Binary Power Allocation in Symmetric Wyner-Type Interference Networks

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***Abstract*—The Wyner interference network is a popular model used in research on cellular networks due to its simplicity and analytical tractability. In this paper, the optimal power allocation strategies in symmetric one- and two-sided Wyner models are investigated. We determine a sufficient condition for binary power control (BPC) to be optimal that can be applied to the one-sided symmetric model. We consider binary power schemes for the symmetric two-sided Wyner network. Using a method of grouping links and performing a piecewise comparison of the group rates, we are able to determine the optimal power policy that maximizes the network sum rate. The result of the optimization can be expressed as follows for both types of networks: When the interfering channel gain $\sqrt{\epsilon}$ is small, it is optimal (in the class of binary schemes) to have all links on; otherwise, alternate links are switched off to remove interference. We characterize the critical values of ϵ where the transitions occur.**

***Index Terms*—Wireless networks, multiple-access interference, power control.**

I. Introduction

In the past two decades or so, the interest in cellular and wireless networks has grown, as traffic shifts from voice-centric to data-centric. To this end, two broad approaches have been taken by researchers in this area. The first is to examine models that incorporate effects like path loss, fading and user locations in as realistic a manner as possible. However, such an approach tends to rely heavily on numerical experiments and simulations, where results can be very case-specific and from which general conclusions are not easy to draw. The other approach is to investigate simplified models, which may provide analytical insights into optimal network behavior, although further investigation is then needed to test these insights against more realistic models.

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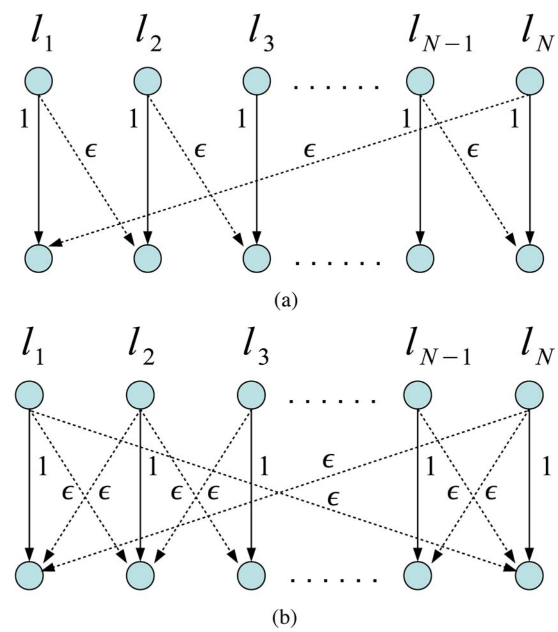


Fig. 1. N-link symmetric Wyner interference networks, with direct channel power gains normalized to 1 and interfering channel power gains represented by ϵ. (a) One-sided Wyner model. (b) Two-sided Wyner model.

One problem that is particularly difficult to handle analytically is rate maximization in interference networks where each transmitter has a power constraint. This problem is non-convex, so recent attempts have been made to study simplified models with additional structure to render them amenable to optimization. Recently, analytical sum-rate maximization was achieved for a symmetric network of interfering links, in which all links interfere equally with each other, and the power constraints on each transmitter are identical [1].

In the present paper, we consider simplified symmetric models, but this time we only allow local interference coupling between the links. This local coupling reflects the reality that in real-world networks the main interference comes from nearby links, not from links that are far away. We consider two different symmetric models of linear array type: In the first, the interference only comes from one other link (say, from the left side), in the second model the interference is two-sided. The link gains for symmetric one-sided and two-sided networks are depicted in Fig. 1(a) and (b), respectively. For both models, we assume that all transmitters have the same power budget.

One question that we would like to answer is whether or not so-called *binary power control* is optimal for these symmetric networks. Binary power control refers to the restriction that each link can only use one of two available power levels: either the link is completely switched off, or the link is allowed to be on, but only at full power. It was shown in [2] that for *arbitrary* two link interfering networks, with arbitrary power constraints, the optimal power control is binary, but they also showed that this is no longer the case in general for an arbitrary number of links. In [1] we showed that binary power control is optimal for an arbitrary number of links in the very restricted class of a symmetric network of interfering links. In fact, it is optimal for either all links to be on (full re-use) or just one link to be on (a scheduled link) in such a network. Clearly, such a result will not hold for the one-sided and two-sided models investigated in the present paper, since these models allow re-use: In both models it is clearly better to have every even-numbered link on, with odd-numbered links off (or vice versa) than to have just one link on.

In the present paper, we wish to ascertain whether or not the optimal power control is binary, and, furthermore, we want to find the optimal power control analytically. Unfortunately, even for these simple networks, proving optimality is not straightforward, and we have only had partial success: We do prove the optimality of binary power control for the one-sided model. For the two-sided model, all the numerical evidence we have accumulated points to the optimality of binary power control, but we have no mathematical proof of this result. Given the one-sided result, and the favorable numerical evidence for the two-sided case, we restrict attention to finding the optimal *binary* power controls for these networks. Note that the assumption of binary power control makes the problem combinatorial, but not necessarily simple: in a network with N links there are 2^N potentially optimal combinations of on–off allocations to consider.[1] Under the assumption of binary power control, we solve the rate maximization problems for both the one-sided and two-sided symmetric networks.

In this paper, we use the term "Wyner networks" to denote one-sided and two-sided linear array interference models. This is in respect to Wyner's seminal work on the Shannon-capacity of cellular networks [4], and it is by now established practice to describe such simplified networks as "Wyner models": Many authors have since worked on problems related to Wyner-type interference networks. It is important to emphasize here that in the present paper we are not considering the Shannon-theoretic capacity under base station cooperation of such models *a la* Wyner, but instead we treat the interference from adjacent links as *Gaussian noise*. Also, we do not explicitly treat cellular networks at all: If we are to think of each link as being a cell, then it is a model for a single *channel* in a cellular network, with co-channel interference from the nearby cells.

It has been shown in [5] that binary power control is optimal in multiple access channels (when interference is treated as noise). This is true regardless of the channel gains in the network. This result does not generalize to arbitrary interference networks [2], and nor even to non-symmetric Wyner models [6], but in this paper we show that it is true for the symmetric one-sided Wyner models, and we conjecture that it is true for the symmetric two-sided model also.

[1]But clearly there is a trellis structure that can be exploited in numerical optimization, see [3].

The paper is organized as follows. In Section II, we describe the two types of Wyner models investigated, assumptions made and the optimization problems to be solved. In Section III we look at the specialized problem of binary power optimization where the transmit powers come from a set of two power values: on or off. We present the optimal binary power control policies for the one and two-sided models in Section III. In Section IV we investigate the question of whether binary power control is optimal for these models, and in Section V, we consider some possible extensions. Section VI presents our conclusions.

II. Network Description and Problem Formulation

A. The Symmetric One-Sided Wyner Model

The network we first consider is an extension to the one-sided interference channel [7] consisting of a set of N links, $\boldsymbol{L} = (l_1, \ldots, l_N)$ where $N \geq 3$. The links are arranged in a ring, with each link interfering with the link immediately to the right of it. It is assumed that interference is treated as Gaussian noise, and each link has an individual maximum power constraint, $P_{\max}$. The network is depicted in Fig. 1(a).

In a symmetric one-sided Wyner interference network, the direct channel gains are assumed to be the same for all the links, and so are the interfering channel gains (usually taken to be smaller than the direct channel gain). Without loss of generality, we take the direct link gain to be unity, so $P_{\max}$ is also the maximum allowable *transmit* power (in appropriate units), whereas the interfering channel gain is denoted by the parameter, $\sqrt{\epsilon}$. Therefore, the received signal of l_i can be expressed as

$$y_i = x_i + \sqrt{\epsilon} x_{i-1} + z_i \tag{1}$$

where x_i is the signal from its own transmitter, x_{i-1} is the signal from its interfering neighbor, and z_i is white Gaussian noise with unit variance. In Fig. 1(a) the direct and interfering transmissions are shown using continuous and dashed arrows, respectively.

Denoting the transmit power vector as $\mathbf{P} = (P_1, P_2, \ldots, P_N)$, the problem to be investigated is finding the optimal power scheme which maximizes the sumrate of the network,

$$\begin{aligned} &\max_{\mathbf{P}} \quad C_{\text{sum}}(N, \epsilon, \mathbf{P}) \\ &\text{s.t.} \quad 0 \leq P_i \leq P_{\max}, \qquad \forall i = 1, \ldots, N. \end{aligned} \tag{2}$$

For the symmetric one-sided Wyner interference network, the sumrate is defined as

$$C_{\text{sum}}(N, \epsilon, \mathbf{P}) = \sum_{i=1}^{N} C\left(\frac{P_i}{1 + \epsilon P_i^-}\right), \tag{3}$$

where P_i is the power of l_i, while P_i^- represents the power of link $(i-1) \bmod N$. Throughout this paper, we also use the notation: $C(x) = \log(1 + x)$.

