Dissecting OFDM: The Independent Roles of the Cyclic Prefix and the IDFT Operation

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***Abstract*—An orthogonal frequency-division multiplex (OFDM) system encodes each block of data by first performing an inverse discrete Fourier transform operation and then adding a cyclic prefix. This letter uses the Cramer–Rao Bound (CRB) to study the roles of these two operations. The main result is that a cyclic prefix allows almost all channels, including channels with otherwise unstable inverses, to be inverted accurately and moreover, it is the most efficient way of achieving this stability.**

***Index Terms*—CP-OFDM, Cramer–Rao bound, cyclic prefix, orthogonal frequency division multiplex, TZ-OFDM.**

I. Introduction

ORTHOGONAL multiplexing was proposed in 1966 [1]. In 1971 it was realized that orthogonal frequency division multiplex (OFDM) systems can be implemented by first performing an Inverse Discrete Fourier Transform (IDFT) operation on the source symbols and then adding a cyclic prefix [8]. This paper shows that these two operations perform two distinct tasks and more importantly, that the cyclic prefix allows almost all channels to be inverted accurately.

The main tool used in this paper to study the performance of various linear precoders is the Cramer-Rao Bound (CRB). It is proved in Section III that the CRB of any precoding scheme which uses a cyclic prefix is invariant to the phase of the channel spectrum and moreover, using a cyclic prefix is the most efficient way of attaining this property. An important consequence of this property is that channels which would otherwise have an unstable inverse become stably invertible.

The role of the IDFT in OFDM systems is studied in Section IV while Section V studies TZ-OFDM systems and shows that they implicitly use a cyclic prefix. Proofs have been omitted to save space.

II. The CRB as a Figure of Merit

Consider using an arbitrary matrix $P \in \mathbb{C}^{n\times p}$ to encode a finite number of unknown source symbols $\boldsymbol{s} \in \mathbb{C}^p$ prior to transmission through a finite impulse response (FIR) channel $\boldsymbol{h} = [h_0, \dots, h_{L-1}]^T \in \mathbb{C}^L$. The received vector $\boldsymbol{y} \in \mathbb{C}^{n-L+1}$ is given by

$$\boldsymbol{y} = \mathcal{H}P\boldsymbol{s} + \boldsymbol{n}, \qquad \boldsymbol{n} \sim N(0, I) \tag{1}$$

where $\mathcal{H}$ is the upper triangular $(n-L+1)\times n$ Toeplitz channel matrix with first row equal to $[h_{L-1}, \dots, h_0, 0, \dots, 0]$ and $\boldsymbol{n}$ denotes additive white Gaussian noise (AWGN) with unit variance.

For simplicity, it is assumed throughout that the receiver has perfect knowledge of the channel parameters $\boldsymbol{h}$. Then, it is well-known that, for any unbiased estimate $\hat{\boldsymbol{s}}$ of $\boldsymbol{s}$, the error covariance matrix $\boldsymbol{E}\left[(\boldsymbol{s}-\hat{\boldsymbol{s}})(\boldsymbol{s}-\hat{\boldsymbol{s}})^H\right]$ is lower bounded by the CRB

$$R = \left(P^H\mathcal{H}^H\mathcal{H}P\right)^{-1}. \tag{2}$$

In fact, this lower bound is met with equality if the maximum-likelihood (ML) decoder

$$\hat{\boldsymbol{s}} = \left(P^H\mathcal{H}^H\mathcal{H}P\right)^{-1}P^H\mathcal{H}^H\boldsymbol{y} \tag{3}$$

is used.

Throughout, this letter uses R as a figure of merit for the precoder P. Indeed, for any given channel $\boldsymbol{h}$, the diagonal element R_{ii} is the mean-square error (MSE) of the estimate of the ith element of $\boldsymbol{s}$ if the ML-decoder is used, while the off-diagonal element R_{ij} is the correlation between the estimates of the ith and jth elements of $\boldsymbol{s}$.

The relevance of (1) and (2) to practical OFDM and TZ-OFDM systems is now explained. An OFDM system uses a precoder $\tilde{P}$ which first IDFT's the block and then adds a cyclic prefix of length $L-1$. Since an OFDM receiver operates on blocks separately, it does not keep any information about the previously decoded block. Therefore, the first $L-1$ received symbols of the current block provide no information about the transmitted symbols and are discarded by the receiver. Thus, for an OFDM system, if $P = \tilde{P}$ then (1) correctly models all the information available to the receiver about the current block $\boldsymbol{s}$ and hence (2) is the best achievable performance of any unbiased equalizer in an OFDM system.

