Commonwealth of Australia

COMMONWEALTH SCIENTIFIC AND INDUSTRIAL RESEARCH ORGANISATION

Reprinted from: Boundary-Layer Meteorology

Vol. 6 No. 1/2 Page Nos. 299–303 Date March/April 1974

ON THE DRAG COEFFICIENT OVER SHALLOW WATER

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(Received 15 January, 1973)

Abstract. It is shown that if the wind-wave spectrum in shallow water is approximately independent of wind speed due to the combined effects of white-capping and bottom friction, then the wave-induced drag coefficient has a maximum value when the wind speed is twice the maximum wave-speed; as the wind speed increases further, the drag coefficient slowly decreases. This result is consistent with the observations of Hicks *et al.* (1974).

1. Introduction

The drag coefficients measured by Hicks *et al.* (1974) over very shallow water (1–2 m) are markedly less than those measured with the same equipment over much deeper water. It seems likely that this result is due to the change in depth, which will have a very significant effect on the wave field. Further, the results of Dobson (1971) show that (at least under the conditions of his experiments) a very high percentage of the total momentum flux from air to water is transmitted directly to the wave field. The object of this note is to suggest that the drag-coefficient results may be understood in terms of the wave-generation results.

2. The Wave Spectrum

In shallow water of approximately uniform depth d, waves with wave-number k such that $kd>1$ (frequency $\sigma>(g/d)^{1/2}$) are deep water waves; waves with $kd<1$ ($\sigma<(g/d)^{1/2}$) are shallow water waves and their dynamics will be strongly influenced by dissipation at the bottom surface. A well-known expression for the bottom stress due to a turbulent boundary layer is

$$\boldsymbol{\tau} = -\varrho_w c_f |\mathbf{u}|\,\mathbf{u}, \tag{2.1}$$

where ϱ_w is the density of water, $\mathbf{u}$ is the fluid velocity and a typical value for c_f is 0.015 (Hasselmann, 1968). The rate of working against the wave motion is

$$\tau\cdot\mathbf{u} = \varrho_w c_f |\mathbf{u}^3|. \tag{2.2}$$

For a single shallow water wave of displacement ζ, energy density E and horizontal fluid velocity u, we have

$$\zeta = a\cos(kx-\sigma t), \qquad u = a\left(\frac{g}{d}\right)^{1/2}\sin(kx-\sigma t),$$
$$E = \tfrac{1}{2}\varrho_w g a^2. \tag{2.3}$$

Hence the rate of loss of energy due to bottom friction is

$$\frac{dE}{dt} = -\varrho_w c_f a^3 \left(\frac{g}{d}\right)^{3/2} \overline{|\sin^3(kx - \sigma t)|}, \tag{2.4}$$

where the overbar denotes an average over a wave-period.

Hence

$$\frac{da}{dt} = \frac{-4c_f}{3\pi} \frac{(g/d)^{1/2}}{d} a^2. \tag{2.5}$$

If a has the value a_0 when $t=0$ then this equation has the solution

$$a = a_0 \Big/ \left(1 + \frac{4c_f}{3\pi}\left(\frac{g}{d}\right)^{1/2} \frac{a_0}{d} t\right), \tag{2.6}$$

yielding

$$\frac{E(t)}{E_0} = \frac{1}{\left(1 + \frac{4c_f}{3\pi}\left(\frac{g}{d}\right)^{1/2} \frac{a_0}{d} t\right)^2}. \tag{2.7}$$

As an example we consider a wave of length 10 m with amplitude 10 cm in a depth of 1 m (wave period 3.16 s), and for comparison with linear decay processes, when $E/E_0 \simeq 2.72$ we have $t \simeq 325$ s $= 103$ wave periods. This time scale is comparable with that for the growth of deep-water wind waves obtained from the Snyder and Cox (1966) relation.

We may therefore expect that the wave-dissipation processes controlling a wind-generated wave-field will be wave-breaking and white-capping for $\sigma > (g/d)^{1/2}$, and bottom friction for $\sigma < (g/d)^{1/2}$.