B. The Symmetric Two-Sided Wyner Model

The other network studied in this paper is the symmetric two-sided Wyner interference network, shown in Fig. 1(b). Unlike the one-sided Wyner model, each link in the two-sided version causes interference to and receives interference from two neighboring links. We make the same assumptions as those made in the one-sided Wyner model.

The received signal at l_i has an additional term to that in (1) to account for the additional interfering signal from l_{i+1}:

$$y_i = x_i + \sqrt{\epsilon}\,(x_{i-1} + x_{i+1}) + z_i. \tag{4}$$

The power allocation problem is also (2), but with the sumrate function for the two-sided Wyner network defined as

$$C_{\text{sum}}(N, \epsilon, \mathbf{P}) = \sum_{i=1}^{N} C\left(\frac{P_i}{1 + \epsilon\left(P_i^- + P_i^+\right)}\right). \tag{5}$$

Notice that the SINR term in (5) has an additional interference term in the SINR, where P_i^+ stands for the power from link $(i+1) \bmod N$.

III. Binary Power Control (BPC)

The problem defined in (2) is to maximize the sumrate over all continuous values of power in the range $[0, P_{\max}]$. In this section we define a more specialized optimization problem where we restrict the allowed power to come from the binary set $\{0, P_{\max}\}$. We use the term *binary power control* (BPC) to refer to this kind of power control and a power scheme which only uses these two values (on and off) is called a *binary power scheme* [2]. If we restrict to consider only BPC schemes, the optimization problem of (2) reduces to:

$$\begin{aligned} &\max_{\mathbf{P}} \quad C_{\text{sum}}(N, \epsilon, \mathbf{P}) \\ &\text{s.t.} \quad P_i \in \{0, P_{\max}\}, \qquad \forall i = 1, \ldots, N \end{aligned} \tag{6}$$

which is a *combinatorial* optimization problem. When a link is switched on, its rate depends only on the *number* of interfering neighbors which are simultaneously switched on. Therefore, the link rate can be expressed as

$$r_k(\epsilon, P_{\max}) = C\left(\frac{P_{\max}}{1 + k\epsilon P_{\max}}\right), \tag{7}$$

where k represents the number of active interfering links. In the case of the one-sided Wyner network, $k \in \{0, 1\}$ as each link can be subjected to interference from at most one other link, whereas for the two-sided model, $k \in \{0, 1, 2\}$. Note that $r_0(\epsilon, P_{\max}) = r_0(P_{\max})$ is constant with respect to ϵ.

The optimality of the different power schemes will depend on the values of ϵ and N. Since ϵ affects which power scheme is optimal, it is useful to define some critical values, about which the nature of the optimal solutions will be seen to change. We consider the solutions in ϵ to the following two equations:

$$2r_1(\epsilon, P_{\max}) - r_0(P_{\max}) = 0, \tag{8}$$

$$2r_2(\epsilon, P_{\max}) - r_0(P_{\max}) = 0. \tag{9}$$

Both $r_1(\epsilon, P_{\max})$ and $r_2(\epsilon, P_{\max})$ are monotonically decreasing functions of ϵ with $2r_1(0, P_{\max}) > r_0(P_{\max})$ and $2r_2(0, P_{\max}) > r_0(P_{\max})$. Hence, the solutions of (8) and (9) are unique and can be explicitly computed. We denote the solution to (8) by $\epsilon^\star(P_{\max})$, which is given by

$$\epsilon^\star(P_{\max}) = \frac{\sqrt{1 + P_{\max}}}{P_{\max}}. \tag{10}$$

The solution to (9) can be computed to be

$$\epsilon = \frac{1}{2}\epsilon^\star(P_{\max}). \tag{11}$$

These two values will be shown to be critical values of ϵ for the one-sided and two-sided network models, respectively. Clearly, $\epsilon^\star(P_{\max})$ is the critical value of ϵ when the alternating "on–off" pattern of link activation exactly equals the all-links-on activation, in the case of N even. The critical value of ϵ in the two-sided model is exactly half this value.

A. Factoring Schemes Into Groups

In an N-link network, even with just two power choices per link, there are still 2^N possible power vectors, which makes exhaustive search computationally intractable. Instead, we look for structure in the problem to reduce the search space down to a small number of choices that can readily be compared.

We will represent possible power allocation vectors by strings of length N taken from the binary alphabet {0,1}, where "0" represents the corresponding link being "off" and "1" represents it being "on" and operating at full power. We will also consider strings of smaller length that represent power allocations to subsets of links. For a string X of 1s and 0s, X^n is a string of n X's in a row, where $n \geq 0$. So X^0 is the null string, X^1 is the string X, X^2 is the concatenation of two X's in a row, etc.

Amongst all the possible strings of length N, we will single out three particular strings that we will see later contain all the potentially optimal schemes. Power Scheme 1 is the string 1^N which represents all links being on. Power Scheme 2 has two representations depending on whether N is even or odd. When N is even, Power Scheme 2 is the string $(10)^{N/2}$; when N is odd $(N \geq 3)$ it is the string $(10)^{(N-3)/2}110$. Power Scheme 3 is the string $(10)^{(N-3)/2}100$ which is defined when N is odd, $N \geq 3$. Both Schemes 2 and 3 can be factored into substrings of the form 10, 110, and 100. Note that re-arranging these substrings in different orders does not affect the sum rate, and we will consider all such re-arrangements as being equivalent to each other. Thus, Power Schemes 2 and 3 are unique (respectively) modulo any such rearrangement of substrings 10, 110, and 100. It is clear that Power Scheme 1 corresponds to "full reuse" and Power Scheme 2 represents "half reuse" when N is even, and Power Schemes 2 and 3 represent two different versions of "half reuse" when N is odd.

A way to show that a particular power allocation is suboptimal is to construct a similar power allocation, using a small change to the first one, that improves the sum-rate. This enables us to obtain necessary conditions for optimality, which we enumerate below.

1) Necessary Conditions for Optimality: All these conditions are valid for both the one-sided and two-sided interference models.

1) *No long substring of consecutive 0s:* A scheme with more than two consecutive zeros (more than two adjacent links switched off) cannot be optimal. The substring 000 can be replaced by 010 improving the overall sum-rate.
2) *No 00 and 11 in the same optimal scheme:* Consider a power allocation that contains two substrings, namely 00 and 11. We can construct a new power allocation from this one, by removing one of the links that are 0 (switched off) and adding a link, also 0 (switched off) placed between the two previously adjacent links that were both 1 (switched on). Thus, the new network has a substring 0, and another substring 101. In this case, the sum-rate has clearly increased, because the interference between the two 1s has been removed.
3) *No duplicate of substring 100:* Consider a string with two substrings, both 100. If they are located adjacent to each other, then there is a substring 100100, which cannot be part of an optimal allocation, since we can replace it with 101010, for a higher sum-rate. On the other hand, assume that there are two substrings, both 100, but *not* adjacent, in an optimum power allocation. Then by the necessary condition 2), both substrings must be preceded by a 0. The second substring 100 can therefore be moved to immediately follow the first, without affecting the sum-rate of the network. So without loss of generality, there is a substring 100100. But, as shown above, this cannot be part of an optimal allocation, which is a contradiction to the optimality of the original scheme.

These conditions do not rule out arbitrarily long substrings of consecutive 1s in an optimal allocation. For example, Power Scheme 1 satisfies these conditions. However, a substring of consecutive 1s, of length 2 or more, can only be followed by at most a single 0. A single 1 can be followed by a 00, forming a substring 100, but there can be no repetition of this substring anywhere else in the optimal scheme.

2) Reducing the Set of Potentially Optimal Schemes: These conditions actually tell us quite a lot about optimal power allocations. Apart from the possibility that Power Scheme 1 is optimal, all other potentially optimal schemes can be factored into substrings of the form 100 or of the form 11^n0 for $n \geq 0$. The factor 100 can appear at most once in an optimal scheme. We use the term "factor" to denote these substrings because they can be re-arranged without affecting the sum-rate. This is because the zero at the end of each factor protects it, and the following factor, from "inter-factor" interference. We view all schemes with the same factors as equivalent, modulo such re-arrangements.

We denote power allocation schemes that can be factored in this way as "composite" schemes, precisely because they can be so factored. An important feature of a composite scheme is that its sum-rate can be calculated by adding up the sum-rates of the component factors. Power Schemes 2 and 3 are both composite schemes. The following lemma expresses what we have shown thus far.

Lemma 3.1: An optimal power allocation is either Power Scheme 1 or it is a composite scheme.

To reduce the set of potentially optimal schemes further, we note that a simple way to show the sub-optimality of a particular composite scheme is to take one of its factors and replace it with another substring of the same length, also ending in zero, with a higher sum-rate than the factor it is replacing. It turns out we can also replace the factor with a group of all 1s (corresponding to Power Scheme 1), as we show in the Proof of Lemma 3.3. Using such comparisons, we can prove the following three lemmas:

Lemma 3.2:

(i) *One-sided model:* An optimal composite scheme cannot contain the factor 100. For $\epsilon \neq \epsilon^\star(P_{\max})$, an optimal composite scheme can have at most one factor 110.

