.11}$$

Since Ψ_{p1} is $C_1 \sin\phi/R$ by (4.2), the $n=1$ term in the outer expansion is the response to a dipole placed at the origin. This problem (4.9) with $n=1$ may be solved in a conventional manner using Laplace and Fourier transforms, and has been considered by Miles (1969) for the analogous problem for a dipole in a rotating fluid. It may be shown that the steady-state limit ($T\to\infty$) is given by

$$\Psi_{1s} = -\Psi_{p1} + \Psi_L, \quad S_{1s} = \Psi_{1s}, \tag{4.12}$$

where

$$\Psi_L = C_1 \int_0^1 \cos[kA + (1-k^2)^{\frac{1}{2}}B]dk + C_1 \int_1^\infty \exp[-(k^2-1)^{\frac{1}{2}}B]\cos kA\, dk. \tag{4.13}$$

Here Ψ_L satisfies the equation

$$\nabla^2\Psi_L + \Psi_L = 0, \tag{4.14}$$

and describes the lee-wave field due to a horizontal dipole placed at the origin (Miles and Huppert, 1969). Following the method used by Miles (1969), it may be shown that the transient terms are $O(T^{-1})$. Miles and Huppert (1969) have shown that Ψ_L may be expressed in the form

$$\Psi_L = C_1\{-\tfrac{1}{2}\pi Y_1(R)\sin\phi + \tfrac{1}{2}\int_0^\pi \cos[R\sin(t+\phi)]\cos t\, dt\}, \tag{4.15}$$

where $Y_1(R)$ is the Bessel function of the second kind of order one. As $R\to\infty$, the integral in (4.15) may be evaluated by the method of stationary phase; using the known asymptotic expansions for $Y_1(R)$ it may be shown that

$$\Psi_L = C_1(2\pi/R)^{1/2}\cos(R-\tfrac{1}{4}\pi)\sin\phi H(A) + C_1 \sin B/A + O|A|^{-3/2}). \quad \text{as } |A|\to\infty, \tag{4.16}$$

where $H(A)$ is the Heaviside function (zero for $A<0$ and one for $A>0$). The first term in (4.16) describes the downstream lee-wave field. As $R\to 0$, we may use the known formulae for $Y_1(R)$, while the integral in (4.15) can be evaluated by expanding in powers of R. We find that,

$$\Psi_{1s}=C_1 \sin\phi\{-(\gamma+\log\tfrac{1}{2}R)[\tfrac{1}{2}R-\tfrac{1}{16}R^3+O(R^5)]+\tfrac{1}{4}R-\tfrac{5}{64}R^3+O(R^5)\}$$
$$-\tfrac{1}{6}C_1R^2\sin 2\phi+O(R^4), \qquad (4.17)$$

where γ is Euler's constant (0.5772...).

Next we put $n=2$ in (4.8), and we see from (4.2), (4.3), (4.4) and (4.7) that $F_2=-S_{02A}$, $G_2=0$, and hence from (4.4) and (4.7)

$$H_2=-\partial^2\Psi_{p2}/\partial A^2. \qquad (4.18)$$

But $\Psi_{p2}=-(\partial/\partial A)\ (C_2\sin\phi/R)$, and hence $\Psi_2=-(C_2/C_1)\partial\Psi_1/\partial A$. In particular, the steady-state limit is

$$\Psi_{2s}=-(C_2/C_1)\partial\Psi_{1s}/\partial A,\quad S_{2s}=\Psi_{2s}. \qquad (4.19)$$

For $n=3$, we shall only calculate the steady-state parts F_{2s}, G_{2s} of the forcing terms F_2, G_2; it may be shown that the transient parts are $O(T^{-2})$ as $T\to\infty$, and they contribute only a transient term $O(T^{-1})$ to Ψ_3. After some manipulation, using (4.2), (4.3), (4.4), (4.12) and (4.14) in (4.7), we find that

$$\left.\begin{aligned} &F_{2s}=-\Psi_{p3A}-2\varepsilon(\partial/\partial A)(\Psi_{p1B}\Psi_L),\quad G_{2s}=0,\\ \text{and}\\ &H_{2s}=F_{2sA}.\end{aligned}\right\} \qquad (4.20)$$