A TZ-OFDM system uses a $\tilde{P}$ which first IDFT's the block and then appends $L-1$ trailing zeros. Unlike in an OFDM system though, the first $L-1$ received symbols of the current block do provide information about the transmitted symbols. This is because the receiver knows that $L-1$ zeros (corresponding to the trailing zeros of the previous block) were transmitted just prior to the current block. In order to incorporate this exra information into (1), it is necessary to use a P different from $\tilde{P}$. Specifically, the correct P is one which first IDFT's the block and then adds both $L-1$ leading zeros and $L-1$ trailing zeros to the block. Then, (1) correctly models all the information available to the receiver about the current block $\boldsymbol{s}$ and thus (2) is the best achievable performance of any unbiased equalizer in a TZ-OFDM system.

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III. Cyclic Prefix

Consider sending the symbols 1,2,3 over a length $L = 2$ FIR channel by first precoding the symbols to form one of: (A) 0,1,2,3, (B) 1,2,3,0, or (C) 3,1,2,3. Consider the following four test channels:

$$\boldsymbol{h}_1 = [1\ 0]^T, \quad \boldsymbol{h}_2 = [0\ 1]^T, \quad \boldsymbol{h}_3 = [1 - 5]^T, \quad \boldsymbol{h}_4 = [-5\ 1]^T. \tag{4}$$

The resulting CRB, given by R in (2), can be calculated for any combination of precoder and channel. Of most interest are the following combinations:

$$R_{A3} = \begin{bmatrix} 1 & 5 & 25 \\ 5 & 26 & 130 \\ 25 & 130 & 651 \end{bmatrix}, \quad R_{B4} = \begin{bmatrix} 651 & 130 & 25 \\ 130 & 26 & 5 \\ 25 & 5 & 1 \end{bmatrix} \tag{5}$$

where R_{A3} denotes the combination of precoder A and channel $\boldsymbol{h}_3$ and similarly for R_{B4}. Furthermore, the CRB is not defined (implying that not all the symbols can be recovered) if precoder A is used over channel 2, or if precoder B is used over channel 1. It can be verified that all other combinations lead to reasonable values of R. For example

$$R_{C3} = R_{C4} = \begin{bmatrix} 0.04 & 0.01 & 0.01 \\ 0.01 & 0.04 & 0.01 \\ 0.01 & 0.01 & 0.04 \end{bmatrix}. \tag{6}$$

There is a simple explanation for (5). The channel $\boldsymbol{h}_3$ is nonminimum phase and hence has an unstable inverse. This is reflected by the exponential growth in the diagonal elements of R_{A3}. Precoder B puts a known symbol at the end and hence performs poorly over channels which have an unstable inverse when run backward (such as any nonmaximum phase channel). Since a channel generated at random has a reasonable chance of being nonminimum phase, it is clear that Precoder A is unsuitable for use in practice and similarly for Precoder B.

This observation can be generalized as follows. Any precoder which does not make the last $L-1$ transmitted symbols in a block[1] a known function of the first $L-1$ transmitted symbols will perform badly if the channel is nonminimum phase. This is because errors introduced by noise near the start of the block build up exponentially in magnitude and go unchecked unless the receiver is able to reconcile the last $L-1$ symbols with their true values (both Precoders B and C allow this reconciliation, for instance). Similarly, if the first $L-1$ transmitted symbols are not a known function of the last $L-1$ transmitted symbols in a block, then channels which are nonmaximum phase will lead to an exponential growth of errors in the reverse direction (as shown by R_{B4}). A cyclic prefix precoder is distinguished by the fact that it satisfies both properties, the first $L-1$ symbols of each block are a function of the last $L-1$ symbols, and the last $L-1$ symbols of each block are a function of the first $L-1$ symbols. Therefore, a cyclic prefix prevents an exponential growth of errors regardless of the phase of the channel.

[1]Here, "block" must be interpreted with care. In the notation of Section II, it refers to $P\boldsymbol{s}$ and not to $\tilde{P}\boldsymbol{s}$. In this sense then, a TZ-OFDM block has $L-1$ zeros at the start and at the end.