(ii) *Two-sided model:* An optimal composite scheme can have at most one of the factors 100 and 110 and cannot have both 100 and 110 factors.

Proof: See Appendix A.

Lemma 3.3:

(i) *One-sided model:* For $\epsilon \neq \epsilon^\star(P_{\max})$, an optimal composite scheme has no factor of the form 1^n0 for $n \geq 3$.

(ii) *Two-sided model:* An optimal composite scheme has no factor of the form 1^n0 for $n \geq 3$.

Proof: See Appendix A.

Lemma 3.4:

(i) *One-sided model:* For $\epsilon \neq \epsilon^\star(P_{\max})$, the optimal scheme is either Power Scheme 1 or Power Scheme 2.

(ii) *Two-sided model:* The optimal scheme is either Power Scheme 1, Power Scheme 2, or Power Scheme 3.

Proof: In either interference model, Lemma 3.1 shows that an optimal scheme is either Power Scheme 1 or it is composite. By Lemma 3.3 an optimal composite scheme can have only factors 10, 110 or 100. In the one-sided model, Lemma 3.2 shows that it cannot have factor 100, and 110 can appear at most once (when $\epsilon \neq \epsilon^\star(P_{\max})$). Thus, if N is even, the only possibility is $(10)^{N/2}$. If N is odd, the only possibility is $(10)^{(N-3)/2}110$. In both cases,[2] these are Power Scheme 2. In the two-sided model, Lemma 3.2 shows that it can have at most one of 100 or 110. Thus, if N is even, the only possibility is $(10)^{N/2}$, which is Power Scheme 2. If N is odd, the only possibilities are $(10)^{(N-3)/2}110$ (Power Scheme 2) or $(10)^{(N-3)/2}100$ (Power Scheme 3). ■

B. Optimal Binary Schemes

The results of Section III-A reduce the problem down to a comparison of Schemes 1 and 2 in the case of the one-sided network. The one-sided network case is particularly simple because in this model there is a critical value of ϵ, namely $\epsilon^\star(P_{\max})$, independent of N, which governs which of Power Scheme 1 or 2 is better. The symmetric two-sided network case has been reduced to a comparison of Schemes 1, 2, and 3, when N is odd, and a comparison of Schemes 1 and 2, when N is even. The even N case is equally simple: there is again a critical

[2]In the one-sided case, we are assuming that $\epsilon \neq \epsilon^\star(P_{\max})$.

value of ϵ, namely $(1/2)\epsilon^\star(P_{\max})$, independent of N, which governs which of Power Scheme 1 or 2 is better. These results are summarized in the theorem below which follows almost immediately from Lemma 3.4. The slight technicality is for the one-sided model when $\epsilon = \epsilon^\star(P_{\max})$ it is possible that schemes other than Power Scheme 1 or Power Scheme 2 are optimal (schemes containing two 110 factors for example). However given the continuity of the objective function with respect to ϵ, it is clear than Power Schemes 1 and 2 must be optimal at this critical value of ϵ, even if they are not the only two possible optimal schemes.

Theorem 3.1:

1) In the one-sided model, the optimal binary scheme depends on the value ϵ in the following way:
   a) For $\epsilon < \epsilon^\star(P_{\max})$ the unique optimal scheme is Power Scheme 1.
   b) For $\epsilon > \epsilon^\star(P_{\max})$ the unique optimal scheme is Power Scheme 2.
   c) For $\epsilon = \epsilon^\star(P_{\max})$ both Power Scheme 1 and Power Scheme 2 achieve the maximum sum rate.
2) In the two-sided model, for the case N even, the optimal binary scheme depends on the value ϵ in the following way:
   a) For $\epsilon < (1/2)\epsilon^\star(P_{\max})$ the unique optimal scheme is Power Scheme 1.
   b) For $\epsilon > (1/2)\epsilon^\star(P_{\max})$ the unique optimal scheme is Power Scheme 2.
   c) For $\epsilon = (1/2)\epsilon^\star(P_{\max})$ both Power Scheme 1 and Power Scheme 2 achieve the maximum sum rate.

The two sided, odd N case is more complicated, and we elaborate on this case now. We compare Power Schemes 1–3. Each scheme provides a sum-rate that is decreasing in ϵ. Therefore, we can easily find the cross-over values of ϵ for each of the schemes. To this end, define $\epsilon_{S1,S2}(N,P_{\max})$ to be the unique solution in ϵ to

$$Nr_2(\epsilon,P_{\max}) = \frac{1}{2}(N-3)r_0(P_{\max}) + 2r_1(\epsilon,P_{\max})$$

and define $\epsilon_{S1,S3}(N,P_{\max})$ to be the unique solution in ϵ to

$$Nr_2(\epsilon,P_{\max}) = \frac{1}{2}(N-1)r_0(P_{\max}).$$

We can also consider the cross-over ϵ for Schemes 2 and 3, $\epsilon_{S2,S3}(N,P_{\max})$, by solving

$$\frac{1}{2}(N-3)r_0(P_{\max}) + 2r_1(\epsilon,P_{\max}) = \frac{1}{2}(N-1)r_0(P_{\max}). \tag{12}$$

However (12) reduces to (8), so $\epsilon_{S2,S3}(N,P_{\max}) = \epsilon^\star(P_{\max})$.

Note that $\epsilon^\star(P_{\max})$ is independent of N, but $\epsilon_{S1,S2}(N,P_{\max})$ and $\epsilon_{S1,S3}(N,P_{\max})$ do depend on N. The cross-over values are such that for $i=1,2$, $j=2,3$, $i<j$, if $\epsilon < \epsilon_{Si,Sj}(N,P_{\max})$ then Scheme i beats Scheme j, but if $\epsilon > \epsilon_{Si,Sj}(N,P_{\max})$ then Scheme j beats Scheme i.

Now define the function

$$\tilde{x}(P_{\max}) = \frac{\log\sqrt{1+P_{\max}}}{\log\sqrt{1+P_{\max}} - \log\left(1+\frac{P_{\max}}{1+2\sqrt{1+P_{\max}}}\right)}. \tag{13}$$

The following lemma characterizes the ordering relationship between $\epsilon_{S1,S2}(N,P_{\max})$ and $\epsilon_{S1,S3}(N,P_{\max})$.

Lemma 3.5: If N is odd and $N < \tilde{x}(P_{\max})$ then

$$\epsilon^\star(P_{\max}) < \epsilon_{S1,S3}(N,P_{\max}) < \epsilon_{S1,S2}(N,P_{\max}). \tag{14}$$

If N is odd and $N > \tilde{x}(P_{\max})$ then

$$\begin{aligned}\frac{1}{2}\epsilon^\star(P_{\max}) &< \epsilon_{S1,S2}(N,P_{\max})\\ &< \epsilon_{S1,S3}(N,P_{\max})\\ &< \epsilon^\star(P_{\max}).\end{aligned} \tag{15}$$

Proof: See Appendix B. ■

A consequence of Lemma 3.5 is the following theorem:

Theorem 3.2: For the two-sided network, N odd: If $N < \tilde{x}(P_{\max})$ then

1) for $\epsilon < \epsilon_{S1,S3}(N,P_{\max})$, Power Scheme 1 is optimal
2) for $\epsilon > \epsilon_{S1,S3}(N,P_{\max})$, Power Scheme 3 is optimal

If instead $N > \tilde{x}(P_{\max})$ then

1) for $\epsilon < \epsilon_{S1,S2}(N,P_{\max})$, Power Scheme 1 is optimal
2) for $\epsilon_{S1,S2}(N,P_{\max}) < \epsilon < \epsilon^\star(P_{\max})$, Power Scheme 2 is optimal
3) for $\epsilon > \epsilon^\star(P_{\max})$, Power Scheme 3 is optimal

Proof: If N is odd, and $N < \tilde{x}(P_{\max})$, we have from (14) that

$$\begin{aligned}\epsilon_{S2,S3}(N,P_{\max}) &< \epsilon_{S1,S3}(N,P_{\max})\\ &< \epsilon_{S1,S2}(N,P_{\max})\end{aligned}$$

so if $\epsilon < \epsilon_{S1,S3}(N,P_{\max})$ then Scheme 1 beats both Schemes 2 and 3, and if $\epsilon > \epsilon_{S1,S3}(N,P_{\max})$ then Scheme 3 beats both Schemes 1 and 2.