Hence, since $\Psi_{p3}=\tfrac{1}{2}(\partial^2/\partial A^2)(C_3\sin\phi/R)$, we may put

$$\Psi_3=(C_3/2C_1)\partial^2\Psi_1/\partial A^2+\hat{\Psi}_3, \qquad (4.21)$$

where $\hat{\Psi}_3$ is the solution of the lee-wave equation (4.9) when the forcing term is $-2\varepsilon(\Psi_{p1B}\Psi_L)_{AA}$. The solution for $\hat{\Psi}_3$ may be obtained in a conventional manner, using Laplace and Fourier transforms and the representation (4.13) for Ψ_L. We find that the steady-state limit ($T\to\infty$) is given by

$$\Psi_{3s}=(C_3/2C_1)\partial^2\Psi_{1s}/\partial A^2+\hat{\Psi}_{3s},\quad S_{3s}=\Psi_{3s},$$

where

$$\begin{aligned}\hat{\Psi}_{3s}=\tfrac{1}{2}\varepsilon C_1^2\{&\textstyle\int_0^1 k^2\cos[kA+(1-k^2)^{1/2}B]dk\\ &+\textstyle\int_1^\infty K^2\exp[-(k^2-1)^{1/2}B]\cos kA dk\\ &+(2/R)\textstyle\int_0^1[k\cos\phi-(1-k^2)^{1/2}\sin\phi]\sin[kA+(1-k^2)^{1/2}B]dk\\ &+(2/R)\textstyle\int_1^\infty[k\cos\phi\sin kA-(k^2-1)^{1/2}\sin\phi\cos kA]\\ &\times\exp[-(k^2-1)^{1/2}B]dk\}. \end{aligned} \qquad (4.22)$$

The transient terms are $O(T^{-1})$. The method used by Miles and Huppert (1969) to transform Ψ_L into the form (4.15) may be employed on $\hat{\Psi}_{3s}$, and we can show that

$$\hat{\Psi}_{3s}=\varepsilon C_1^2\left\{-\frac{\pi}{8}Y_1(R)\sin 2\phi\cos\phi-\frac{\pi}{4}R^{-1}Y_0(R)\sin\phi\right.$$
$$+(1/2R)\int_0^\pi \sin[R\sin(t+\phi)]\sin(t-\phi)\cos t\,dt$$
$$\left.+(1/8)\int_0^\pi \cos[R\sin(t+\phi)]\sin 2t\sin t\,dt\right\}. \tag{4.23}$$

Using the representation (4.15) for Ψ_L it may now be directly verified that Ψ_{3s} satisfies the equation

$$\nabla^2\hat{\Psi}_{3s}+\hat{\Psi}_{3s}=-2\varepsilon\Psi_{p1B}\Psi_L. \tag{4.24}$$

As $R\to\infty$, the integrals in (4.22) may be evaluated by the method of stationary phase; using the known asymptotic expansions for $Y_0(R)$ and $Y_1(R)$ we can show that

$$\hat{\Psi}_{3s}=\tfrac{1}{4}\varepsilon C_1^2(2\pi/R)^{1/2}\cos\left(R-\frac{\pi}{4}\right)\sin 2\phi\cos\phi\, H(A)+O(|A|^{-3/2}),$$
$$\text{as } |A|\to\infty. \tag{4.25}$$