In fact, the key property of the cyclic prefix is that its performance is invariant to the phase of the channel spectrum (see Theorem 1 below), and moreover, it keeps this property regardless of what other linear operations are performed prior to adding the cyclic prefix. Note too that a cyclic prefix is the most efficient way of attaining the phase invariance property because a necessary condition for the inverse in (2) to exist is for the precoder to introduce at least $L-1$ redundant symbols.

Theorem 1: In (1), assume that the linear precoder P factorizes as $P = CA$ where $A \in \mathbb{C}^{(n-L+1)\times p}$ is an arbitrary linear precoder and

$$C = \begin{bmatrix} 0_{(L-1)\times(n-2L+2)} & I_{L-1} \\ & I_{n-L+1} \end{bmatrix} \tag{7}$$

is the circulant matrix introducing the cyclic prefix. Then, the CRB R, defined in (2), depends on $\boldsymbol{h}$ only through the magnitude of the channel spectrum and in particular, is invariant to the phase of the channel spectrum.

IV. The IDFT Operation

An OFDM system uses the precoder $P = CD^H$, where C is the cyclic prefix defined in (7) and D is the DFT matrix. Dropping the IDFT operation (that is, choosing $P = C$) results in a Single Carrier with Cyclic Prefix system [7]. It is well known [2], [5], that Single Carrier with Cyclic Prefix systems have a better BER than OFDM systems do at high SNR's. This section uses the CRB to explain why this is so.

It is first noted that the total MSE in estimating the source symbols $\boldsymbol{s}$ in (1) is given by tr$\{R\}$, the trace of the CRB matrix defined in (2) and moreover, it is straightforward to show that if the precoder matrix P is replaced by PQ, where Q is any unitary matrix, then the total MSE remains the same. In particular, the total MSE of an OFDM system ($P = CD^H$) is the same as that of a single carrier with cyclic prefix system ($P = C$). However, unitary transforms can affect the BER even though they do not alter the total MSE. The form of the CRB matrix for the two systems is given in the following two theorems, the first of which is well-known since it is the basis of OFDM systems.

Theorem 2: Let $P = CD^H$ in (1), where C is defined in (7) and D is the DFT matrix. Then, for any channel $\boldsymbol{h}$, the CRB matrix R, defined in (2), is diagonal.

Theorem 3: Choose $P = C$ in (1), where C is defined in (7). Then, for any channel $\boldsymbol{h}$, the CRB R, defined in (2), has a constant diagonal, meaning that every source symbol can be recovered with the same accuracy.

Theorems 2 and 3 imply that the precoders $P = C$ and $P = CD^H$ are at two extremes; choosing $P = CD^H$ means that each symbol sees an independent sub-channel and hence its MSE can range from very good to very bad, whereas choosing $P = C$ spreads the symbols over the sub-channels so as to average out the performance. This explains why there is an advantage to using only a cyclic prefix at high SNR; each source symbol will be recovered with the same high accuracy, meaning that the BER will be close to zero. However, at low SNR, all the source symbols are subject to the same high amount of error, meaning that the BER is likely to be very high.

V. Trailing Zero OFDM

Referring to Precoders A, B and C in Section III, it is noted that in all cases, four symbols are transmitted yet the equalizer (3) sees only three symbols due to the memory of the channel. If multiple blocks are transmitted, this corresponds to the well known fact that an OFDM equalizer discards the guard interval (that is, the cyclic prefix) between blocks.

Scrutiny of [4], [7] reveals that a TZ-OFDM equalizer does not discard the guard interval. That is, a TZ-OFDM equalizer makes use of more symbols per block than an OFDM equalizer does. This is indeed fortunate; Precoders A and B in Section III correspond to the two possible configurations of a TZ-OFDM system whose equalizer discards the guard interval and both configurations have an unacceptable performance over a wide range of channels.

Building on the example in Section III, let Precoder D map 1,2,3 to 0,1,2,3,0. Because a TZ-OFDM equalizer not only relies on the fact that every block has $L-1$ trailing zeros, but also relies on the fact that an all-zero guard interval of length $L-1$ is transmitted prior to every block (this is merely the $L-1$ trailing zeros of the previous block), even though the TZ-OFDM transmitter uses Precoder[2] B to encode each block, in order for (3) to model the TZ-OFDM equalizer, it is necessary to choose P in (1) to correspond to Precoder D; see Section II.