If N is odd, and $N > \tilde{x}(P_{\max})$, we have from (15) that

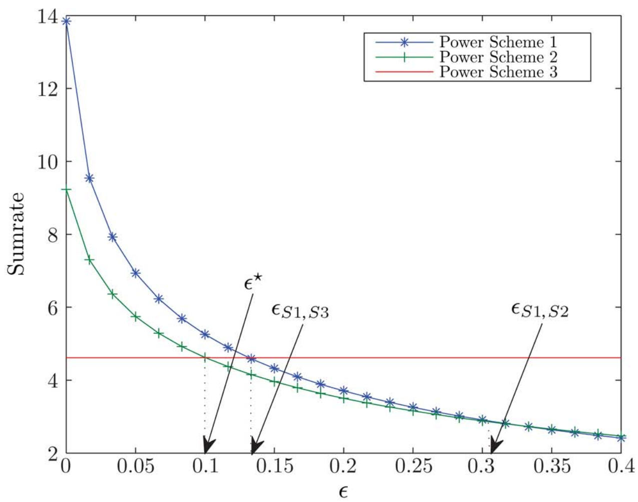
$$\begin{aligned}\epsilon_{S1,S2}(N,P_{\max}) &< \epsilon_{S1,S3}(N,P_{\max})\\ &< \epsilon_{S2,S3}(N,P_{\max}),\end{aligned}$$

so if $\epsilon < \epsilon_{S1,S2}(N,P_{\max})$ then Scheme 1 beats both Schemes 2 and 3; if $\epsilon_{S1,S2} < \epsilon < \epsilon_{S2,S3}(N,P_{\max})$ then Scheme 2 beats both Schemes 1 and 3; and if $\epsilon > \epsilon_{S2,S3}(N,P_{\max})$ then Scheme 3 beats both Schemes 1 and 2. ■

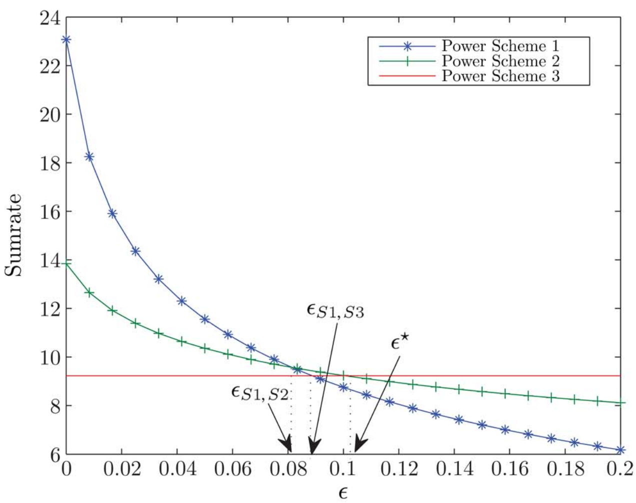
The two different types of behavior, depending on whether N is smaller or larger, are illustrated in Fig. 2(a) and (b). For example, if $P_{\max} = 100$, then $\epsilon^\star = 0.1005$, and $\tilde{x} = 4.1191$. When $N = 3 < \tilde{x}$, then $\epsilon_{S1,S3} = 0.1317 > \epsilon^\star$ and the optimal power scheme goes from Power Scheme 1 to Power Scheme 3 as ϵ is increased above $\epsilon_{S1,S3}$. On the other hand, if we look at the case of $N = 5 > \tilde{x}$, $\epsilon_{S1,S2} = 0.0887 < \epsilon^\star$, and the transition of the optimal power scheme as ϵ is increased, is Power Scheme 1 to Power Scheme 2 (at $\epsilon = \epsilon_{S1,S2}$) to Power Scheme 3 at $\epsilon = \epsilon^\star$.

C. Discussion of the Odd Links Case for the Two-Sided Network Model

Recall that in the even-N two-sided model, there is a single switch from Power Scheme 1 (full re-use) to Power Scheme 2



(a)



(b)

Fig. 2. Sumrate of different power schemes when $P_{\max} = 100$. In (a), $N = 3$, $\epsilon_{S1,S3} = 0.1317 > \epsilon^\star$ and the optimal power scheme transitions from Power Scheme 1 to Power Scheme 3. In (b), $N = 5$, $\epsilon_{S1,S2} = 0.0887 < \epsilon^\star$ and the optimal power scheme transitions from Power Scheme 1 to Power Scheme 2 to Power Scheme 3.

(half reuse) at $\epsilon = (1/2)\epsilon^\star(P_{\max})$. When N is odd there are two half re-use schemes: Power Schemes 2 and 3. We would expect that as N grows large, there should be only weak dependency on the polarity of N, and therefore that the switchover value from full reuse to half re-use should occur at approximately the same critical value. Indeed, we have the following lemma:

Lemma 3.6: $\epsilon_{S1,S3}(N, P_{\max}) \to (1/2)\epsilon^\star(P_{\max})$ as $N \uparrow \infty$.

Proof: See Appendix C. ∎

From (15), we obtain

Corollary 3.1: $\epsilon_{S1,S2}(N, P_{\max}) \to (1/2)\epsilon^\star(P_{\max})$ as $N \uparrow \infty$.

Thus, for large N, N odd, we can say that full reuse is optimal for $\epsilon \lesssim (1/2)\epsilon^\star(P_{\max})$ and half-reuse is optimal for $\epsilon \gtrsim (1/2)\epsilon^\star(P_{\max})$, very similarly to the even N case. Note, though, that for $\epsilon_{S1,S2}(N, P_{\max}) < \epsilon < \epsilon^\star(P_{\max})$, Power Scheme 2 is optimal, but for $\epsilon > \epsilon^\star(P_{\max})$ Power Scheme 3 is optimal, so the *type* of half-reuse scheme has a transition at $\epsilon = \epsilon^\star(P_{\max})$.

However, if N is fixed, and $P_{\max}$ is large, the situation is different. When $P_{\max}$ is large, we have

$$\tilde{x}(P_{\max}) \sim \frac{1}{2\ln 2}\ln(1 + P_{\max})$$

which grows to infinity with $P_{\max}$. Thus, for N odd, with $P_{\max}$ large, we have Power Scheme 1 optimal for $0 < \epsilon < \epsilon_{S1,S3}(N, P_{\max})$ and Power Scheme 3 optimal for $\epsilon > \epsilon_{S1,S3}(N, P_{\max})$. Note that by Lemma 3.5, $\epsilon^\star(P_{\max}) < \epsilon_{S1,S3}(N, P_{\max})$. If N is also quite large, then we can say that the cross-over from full reuse to half reuse occurs at $\epsilon \approx \epsilon^\star(P_{\max})$ which is *double* the value that occurs if we just increase or decrease N by 1, when the cross-over value to half reuse occurs at $(1/2)\epsilon^\star(P_{\max})$. Thus, there is a discontinuity in behavior as we vary N, due to the effect of even or odd parity of the network as a whole, when the SNR is very large. This fact is interesting in itself, and is a pointer as to why the general problem (when P_i is not restricted to be binary) might be difficult to solve.

IV. The Optimality of Binary Power Control

So far in this paper we have assumed that binary power control is to be employed. But is it optimal? The answer is firmly positive in the case of the symmetric one-sided model, as we now show. We present a more general result: a sufficient condition for BPC to be optimal, which clearly implies the optimality of BPC for the one-sided model.

A. A Sufficient Condition for BPC to be Optimal and Application to the One-Sided Model

In this section, we consider more general models than just the one-sided or two-sided symmetric models: Consider a network consisting of N links, and let $\mathcal{N} = \{1, 2, \ldots, N\}$ be the index set of the links in the network. Also, denote link k and its corresponding power as l_k and P_k, respectively, with $k \in \mathcal{N}$. We wish to optimize the power of the links in the network, $\mathbf{P} = \{P_1, \ldots, P_N\}$ to maximize the sumrate of the network given that each link, has a maximum power constraint, i.e., $0 \le P_k \le P_{k,\max}$. The optimization problem is given as

$$\max_{\mathbf{P}} \quad C_{\text{sum}}(\mathbf{P}) \qquad \text{s.t.} \quad \mathbf{0} \le \mathbf{P} \le \mathbf{P}_{\max}, \tag{16}$$

where

$$C_{\text{sum}}(\mathbf{P}) = \sum_{i \in \mathcal{N}} \log\left(1 + \frac{\eta_i P_i}{\sum_{j \in \mathcal{J}_i} \epsilon_{j,i} P_j + \sigma_i^2}\right). \tag{17}$$

The parameters $\sqrt{\eta_i}$ and σ_i^2 are the direct channel gain and noise power of link i, respectively. The set $\mathcal{J}_i$ is an index set of all the links interfering with link i, with $\sqrt{\epsilon_{j,i}}$ being the interfering channel gain of link j to link i. The vector $\mathbf{P}_{\max}$ is the maximum power vector, i.e., $\mathbf{P}_{\max} = \{P_{1,\max}, \ldots, P_{N,\max}\}$, where link i has maximum power level $P_{i,\max}$.