As $R\to 0$, we may use the known formulae for $Y_1(R)$ and $Y_0(R)$ while the integrals in (4.23) can be evaluated by expanding in powers of R. We find that

$$\hat{\Psi}_{3s}=\varepsilon C_1^2\{-(\gamma+\log\tfrac{1}{2}R)(\tfrac{1}{2}R^{-1}\sin\phi-\tfrac{1}{8}R\sin\phi+\tfrac{1}{8}R\sin 2\phi\cos\phi)$$
$$+\tfrac{1}{4}R^{-1}\sin 2\phi\cos\phi+\tfrac{1}{16}R\sin 2\phi\cos\phi-\tfrac{1}{8}R\sin\phi+O(R^2)\}. \tag{4.26}$$

In summary, the outer expansion describes the far-field response to the obstacle. In the steady-state limit the leading terms for ψ in (4.5) are

$$\psi=\delta\Psi_L-\delta^2\frac{C_2}{C_1}\Psi_{LA}+\delta^3\frac{C_3}{2C_1}\Psi_{LAA}+\delta^3\hat{\Psi}_{3s}+O(\delta^4). \tag{4.27}$$

The first term is the lee-wave field due to a horizontal dipole at the origin. The next two terms are modifications of this dipole field due to a more accurate representation of the obstacle, while $\hat{\Psi}_{3s}$ is a term due to the interaction of the lee-wave field Ψ_L with the obstacle. As $R\to\infty$ upstream (i.e. $A\to-\infty$), the asymptotic expressions (4.16) and (4.25) show that ψ is $O(|A|^{-1})$, and hence, at least to $O(\delta^3)$, there is no upstream influence.

5. THE MATCHING PROCEDURE

It remains to match the outer expansion with the inner expansion. In fact we shall do this only for the steady-state terms, and so match the outer expansion (4.27) to the steady-state limit of the inner expansion. The matching procedure will show that the outer expansion (4.27) is complete, and will provide the boundary conditions as $\rho\to\infty$ which will enable the inner expansion to be completely determined. Strictly speaking, the matching should be accomplished using intermediate variables (Cole, 1968), but it will suffice here to write the outer expansion (4.27) in terms of the inner variables, and compare the result with the limit as $\rho\to\infty$ of the inner expansion (cf. Van Dyke, 1964).

Thus we substitute $R=\delta\rho$ in (4.27), and expand the result in δ, using (4.17) and (4.26). Recalling that $\psi'=\delta\Psi$ by (2.17) and (4.5), we find that the inner limit of the outer expansion is

$$\psi'=\delta^2\log\delta\,\psi_{2l}^{+}+\delta^2\psi_{2l}+\delta^3\psi_{3l}+\delta^4\log\delta\,\psi_{4l}^{+}+\delta^4\psi_{4l}+\mathrm{O}(\delta^5). \tag{5.1}$$

where the expressions ψ_{2l}^{+}, ψ_{2l} ... are given by

$$\psi_{2l}^{+}=-\tfrac{1}{2}C_1\rho\sin\phi-\tfrac{1}{2}\varepsilon C_1^2\rho^{-1}\sin\phi+\mathrm{O}(\rho^{-2}), \tag{5.2}$$

$$\left.\begin{aligned}\psi_{2l}=&-\tfrac{1}{2}C_1\rho\sin\phi\log\tfrac{1}{2}\rho-\tfrac{1}{2}C_1\rho\sin\phi(\gamma-\tfrac{1}{2})\\&+\tfrac{1}{4}C_2\sin2\phi+\tfrac{1}{4}C_3\rho^{-1}\cos2\phi\sin\phi\\&+\varepsilon C_1^2\{\tfrac{1}{4}\rho^{-1}\sin2\phi\cos\phi-\tfrac{1}{2}(\gamma+\log\tfrac{1}{2}\rho)\rho^{-1}\sin\phi\}+\mathrm{O}(\rho^{-2}),\end{aligned}\right\} \tag{5.3}$$

$$\psi_{3l}=-\tfrac{1}{6}C_1\rho^2\sin2\phi+\tfrac{1}{3}C_2\rho\sin\phi+\mathrm{O}(\rho^{-1}), \tag{5.4}$$