Since the TZ-OFDM equalizer behaves as if Precoder D is used to encode each block, the good performance of a TZ-OFDM system can be understood by studying the properties of Precoder D. To this end, observe that Precoder D decomposes as the product of two precoders; the inner precoder adds $L-1$ zeros to the end of the block and the outer precoder adds a cyclic prefix of length $L-1$. (The effect of appending $L-1$ zeros in the time domain is proved in [4] to correspond to spreading the source symbols out over the frequency domain.)

Thus, a TZ-OFDM system achieves the same performance as a system which encodes each block by first adding $L-1$ trailing zeros and then adding a cyclic prefix of length $L-1$, but it is able to achieve this performance by encoding all but the first block by simply adding $L-1$ trailing zeros. (The very first block must be preceeded by $L-1$ zeros.) This is because the trailing zeros of the previous block double as the cyclic prefix of the current block.

Even though a TZ-OFDM system introduces redundancy, the CRB is not necessarily better than that of an OFDM system for all channels $\boldsymbol{h}$.

[2]Where appropriate, Precoder B is used to denote a precoder which introduces $L-1$ trailing zeros and Precoder D is used to denote a precoder which introduces $L-1$ leading and $L-1$ trailing zeros. The fact that a TZ-OFDM system IDFT's each block is omitted from this discussion for simplicity.

Theorem 4: Consider sending a single block of p symbols over an FIR channel of length L, using either an OFDM precoder P_1 or a TZ-OFDM precoder P_2. Let R_1 and R_2 be the associated CRB matrices (2) for a given channel vector $\boldsymbol{h}$. There exist values of p, L and $\boldsymbol{h}$ for which $\mathrm{tr}\{\mathrm{R}_1\}<\mathrm{tr}\{\mathrm{R}_2\}$.

Proof: Choose $p=3$ and $L=3$. The TZ-OFDM system spreads the $p=3$ symbols over $p+L-1=5$ sub-channels. Choose $\boldsymbol{h}$ to have spectral nulls on the second and fifth sub-channels, that is, $\boldsymbol{h}=[1 - 0.618\ 1]^T$. Then $\mathrm{tr}\{\mathrm{R}_1\}=1.29$ and $\mathrm{tr}\{\mathrm{R}_2\}=1.88$. □

VI. Conclusion

This letter showed that a cyclic prefix is the most efficient way of ensuring that almost all channels can be inverted accurately. Furthermore, it was shown that the good performance of TZ-OFDM systems does not contradict this recommendation because TZ-OFDM systems implicitly use a cyclic prefix. It was also explained why the IDFT operation in OFDM systems, even though it does not alter the total MSE, improves the BER at low SNR, and it was proved that there exist channels for which TZ-OFDM systems have a worse CRB than OFDM systems.

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References

[1] R. W. Chang, "Synthesis of band-limited orthogonal signals for multichannel data transmission," *Bell Syst. Tech. J.*, Dec. 1966.

[2] A. Czylwik, "Comparison between adaptive OFDM and single carrier modulation frequency domain equalization," in *IEEE Vehicular Technology Conf.*, vol. 2, 1997, pp. 865–869.

[3] J. H. Manton and Y. Hua, "A frequency domain deterministic approach to channel identification," *IEEE Signal Processing Lett.*, vol. 6, pp. 323–326, Dec. 1999.

[4] B. Muquet, Z. Wang, G. B. Giannakis, M. de Courville, and P. Duhamel, "Cyclic-prefixed or zero-padded multicarrier transmissions," IEEE Trans. Commun., vol. 48, pp. ???–???, which month??? 2000, submitted for publication.

[5] H. Sari, G. Karam, and I. Jeanclaude, "Transmission techniques for digital terrestrial TV broadcasting," *IEEE Commun. Mag.*, vol. 33, pp. 100–109, Feb. 1995.

[6] A. Scaglione, G. B. Giannakis, and S. Barbarossa, "Redundant filterbank precoders and equalizers: Parts I and II," *IEEE Trans. Signal Processing*, vol. 47, pp. 1988–2022, July 1999.

[7] W. Wang and G. B. Giannakis, "Wireless multicarrier communications: Where Fourier meets Shannon," *IEEE Signal Processing Mag.*, May 2000.

[8] S. B. Weinstein and P. M. Ebert, "Data transmission by frequency division multiplexing using the discrete Fourier transform," *IEEE Trans. Commun.*, vol. COM-19, pp. 628–634, Oct. 1971.