Further assume that the network has a subset of links with index set $\mathcal{L} \subseteq \mathcal{N}$, such that a link whose index is in $\mathcal{L}$ causes interference to only one other link. In other words, we define a mapping $m : \mathcal{L} \to \mathcal{N}$, with the interpretation that $t = m(s)$ is the index of the link that suffers interference from link s. There are only two link-rates affected by P_s: the rate of l_s and the rate of l_t. Denote the sum of the two link rates affected by P_s as

$$R_s(\mathbf{P}) = \log\left(1 + \frac{\eta_s P_s}{\sum_{i\in\mathcal{J}_s} \epsilon_{i,s} P_i + \sigma_s^2}\right) + \log\left(1 + \frac{\eta_t P_t}{\epsilon_{s,t} P_s + \sum_{j\in\mathcal{J}_t \setminus s} \epsilon_{j,t} P_j + \sigma_t^2}\right) \tag{18}$$

where $\mathcal{J}_s$ is an index set of links interfering with l_s and $\mathcal{J}_t$ is the index set of links interfering with l_t. Referring back to our original objective function, we can write (17) as

$$C_{\text{sum}}(\mathbf{P}) = R_s(\mathbf{P}) + \sum_{i\in\mathcal{N}\setminus\{s,t\}} \log\left(1 + \frac{\eta_i P_i}{\sum_{j\in\mathcal{J}_i} \epsilon_{j,i} P_j + \sigma_i^2}\right). \tag{19}$$

Now let's consider the following single-variable optimization problem

Problem 4.1:

$$\max_{P_s} \quad C_{\text{sum}}(\mathbf{P}) \qquad \text{s.t.} \quad 0 \le P_s \le P_{s,\max} \tag{20}$$

where $C_{\text{sum}}(\mathbf{P})$ is given in (19).

The solution to Problem 4.1 is given in the following lemma.

Lemma 4.1: Let $P_s^\star$ be the solution to Problem 4.1, then $P_s^\star \in \{0, P_{s,\max}\}$.

Proof: See Appendix D.

This leads to the following corollary which provides a sufficient condition for binary power allocation to be optimal in network sumrate maximization.

Corollary 4.1: If a link, l_s causes interference to only one other link and the objective is to maximize the sumrate of the network, then the optimal value for P_s subject to $0 \le P_s \le P_{s,\max}$ is either 0 or $P_{s,\max}$ (OFF or ON).

Proof: A direct corollary of Lemma 4.1. ■

Corollary 4.2: If each and every link in a network only causes interference to one other link in the network, then the optimal power scheme for the whole network is binary.

Corollary 4.2 implies that binary power control is indeed optimal for the one-sided Wyner network.

B. The Two-Sided Symmetric Model

Unfortunately, we cannot obtain the optimality of binary power control for the two-sided symmetric model from Lemma 4.1. Nevertheless, we believe this to be the case: We first attempted to solve the optimization problem using the KKT conditions, similar to our previous work for the two-link interference channel in [8], since the two-link interference channel is a special case of the two-sided Wyner network, with $N = 2$. However, we were not successful in obtaining a solution using the KKT conditions: The general case is a non-convex optimization problem, and we have been unsuccessful in finding any special structure to solve it. However, we have conducted fairly exhaustive numerical searches to find the optimal power vector, using finite, discretized power values for various-sized networks.[3] Numerical experiments using gradient search were conducted to determine the existence of local maxima. From these experiments, we found that all local maximum points which are not one of Schemes 1–3 are still binary.

We therefore present the following conjecture.

Conjecture 4.1: The optimal power scheme which maximizes the sumrate of the symmetric two-sided Wyner network is binary in nature.

We support this conjecture further here by proving it to be true in the special cases of $N = 3$ and $N = 4$ links.

1) $N = 3$ and $N = 4$ Cases: The 3-link two-sided Wyner model is exactly the same as the 3-link all interfering network [1], since each link interferes will all other links within the network. From that investigation it is found that for any number of links, the optimal power scheme is either to have all links on at maximum power or to have only one link on while all others are switched off. Hence, we can conclude that the optimal power scheme for a 3-link symmetric two-sided Wyner network is binary.

For the 4-link Wyner model, we exploit the concavity of the function $h(x) = C(x/a) = \log(1 + (x/a))$ and the two link results from [1], [9] to obtain the upper bound on the sumrate

$$C_{\text{sum}}(4, \epsilon, \mathbf{P}) = 2\left[\frac{1}{2}\sum_{i=1}^{4} C\left(\frac{P_i}{1 + \epsilon\left(P_i^- + P_i^+\right)}\right)\right] \le \max\left\{4C\left(\frac{P_{\max}}{1 + 2\epsilon P_{\max}}\right), 2C(P_{\max})\right\}. \tag{21}$$

When $\epsilon < (1/2)\epsilon^\star(P_{\max})$, the upper bound is $4C(P_{\max}/(1 + 2\epsilon P_{\max}))$, which is achievable by having all transmitters using power $P_{\max}$. If $\epsilon > (1/2)\epsilon^\star(P_{\max})$, the upper bound is $2C(P_{\max})$, but this bound is achievable with the Alternate On-Off power scheme. At $\epsilon = (1/2)\epsilon^\star$, the All-On and Alternate On-Off power schemes are equally good. Hence, the optimal power scheme for the 4-link two-sided Wyner network is also binary. Further details can be found in [10].

2) Binary is Optimal When Interference is Strong Enough: In the case of symmetric two-sided Wyner networks where the number of links, N, is an even number, we can establish an upper bound to the network sumrate using the results from the symmetric one-sided Wyner network:

$$\sum_{i=1}^{N} C\left(\frac{P_i}{1 + \epsilon(P_i^- + P_i^+)}\right) \le \max\left\{NC\left(\frac{P_{\max}}{1 + \epsilon P_{\max}}\right), \frac{N}{2}C(P_{\max})\right\}. \tag{22}$$

When $\epsilon \ge \epsilon^\star(P_{\max})$, the maximum sumrate for the symmetric one-sided Wyner network is $(N/2)C(P_{\max})$, which is also achievable in the two-sided model using the Alternate

[3]It should be noted that these one and two-sided models have a certain Trellis structure which makes them very amenable to numerical optimization via dynamic programming [3].

On-Off power scheme. Therefore we can conclude that for an Even-N symmetric two-sided Wyner network, when $\epsilon \geq \epsilon^\star(P_{\max})$, the maximum achievable sumrate is obtained with binary power control.

V. Extensions

The models considered in this paper are very specialized, and it is of interest to know how the results might extend to more general networks. The underlying assumption is that the network is symmetric in that the cross-gain is the same between any two adjacent links.

A natural extension is to allow the cross-gains to be different. If each cross-gain is chosen independently at random, then the chance of getting a symmetric network is zero. What general conclusions can be drawn from the symmetric case, if any?

One simple extension is to allow the cross-gains to deviate slightly from the symmetric case. If we allow each cross-gain to be different, then we can represent the network by a vector of cross gains $(\epsilon_{i,i-1}, \epsilon_{i,i+1})_{i=1}^{N}$. In the symmetric case, all the parameters in the vector are the same, with common value ϵ.

If ϵ in our symmetric model is chosen at random from a continuous distribution, then with probability 1, all 2^N binary power schemes achieve distinct sum-rates. We have characterized the particular binary power scheme that is optimal in this paper, so all other $2^N - 1$ other binary power schemes achieve strictly less sum-rate than the optimal binary scheme.

Now, if we continuously vary the channel gains of the N links, the sum-rate of *any* of the binary power schemes will vary continuously. For small perturbations of the channel parameters, it follows that the optimal binary power scheme in the symmetric case will remain the optimal binary power scheme after the small perturbation of the channel parameters.

In the one-sided model, we can make a stronger statement since we know from Corollary 4.1 that binary power control is optimal for this model, even when the cross-gains are distinct. For small perturbations of the channel parameters, it follows that the optimal power scheme in the symmetric case will remain the optimal power scheme after the small perturbation.

These arguments show that our results are valid for scenarios beyond the symmetric case that is the focus of this paper. There can be positive probability of realizing such scenarios if link gains are drawn at random from continuous distributions.

Another extension is to a fading model. For example, suppose that the complex-valued, direct gain on link i, $H_{i,i}$, is Rayleigh distributed, with $|H_{i,i}|^2$ having mean 1. Suppose the cross-gains $H_{i-1,i}$ and $H_{i+1,i}$ are i.i.d. Rayleigh, with $|H_{i-1,i}|^2$ having mean ϵ. The ergodic channel capacity of link i, without channel knowledge at the transmitter, and treating interference as noise, is

$$\mathbb{E}\left[\log\left(1 + \frac{|H_{i,i}|^2 P_i}{|H_{i-1,i}|^2 P_{i-1} + |H_{i+1,i}|^2 P_{i+1} + \sigma^2}\right)\right].$$

It is of interest to prove similar results for this symmetric, fading model. Most of the arguments in this paper do not rely on the specific form of the sum-rate function; what is more important is the existence, uniqueness and ordering of critical values of ϵ where various curves cross, and it will be of interest to see if similar arguments can be applied in more general settings, including the symmetric fading model mentioned here.