Hence for the one-dimensional wave frequency spectrum when the wind has been blowing for a sufficiently long time, we may write (Phillips, 1966)

$$\phi(\sigma) = mg^2\sigma^{-5}, \quad \sigma > \left(\frac{g}{d}\right)^{1/2}, \tag{2.8}$$

where $m \simeq 1.2 \times 10^{-2}$ and $\mathscr{E}$, the total energy density of the wave field, is given by

$$\mathscr{E} = \varrho_w g \int_0^\infty \phi(\sigma)\, d\sigma. \tag{2.9}$$

For $\sigma < (g/d)^{1/2}$, apparently no accurate observations of the wave spectra have been made; various hypothetical spectra may be constructed, based on assumptions about wave-generation processes and non-linear interactions, but these are too speculative to warrant presentation here. However, it seems reasonable to expect that an equilibrium spectrum will be attained with a spectral peak near $\sigma = (g/d)^{1/2}$, provided the mean wind speed is greater than $(g/d)^{1/2}$, for long fetches and after a sufficiently long time. We also assume that the energy of such a wave-field for $\sigma<(g/d)^{1/2}$ is γ times that for $\sigma>(g/d)^{1/2}$ where γ is of order unity.

Hence

$$\mathscr{E} = \varrho_w \frac{(1+\gamma)}{4} mg^3 \left(\frac{g}{d}\right)^{-4/2} = \varrho_w \frac{(1+\gamma)}{4} mgd^2 . \tag{2.10}$$

3. The Drag Coefficient

The drag of the air on the sea surface may be divided into two parts; firstly, that due to small atmospheric eddies in conjunction with surface roughness elements (in this case, capillaries) in a manner similar to the drag of the air over land, and secondly the drag due to the momentum flux directly into gravity waves (Dobson, 1971). (It is possible that there is significant interaction and interdependence of the motion on these two scales (e.g., Mollo-Christensen, 1970) but this is not the issue here). Hence we may write:

$$\tau = C_D \varrho_a U^2 = (C_{DV} + C_{DW})\, \varrho_a U^2 , \tag{3.1}$$

where C_{DV} is the drag coefficient due to small surface roughness elements, C_{DW} the drag coefficient due to gravity waves, ϱ_a the density of air and U the mean velocity of air at some representative height. Since the sea surface is not rigid but is free to move in the direction of the mean surface wind, one would expect C_{DV} to be less than the corresponding land value for roughness elements of comparable size.

For an irrotational surface gravity wave, we have (e.g., Phillips, 1966),

$$E = Mc_p , \tag{3.2}$$

where E and M denote the energy and momentum densities, respectively, and c_p is the phase speed. Following Manton (1971), we apply this equation to the peak of the wave spectrum, obtaining, approximately,

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = (gd)^{1/2}\, \tau_w , \tag{3.3}$$

where $\tau_w = (\mathrm{d}M/\mathrm{d}t) = C_{DW}\, \varrho_a U^2$ is the momentum flux to the waves, $c_p = \text{const.} = (g/d)^{1/2}$ is the approximate phase speed for most of the energetic waves, and $\mathscr{E}$ represents the total energy density of the wave field. The results of Dobson (1971) indicate that the Snyder and Cox relation is valid for waves in the frequency range being considered, and so we have

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = \frac{\varrho_a}{\varrho_w}\left(\frac{U}{c_p} - 1\right)\sigma\mathscr{E}, \quad \frac{U}{c_p} > 1 , \tag{3.4}$$

which, from Equation (2.10) becomes

$$\mathscr{E} = \varrho_a \left(\frac{U}{(gd)^{1/2}} - 1\right)\left(\frac{g}{d}\right)^{1/2} \frac{(1+\gamma)}{4} mgd^2 . \tag{3.5}$$

From Equations (3.3), (3.5) we have

$$C_{DW} = \frac{(1+\gamma)\,m}{4} \frac{\left(\dfrac{U}{(gd)^{1/2}} - 1\right)}{(U/(gd)^{1/2})^2}, \tag{3.6}$$

so that C_{DW} has a maximum value of $(1+\gamma)\,m/16$. Hence, provided γ is approximately constant, C_{DW} has a maximum value (for $U>(g/d)^{1/2}$) at $U=2(g/d)^{1/2}$ and then slowly decreases with increasing wind speed, as shown in Figure 1. Equation (3.6) is consistent