$$\begin{aligned}\psi_{4l}^{+}=&\tfrac{1}{16}C_1\rho^3\sin\phi-\tfrac{1}{16}C_2\rho^2\sin2\phi+\tfrac{1}{16}C_3\rho\sin\phi\\&+\varepsilon C_1^2\{\tfrac{1}{8}\rho\sin\phi-\tfrac{1}{8}\rho\sin2\phi\cos\phi\}+\mathrm{O}(\log\rho),\end{aligned} \tag{5.5}$$

$$\begin{aligned}\psi_{4l}=&\tfrac{1}{16}C_1\rho^3\sin\phi\log\tfrac{1}{2}\rho-C_1\rho^3\sin\phi\left(\frac{5}{64}-\frac{\gamma}{16}\right)\\&-\tfrac{1}{16}C_2\rho^2\sin2\phi(\log\tfrac{1}{2}\rho+\tfrac{1}{2})+C_2\rho^2\sin2\phi\left(\frac{5}{64}-\frac{\gamma}{16}\right)\\&+\tfrac{1}{16}C_3\rho\sin\phi\log\tfrac{1}{2}\rho+\tfrac{1}{64}C_3\rho\sin3\phi+C_3\rho\sin\phi\left(\frac{\gamma}{16}-\frac{1}{32}\right)\\&+\varepsilon C_1^2\left\{-\tfrac{1}{8}\rho\sin2\phi\cos\phi\log\tfrac{1}{2}\rho+\rho\sin2\phi\cos\phi\left(\frac{1}{16}-\frac{\gamma}{8}\right)\right.\\&\left.+\rho\sin\phi\left(\frac{\gamma}{8}-\frac{1}{8}\right)\right\}+\mathrm{O}(\log\rho).\end{aligned} \tag{5.6}$$

Here the error terms arise from higher-order terms in the outer expansion (i.e. Ψ_{4s} etc). The expressions ψ_{2l}^{+}, ψ_{2l} ... form the boundary conditions as $\rho \to \infty$ for ψ_{2s}^{+}, ψ_{2s} ....

Consider first ψ_{2s}^{+} which is a potential flow (3.7), which satisfies the boundary condition $\psi_{2s}^{+}=0$ on $\beta=0$, and $\psi_{2s}^{+} \sim \psi_{2l}^{+}$ as $\rho \to \infty$. Hence

$$\psi_{2s}^{+} = -\tfrac{1}{2}C_1\rho\sin\phi - \tfrac{1}{2}\varepsilon C_1^2\rho^{-1}\sin\phi + \sum_{n=2}^{\infty} D_n^{+}\rho^{-n}\sin n\phi, \tag{5.7}$$

where the constants D_n^{+} can only be determined by matching to higher-order terms in the outer expansion.

Next ψ_{2s} satisfies the Poisson equation (3.11)

$$\nabla^2\psi_{2s} = -V\psi_p. \tag{5.8}$$

Using the expressions (2.19) and (2.21) for ψ_p and V respectively we can show that

$$\begin{aligned}\psi_{2s} = -\tfrac{1}{2}C_1\rho\sin\phi\log\tfrac{1}{2}\rho + \tfrac{1}{4}C_2\sin 2\phi + \tfrac{1}{8}(C_3+\varepsilon C_1^2)\rho^{-1}\sin 3\phi \\ -\tfrac{1}{2}\varepsilon C_1^2\rho^{-1}\sin\phi\log\tfrac{1}{2}\rho + D_{-1}\rho\sin\phi + D_1\rho^{-1}\sin\phi + O(\rho^{-2}),\end{aligned} \tag{5.9}$$

where the constants D_{-1}, D_1 associated with the potential terms (i.e. solutions of $\nabla^2\psi_{2s}=0$) are to be determined. Comparing (5.9) with (5.3), we see that the matching is achieved by choosing

$$D_{-1} = \tfrac{1}{2}C_1(\tfrac{1}{2}-\gamma), \quad D_1 = -\tfrac{1}{8}C_3 + \varepsilon C_1^2(\tfrac{1}{8}-\tfrac{1}{2}\gamma). \tag{5.10}$$