VI. Conclusion

In this paper, we have presented the results of sumrate maximization in two symmetric Wyner network models. For the symmetric one-sided Wyner model, using Lemma 4.1 and Corollary 4.2, we were able to simplify the original optimization problem by restricting the set of power levels to $\{0, P_{\max}\}$. Using a method of grouping links and doing a piecewise comparison of the rates of each group, we have characterized completely the optimal power vectors for the one-sided model in Theorem 3.1.

For the symmetric two-sided Wyner model, we have determined that in the case of 3- and 4-link models, the optimal power schemes are binary. For models with $N > 4$, N even, and for which ϵ is greater than $\epsilon^*(P_{\max})$, then binary is optimal, and the alternate on–off binary allocation is optimal. Numerical evidence (not provided in this paper) suggests that binary is always optimal for the two-sided symmetric model.

For the two-sided symmetric model we restrict our attention to binary power allocations. If ϵ is less than the threshold $(1/2)\epsilon^\star(P_{\max})$, then the All-On power scheme is optimal. For ϵ above this threshold, it is better to switch alternate links on and off, respectively. This result is exact when N is even, and true in the limit of large N, when N is odd. An exact characterization is also obtained in the N odd case. The exact results for N odd are more complicated, with more critical transitions, the values of which depend on N, and whether or not the SNR is large. The sensitivity to even or odd parity at high SNR suggests that the general problem with continuous power levels may be quite difficult to solve.

Although the models considered in this paper are simple and not very realistic, they extend our understanding of power control in interfering networks, which is a difficult and generally intractable area, yet of practical importance. Our results support the view in [2] that binary power control schemes are generally good for networks of interfering links, even if they are not always optimal, although they do seem to be optimal in the models considered in this paper. Finally, these models provide a simple way to explain why cellular networks tend to be either "CDMA-like", with full re-use between cells, or "FDMA-like", with re-use partitioning between adjacent cells, and that the best approach depends on the level of intercell interference between the cells.

Appendix A
Proof of Lemmas 3.2–3.3

In order to prove these two lemmas, we need to introduce two additional critical values of ϵ, which we denote by $\breve{\epsilon}(P_{\max})$ and $\grave{\epsilon}(P_{\max})$. These are the unique solutions to the equations:

$$3r_2(\epsilon, P_{\max}) = r_0(P_{\max}), \tag{23}$$

$$3r_2(\epsilon, P_{\max}) = 2r_1(\epsilon, P_{\max}) \tag{24}$$

respectively. The uniqueness of $\breve{\epsilon}(P_{\max})$ is immediate from the fact that $3r_2(\cdot, P_{\max})$ is decreasing, and its value can be explicitly computed:

$$\breve{\epsilon}(P_{\max}) = \frac{(1+P_{\max}^{\frac{1}{3}})\left((1+P_{\max})^{\frac{1}{3}}+1\right)}{2P_{\max}}. \tag{25}$$

The uniqueness of $\grave{\epsilon}(P_{\max})$ follows from the following lemma, which also characterizes the ordering of some of the critical values of ϵ in this paper:

Lemma A.1:

1) $\grave{\epsilon}(P_{\max})$ is uniquely defined for $\epsilon > 0$.
2) For $\epsilon < \grave{\epsilon}$, $3r_2(\epsilon, P_{\max}) > 2r_1(\epsilon, P_{\max})$, otherwise for $\epsilon > \grave{\epsilon}$, $3r_2(\epsilon, P_{\max}) < 2r_1(\epsilon, P_{\max})$.
3)

$$\epsilon^\star(P_{\max}) < \breve{\epsilon}(P_{\max}) < \grave{\epsilon}(P_{\max}). \tag{26}$$

Proof: We show that there is only one positive solution in ϵ to (24). Note that (24) is equivalent to the equation:

$$\begin{aligned}(1+2\epsilon P_{\max}+P_{\max})^3(1+\epsilon P_{\max})^2 \\ = (1+\epsilon P_{\max}+P_{\max})^2(1+2\epsilon, P_{\max})^3\end{aligned} \tag{27}$$

which can be written in polynomial form as $a_4\epsilon^4 + a_3\epsilon^3 + a_2\epsilon^2 + a_1\epsilon + a_0 = 0$, where $a_4 = -4P_{\max}^5$, $a_3 = -2P_{\max}^4(P_{\max}+1)$, $a_2 = P_{\max}^3(P_{\max}^2 + 3P_{\max} + P_{\max})$, $a_1 = 2P_{\max}^2(P_{\max}+1)(P_{\max}+2)$ and $a_0 = P_{\max}(P_{\max}+1)^2$. Note that a_4 and a_3 are negative, while a_2, a_1, and a_0 are positive. Using Descartes' Rule of Signs, since there is only one change of sign in the polynomial coefficients, there is exactly one positive root [11] to (27), and hence there can only be one value of $\grave{\epsilon}(P_{\max})$, proving part 1). Part 2) then follows by inspection.

To prove 3), note that

$$\begin{aligned}\breve{\epsilon}(P_{\max}) - \epsilon^\star(P_{\max}) &= \frac{(1+P_{\max}^{\frac{1}{3}})\left((1+P_{\max})^{\frac{1}{3}}+1\right)}{2P_{\max}} \\ &\quad - \frac{(1+P_{\max})^{\frac{1}{2}}}{P_{\max}} \\ &= \frac{(1+P_{\max}^{\frac{1}{3}})\left((1+P_{\max})^{\frac{1}{6}}-1\right)^2}{2P_{\max}} > 0.\end{aligned} \tag{28}$$

If we fix $P_{\max}$ and plot the three functions, $3r_2(\epsilon)$, $2r_1(\epsilon)$, and r_0 with respect to ϵ, clearly $3r_2(0) > 2r_1(0) > r_0$. Since $\epsilon^\star(P_{\max}) < \breve{\epsilon}(P_{\max})$ and $\grave{\epsilon}(P_{\max})$ is uniquely defined, then $\epsilon^\star(P_{\max}) < \breve{\epsilon}(P_{\max}) < \grave{\epsilon}(P_{\max})$ (See Fig. 3). This completes the proof. ■

The ordering of these critical ϵ values is illustrated in Fig. 3.

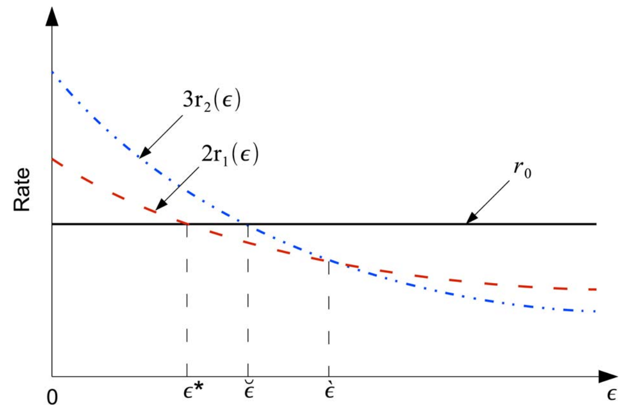


Fig. 3. The relationship between $\epsilon^\star$, $\breve{\epsilon}$, and $\grave{\epsilon}$.

Proof of Lemma 3.2:

One-sided case: It is immediate that 100 cannot be a substring in an optimal scheme, because we can replace it by 110 and increase the sum-rate.

Now suppose that an optimal scheme has 110 repeated twice. Without loss of generality, we can move the two factors together to form the group 110110 without changing the sum-rate. This group achieves a sum-rate of $2r_0(P_{\max}) + 2r_1(\epsilon, P_{\max})$.

Suppose first that $\epsilon > \epsilon^\star$. Since $r_1(\cdot, P_{\max})$ is decreasing, we have $2r_1(\epsilon, P_{\max}) < r_0(P_{\max})$, which implies $2r_0(P_{\max}) + 2r_1(\epsilon, P_{\max}) < 3r_0(P_{\max})$. But $3r_0(P_{\max})$ can be achieved by the group 101010, which contradicts the optimality of the considered scheme.

Suppose instead that $\epsilon < \epsilon^\star$ so that $2r_1(\epsilon, P_{\max}) > r_0(P_{\max})$, which implies

$$2r_0(P_{\max}) + 2r_1(\epsilon, P_{\max}) < r_0(P_{\max}) + 4r_1(\epsilon, P_{\max}).$$

But $r_0(P_{\max}) + 4r_1(\epsilon, P_{\max})$ can be achieved by the factor 111110 which is another contradiction.

Two-sided case: That the factor 100 cannot be repeated more than once has been shown in Section III-A1, condition 2).

Now suppose that an optimal scheme has 110 repeated twice. Without loss of generality, we can move the two factors together to form the group 110110 without changing the sum-rate. This group achieves a sum-rate of $4r_1(\epsilon, P_{\max})$.