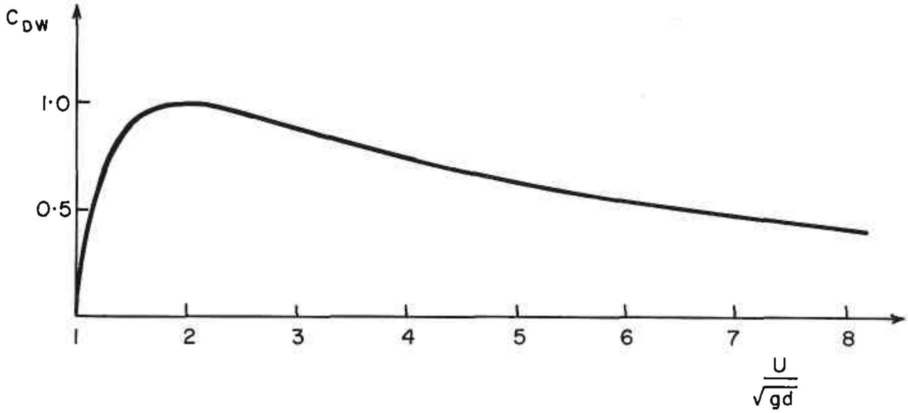

Fig. 1. C_{DW} in units of $(1+\gamma)\,m/16$ as a function of $U/(gd)^{1/2}$.

with the results of Hicks *et al.* (1974), but a meaningful test of the relation as a function of U and d is not possible owing to the uncertainties in the measurements and the small data sample. Also Equation (3.6) is not inconsistent with the measurements of Figure 1 of Hicks *et al.* (1974) since Bass Strait and Lake Michigan are so deep that the speed $2(g/d)^{1/2}$ was not attained by the wind speeds for which measurements were made. Equation (3.6) may also explain other low values of drag coefficients obtained in shallow water, such as those by Smith (1967).

4. Conclusions

Under the assumption that the equilibrium wave energy and spectrum are approximately independent of the wind speed U, when $U>(g/d)^{1/2}$, it has been shown that the wave-induced drag coefficient is a maximum when $U=2(g/d)^{1/2}$ and then slowly decreases as U increases, so that the drag coefficient is a function of depth as well as wind speed. This argument provides a plausible explanation for the different drag coefficients observed by Hicks *et al.* (1974) and may also explain some of the variations in drag coefficients obtained by other investigators.

References

Dobson, F. W.: 1971, 'Measurements of Atmospheric pressure on wind-generated sea waves', *J. Fluid Mech.* **48**, 91–127.

Hasselman, K.: 1968, 'Weak Interaction Theory of Ocean Waves', Basic Developments in Fluid Dynamics, Vol. 2, Academic Press Inc., New York, pp. 117–182.

Hicks, B., Drinkrow, R. L., and Grauze, G.: 1974, 'Drag and bulk transfer coefficients associated with a shallow surface', *Boundary-Layer Meteorol.* this volume p. 287

Manton, M. J.: 1971, 'Wave generation on the air-sea interface'. *Boundary-Layer Meteorol.* **2**, 152–160.

Mollo-Christensen, E.: 1970, *Observations and speculations on mechanics of wave-generation by wind*, Mass. Inst. Tech. Dept. of Meteorol. Report No. 70–2, 32 pp.

Phillips, O. M.: 1966, *The Dynamics of the Upper Ocean*, Cambridge University Press, 261 pp.

Smith, S.: 1967, 'Thrust-anemometer measurements of wind-velocity spectra and of Reynolds stress over a coastal inlet', *J. Mar. Res.* **25**, 239–262.

Snyder, R. L. and Cox, C. S.: 1966, 'A field study of the wind generation of ocean waves', *J. Mar. Res.* **24**, 141–178.