Proceeding in this manner, we see from (3.12) that ψ_{3s} is a potential flow, and its outer boundary condition ψ_{3l} is also a potential flow, so that the matching can be achieved. For ψ_{4s}^{+}, we can use (5.7) to determine a particular integral for ψ_{4s}^{+}, which we can show matches with the non-potential terms in ψ_{4l}^{+}; the remaining potential terms in ψ_{4l}^{+} determine the potential flow part of ψ_{4s}^{+}. Similarly we can show that ψ_{4s} matches to ψ_{4l} as $\rho \to \infty$. In fact the expressions ψ_{2l}^{+}, ψ_{2l}, ... are the leading terms for ψ_{2s}^{+}, ψ_{2s} ... as $\rho \to \infty$.

6. CONCLUSIONS AND COMMENTS

A perturbation expansion in powers and logarithms of $\delta = Na/U$ about the state of potential flow over the obstacle shows that, after starting from rest, the flow develops on two length and time scales. In addition to potential flow, flow on the length scale a develops in the time scale a/U in the vicinity of the obstacle, and consists of a vortex with negative vorticity situated over the obstacle, together with transients which are advected

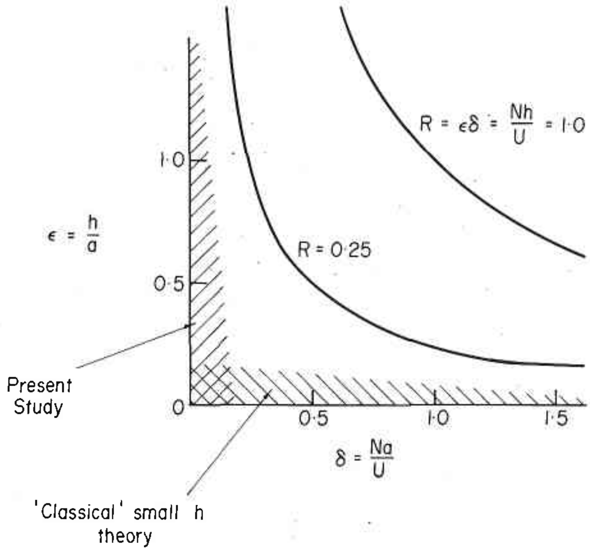

FIGURE 2 Parameter space, showing expected regions of applicability of the "weak stratification" and "small obstacle height" expansions.

downstream. Physically, this motion is due to buoyancy forces acting on the deformed density field caused by potential flow. On the longer time-scale $1/N$, flow develops with the length scale U/N, forced by the smaller-scale motion. This larger-scale motion is mostly composed of the lee-waves and hence is mainly on the downstream side. In fact, the leading order term for this large-scale motion is just that due to a dipole situated at the origin (i.e. the position of the obstacle). The expansion has been carried out to $O(\delta^3)$ in δ [i.e. error of $O(\delta^4)$], and to this order it does not yield any terms which are independent of the horizontal coordinate, or which are found far upstream in the steady state.

These results are fully consistent with those of the corresponding expansion in small obstacle height, and the regions of applicability of the two theories are shown in Figure 2. It seems probable that, in the steady state, both expansions are converging to Long's finite amplitude model (Long, 1955, Miles and Huppert, 1969, etc.) when Nh/U is sufficiently small, and consequently they provide support for the assumptions required by this model. However, experimental observations (Baines, 1977, 1979) in fluid of finite depth with values of δ and/or ε which are substantially less than unity show transient and steady-state effects in the flow (particularly upstream) which are not described by the small δ or small ε expansions, and it appears in view of the results of the present study that these effects are primarily a consequence of the finite depth.

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