Suppose first that $\epsilon \geq \epsilon^\star$. Since $r_1(\cdot, P_{\max})$ is decreasing, we have $2r_1(\epsilon, P_{\max}) \leq r_0(P_{\max})$, which implies $4r_1(\epsilon, P_{\max}) \leq 2r_0(P_{\max}) < 3r_0(P_{\max})$. But $3r_0(P_{\max})$ can be achieved by the group 101010, contradiction.

Suppose instead that $\epsilon < \epsilon^\star$. Lemma A.1 1) implies that $\epsilon < \grave{\epsilon}(P_{\max})$ and Lemma A.1 2) implies that $4r_1(\epsilon, P_{\max}) < 2r_1(\epsilon, P_{\max}) + 3r_2(\epsilon, P_{\max})$. But $2r_1(\epsilon, P_{\max}) + 3r_2(\epsilon, P_{\max})$ can be achieved by the factor 111110, contradiction.

A very similar argument can be given for why 110 and 100 cannot both be in the optimal scheme. This completes the proof of Lemma 3.2.

Proof of Lemma 3.3: Consider the factor $1^{m-1}0$ $m \geq 4$. We will denote its sum-rate by $\rho(m, \epsilon, P_{\max})$. We can compare its sum-rate with the sum-rate of the same-sized group from both Power Scheme 1, and Power Scheme 2.

For comparison with a group taken from Power Scheme 2, the corresponding group is $(10)^{m/2}$, when m is even, and $(10)^{(m-3)/2}110$, when m is odd. We denote the sum-rate of this group by $\grave{\rho}(m, \epsilon, P_{\max})$.

For comparison with a group taken from Power Scheme 1, we are considering the replacement of the final 0 in the factor with a 1. However, this 1 will cause interference to the first link in the factor that follows, which appears to complicate the analysis, since we lose the zero padding between factors. However, we can compensate for this interference by replacing it with interference added to the first 1 in the factor, and then ignoring the interference to the following factor. To see that this works in the one-sided case, note that the original sum-rate of the factor is $r_0 + (m-2)r_1$. The first link of the following factor achieves a rate of r_0. If we change the final link of the considered factor to 1, then the new sum-rate is $r_0 + (m-1)r_1$ and the first link of the following factor achieves r_1. Thus the total rate of all $m+1$ links is $mr_1 + r_0$ which is the same as if there is no inter-factor interference, and the group of m 1's (which replace the original factor) are in a circle, with interference to link 1 from link m. The same argument applies to the two-sided model.

Thus, the corresponding Power Scheme 1 group consists of m 1's, but we should think of this group as wrapped in a circle so that each link receives interference from just one link to its left, in the one-sided model, or from both adjacent links, in the two-sided model. We denote the sum-rate of this circular group by $\breve{\rho}(m, \epsilon, P_{\max})$ in both models, although the value will be different in each case (see below).

One-sided model: In the one-sided model, we have

$$\begin{aligned}\rho(m,\epsilon,P_{\max}) &= r_0(P_{\max}) + (m-2)r_1(\epsilon,P_{\max})\\ \breve{\rho}(m,\epsilon,P_{\max}) &= mr_1(\epsilon,P_{\max})\\ \grave{\rho}(m,\epsilon,P_{\max}) &= \begin{cases}\frac{m}{2}r_0(P_{\max}) & m \textbf{ even}\\ \frac{(m-1)}{2}r_0(P_{\max}) & \\ \quad + r_1(\epsilon,P_{\max}) & m \textbf{ odd}\end{cases}\end{aligned} \tag{29}$$

Note that all these functions are decreasing in ϵ.

To compare with Scheme 2, we compare $\rho(m, \epsilon, P_{\max})$ with the corresponding sumrate for a Scheme 2 group of size m, as in (29). Whether m is even or odd, the unique ϵ that solves the equation $\grave{\rho}(m, \epsilon, P_{\max}) = \rho(m, \epsilon, P_{\max})$ is $\epsilon^\star(P_{\max})$, as given in (10). For $\epsilon > \epsilon^\star(P_{\max})$, $\rho(m, \epsilon, P_{\max}) < \grave{\rho}(m, \epsilon, P_{\max})$.

We can also compare the factor sumrate $\rho(m, \epsilon, P_{\max})$ with the corresponding (circular) group rate of Scheme 1, using the equation $\breve{\rho}(m, \epsilon, P_{\max}) = \rho(m, \epsilon, P_{\max})$. The unique solution to this equation is also $\epsilon^\star(P_{\max})$. For $\epsilon < \epsilon^\star(P_{\max})$, $\breve{\rho}(m, \epsilon, P_{\max}) > \rho(m, \epsilon, P_{\max})$. It follows that the factor of size m cannot be part of the optimal scheme in the one-sided network model, unless $\epsilon = \epsilon^\star(P_{\max})$.

Two-sided case: We can use the same steps as above, except that the corresponding sum-rates must be adjusted to take account of two-sided interference. In this case, when comparing the sum-rate of a factor of size m with the corresponding (circular) group in Power Scheme 1, the sumrate of the circular group is $\breve{\rho}(m) = \breve{\rho}(m, \epsilon, P_{\max}) = mr_2(\epsilon, P_{\max})$. When the factor size, m, is even, the corresponding Scheme-2 group rate is $\grave{\rho}(m) = (m/2)r_0(P_{\max})$. When the factor size, m, is odd, the corresponding Scheme-2 group rate is $\grave{\rho}(m) = \grave{\rho}(m, \epsilon, P_{\max}) = ((m-3)/2)r_0(P_{\max}) + 2r_1(\epsilon, P_{\max})$. Note that, as in the one-sided case, the functions $\rho(m, \cdot, P_{\max})$, $\breve{\rho}(m, \cdot, P_{\max})$, and $\grave{\rho}(m, \cdot, P_{\max})$ are all decreasing functions.

In the case of m even, the equation $\breve{\rho}(m, \epsilon, P_{\max}) = \grave{\rho}(m) = (m/2)r_0(P_{\max})$ reduces to $2r_2(\epsilon, P_{\max}) = r_0(P_{\max})$ which has solution $\epsilon = (1/2)\epsilon^\star(P_{\max})$ defined in (11).

For $m \geq 4$, the equation $\rho(m, \epsilon, P_{\max}) = \breve{\rho}(m, \epsilon, P_{\max})$ reduces to

$$3r_2(\epsilon, P_{\max}) = 2r_1(\epsilon, P_{\max}), \tag{30}$$

which has the unique solution $\grave{\epsilon}(P_{\max})$ defined in Lemma A.1. From (9), (10) and Lemma A.1, we see that $(1/2)\epsilon^\star < \epsilon^\star < \grave{\epsilon}$. It follows that for $\epsilon \leq (1/2)\epsilon^\star(P_{\max})$, we have $\epsilon < \grave{\epsilon}$ and hence $\rho(m, \epsilon, P_{\max}) < \breve{\rho}(m, \epsilon, P_{\max})$. Thus, if $\epsilon \leq (1/2)\epsilon^\star(P_{\max})$ then a factor of size $m \geq 4$ cannot be optimal.

Now consider $m \geq 4$ and m even. It follows from $(1/2)\epsilon^\star < \grave{\epsilon}$ that $\breve{\rho}(m, (1/2)\epsilon^\star, P_{\max}) > \rho(m, (1/2)\epsilon^\star, P_{\max})$. But both $\rho(m, \cdot, P_{\max})$ and $\breve{\rho}(m, \cdot, P_{\max})$ are decreasing functions, so when $\epsilon > (1/2)\epsilon^\star$,

$$\begin{aligned}\rho(m,\epsilon,P_{\max}) &< \rho(m, \frac{1}{2}\epsilon^\star, P_{\max})\\ &< \breve{\rho}(m, \frac{1}{2}\epsilon^\star, P_{\max})\\ &= \grave{\rho}(m)\\ &= \frac{m}{2}r_0(P_{\max})\end{aligned}$$

which shows that the factor of size m is beaten by the corresponding Scheme-2 group, and hence it can't be part of an optimal solution when $\epsilon > (1/2)\epsilon^\star$.

Now consider $m \geq 4$ and m odd. The equation $\rho(m, \epsilon, P_{\max}) = \grave{\rho}(m, \epsilon, P_{\max})$ reduces to

$$2_{r2}(\epsilon, P_{\max}) = r_0(P_{\max}), \tag{31}$$

which has the unique solution $\epsilon = (1/2)\epsilon^\star$. Thus, for $\epsilon > (1/2)\epsilon^\star$, this group is beaten by the corresponding Scheme-2 group, and hence it can't be part of an optimal solution when $\epsilon > (1/2)\epsilon^\star$. We conclude that the factor of size $m \geq 4$ cannot be part of the optimal scheme in the two-sided network model.

Appendix B
Proof of Lemma 3.5

Firstly, it is trivial to show that $\epsilon_{S1,S3}(N, P_{\max})$ is given as

$$\epsilon_{S1,S3}(N,P_{\max}) = \frac{1}{2P_{\max}}\left(\frac{P_{\max}}{(\sqrt{1+P_{\max}})^{(N-1)/N}-1} - 1\right).$$

We begin by considering the case when $N < \tilde{x}(P_{\max})$, i.e.,

$$N < \frac{\log\sqrt{1+P_{\max}}}{\log(\sqrt{1+P_{\max}}) - \log\left(1+\frac{P_{\max}}{1+2\sqrt{1+P_{\max}}}\right)}.$$

After elementary algebra we obtain the equivalent inequality

$$\frac{\sqrt{1+P_{\max}}}{P_{\max}} < \frac{1}{2P_{\max}}\left(\frac{P_{\max}}{(\sqrt{1+P_{\max}})^{(N-1)/N}-1} - 1\right)$$

which implies

$$\begin{aligned}\epsilon_{S2,S3}(N,P_{\max}) &= \epsilon^\star(P_{\max})\\ &< \epsilon_{S1,S3}(N,P_{\max}).\end{aligned} \tag{32}$$

Conversely, if $N > \tilde{x}(P_{\max})$, then $\epsilon^\star(P_{\max}) > \epsilon_{S1,S3}(N, P_{\max})$.

Recall that the cross-over values are such that for $i = 1, 2, \quad j = 2, 3, \quad i < j,$ if $\epsilon < \epsilon_{Si,Sj}(N, P_{\max})$ then Scheme i beats Scheme j, but if $\epsilon > \epsilon_{Si,Sj}(N, P_{\max})$ then Scheme j beats Scheme i. It follows that if $\epsilon_{S2,S3}(N, P_{\max}) < \epsilon_{S1,S3}(N, P_{\max})$ then $\epsilon_{S1,S2}(N, P_{\max}) > \epsilon_{S1,S3}(N, P_{\max})$. For otherwise, we can choose ϵ such that $\max\{\epsilon_{S1,S2}, \epsilon_{S2,S3}\} < \epsilon < \epsilon_{S1,S3}$ which implies that for this ϵ, Power Scheme 1 beats Power Scheme 3, Power Scheme 3 beats Power Scheme 2, and Power Scheme 2 beats Power Scheme 1, which is a contradiction. Conversely, a similar argument shows that if $\epsilon_{S2,S3}(N, P_{\max}) > \epsilon_{S1,S3}(N, P_{\max})$ then $\epsilon_{S1,S2}(N, P_{\max}) < \epsilon_{S1,S3}(N, P_{\max})$.

Hence, if $N < \tilde{x}(P_{\max})$, then

$$\epsilon^\star(P_{\max}) < \epsilon_{S1,S3}(N, P_{\max}) < \epsilon_{S1,S2}(N, P_{\max}),$$

as illustrated in Fig. 2(a). Conversely, for $N > \tilde{x}(P_{\max})$, a similar argument shows

$$\epsilon_{S1,S2}(N, P_{\max}) < \epsilon_{S1,S3}(N, P_{\max}) < \epsilon^\star(P_{\max}),$$

as illustrated in Fig. 2(b). Finally, we show that in this case, $(1/2)\epsilon^\star < \epsilon_{S1,S2}(N, P_{\max})$. First, note that by Lemma A.1 in Appendix A we have that $(1/2)\epsilon^\star < \grave{\epsilon}(P_{\max})$ and hence

$$3r_2\left(\frac{\epsilon^\star}{2}, P_{\max}\right) > 2r_1\left(\frac{\epsilon^\star}{2}, P_{\max}\right). \tag{33}$$

Now, Power Scheme 2 achieves the value

$$\begin{aligned}\frac{1}{2}(N-3)r_0(P_{\max}) &+ 2r_1(\epsilon, P_{\max})\\ &= \frac{1}{2}Nr_0(P_{\max}) - \frac{3}{2}r_0(\epsilon, P_{\max}) + 2r_1(\epsilon, P_{\max}).\end{aligned}$$

From (9), we have that

$$\begin{aligned}\frac{3}{2}r_0\left(\frac{\epsilon^\star}{2}, P_{\max}\right) &= 3r_2\left(\frac{\epsilon^\star}{2}, P_{\max}\right),\\ &> 2r_1\left(\frac{\epsilon^\star}{2}, P_{\max}\right),\end{aligned}$$

the last inequality following from (33). It follows that for $\epsilon = (1/2)\epsilon^\star$, Power Scheme 2 achieves a sum-rate strictly less than $(1/2)Nr_0(P_{\max})$, whereas Power Scheme 1 achieves exactly this value. Thus, for $\epsilon < (1/2)\epsilon^\star$, Power Scheme 1 achieves a larger sum-rate than Power Scheme 2. But Power Scheme 1 and Power Scheme 2 achieve the same sum-rate at $\epsilon = \epsilon_{S1,S2}(N, P_{\max})$. Thus, $(1/2)\epsilon^\star < \epsilon_{S1,S2}(N, P_{\max})$. This completes the proof.

Appendix C
Proof of Lemma 3.6

Using the definition of $\epsilon_{S1,S3}(N, P_{\max})$ given in Appendix B and performing a change of variables by letting $x = 1/N$, we get

$$\begin{aligned}\lim_{N\uparrow\infty} &\frac{1}{2P_{\max}}\left(\frac{P_{\max}}{(\sqrt{1+P_{\max}})^{(N-1)/N}-1} - 1\right)\\ &= \lim_{x\downarrow 0}\frac{1}{2P_{\max}}\left(\frac{(\sqrt{1+P_{\max}})^{2+x} - (\sqrt{1+P_{\max}})}{\sqrt{1+P_{\max}} - (\sqrt{1+P_{\max}})^x}\right)\\ &= \frac{1}{2P_{\max}}\left(\frac{1+P_{\max} - \sqrt{1+P_{\max}}}{\sqrt{1+P_{\max}}-1}\right)\\ &= \frac{\sqrt{1+P_{\max}}}{2P_{\max}} = \frac{1}{2}\epsilon^\star(P_{\max}).\end{aligned} \tag{34}$$

Appendix D
Proof of Lemma 4.1

The objective function to be maximized is given in (19) and we are only interested in optimizing one power variable, i.e., P_s. The following lemma provides the solution to Problem 4.1.

Lemma D.1: Consider the following optimization problem

$$\begin{aligned}&\max_x \left[\log\left(1+\frac{x}{A}\right) + \log\left(1+\frac{C}{x+B}\right)\right]\\ &\text{s.t.} \quad 0 \le x \le x_{\max}.\end{aligned} \tag{35}$$

If $A, B > 0$, and $C \ge 0$, then the above objective function is maximized at either $x = 0$ or $x = x_{\max}$.

Proof: Denote the objective function above as $f(x)$, and take the derivative:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{1}{(1+\frac{x}{A})}\left(\frac{1}{A}\right) + \frac{1}{\left(1+\frac{C}{x+B}\right)}\left(\frac{-C}{(x+B)^2}\right)\\ &= \frac{n(x)}{d(x)},\end{aligned} \tag{36}$$

where $n(x) = x^2 + 2Bx + [B^2 + C(B-A)]$ and $d(x) = (A+x)(x+B+C)(x+B)$. Since $A, B > 0$, $C \ge 0$, and x can only take non-negative values up to $x_{\max}$, hence $d(x) > 0$. Observe that $n(x)$ is a quadratic function, and has at most two zero-crossing points. Also note that $n'(x) > 0$ and the roots of $n(x)$ are $-B \pm \sqrt{C(A-B)}$. For $x \ge 0$, we can make the following deductions based on the roots of $n(x)$.

- If $A < B$, then both roots are complex and so $n(x)$ does not cross zero and is positive. The maximum of $f(x)$ is thus attained at $x = x_{\max}$.
- If $A = B$, then we have a double root at $x = -B$, and the function $n(x)$ is positive. The maximum of $f(x)$ is again at $x = x_{\max}$.
- If $A > B$ and $\sqrt{C(A-B)} \le B$, then both roots are non-positive and $n(x)$ is positive. The maximum value of $f(x)$ is achieved at $x = x_{\max}$.
- If $A > B$ and $\sqrt{C(A-B)} > B$, then we have one negative and one positive root. Evaluating the second derivative

at the positive root (denoted as x^*), we get

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x^*} = \frac{d(x^*)n'(x^*) - n(x^*)d'(x^*)}{(d(x^*))^2} = \frac{d(x^*)n'(x^*)}{(d(x^*))^2} = \frac{n'(x^*)}{d(x^*)} > 0$$

The root x^* corresponds to a minimum point of $f(x)$. If $0 < x^* < x_{\max}$, then the maximum value of $f(x)$ is attained at either $x = 0$ or $x = x_{\max}$. If $x^* \geq x_{\max}$, then $f(x)$ is maximum at $x = 0$.

Thus for all possible combinations of A, B, and C, the maximum of $f(x)$ is achieved at either $x = 0$ or at $x = x_{\max}$. ∎

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