Group Representation Theory in Radar

Jonathan H. Manton

Department of Information Engineering
Research School of Information Sciences and Engineering
The Australian National University

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Outline

Motivation

I'm Interested in Radar...

Group representation theory provides a better understanding of the inherent limitations of the performance of a radar system.

I'm Interested in Group Representation Theory...

Studying radar leads one to the Heisenberg group, the Stone-von Neumann Theorem, the Bargmann-Segal representation, Hermite and Laguerre polynomials, and more.

Radar

RAdio Detection And Ranging

An electronic device designed to detect objects, and calculate their distance and/or speed.

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Measuring Distance

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- When the reflected pulse arrives back at the radar, the timer is stopped.
- A simple calculation involving the speed of light and the time taken for the pulse to return enables the distance from the radar to the object to be determined.

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- By computing the change in frequency (Doppler effect), the speed of the object can be calculated.

Pulse Design

- The received signal will always be corrupted by noise.
- To measure distance accurately in the presence of noise, want the pulse to be localised in time.
- To measure speed accurately in the presence of noise, want the pulse to be localised in frequency.
- Moreover, there are practical limitations on the bandwidth and duration of a pulse.
- What is the best pulse to use?

Narrowband Pulse

- Transmit a pulse $w(t)$ modulated onto a much higher carrier frequency: $s(t) = w(t)\sin(2\pi f_c t)$.
- If it hits an object at distance d with speed v, then approximately $w(t)$ becomes $w(t-2d)$ and the frequency f_c becomes $\frac{1-v}{1+v} f_c \approx (1-2v) f_c$.
- After demodulation (and working in the complex domain), the received signal is $r(t) = w(t-2d)e^{-4\pi i v f_c t}$.

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- Given a noisy version of $r(t)$, estimate d and v.

How to Estimate Range and Doppler

- Given $r(t) = w(t - 2d)e^{-4\pi i v f_c t} + n(t)$ where $n(t)$ is noise.
- To estimate d and v, we correlate ("compare") $r(t)$ with $u(t; d', v') = w(t - 2d')e^{-4\pi i v' f_c t}$.

$$C(d', v') = \left| \int_{-\infty}^{\infty} r(t)\overline{u(t; d', v')}\, dt \right|$$

- By plotting the intensity of $C(d', v')$ on the (d', v') plane, the point (d', v') where $C(d', v')$ is the largest can be found visually. This is our estimate of the distance and speed of the object.

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- Ideally we want $C(d', v')$ to be a sharp pulse centred on (d, v). How do we design the pulse $w(t)$ to achieve this?

Outline

The Ambiguity Function

Definition

Given two waveforms $w(t), v(t)$, (one form of) the radar ambiguity function is

$$A_{w,v}(t,f) = \int_{-\infty}^{\infty} w(\tau) e^{2\pi i f \tau} \overline{v(\tau - t)}\, d\tau.$$

- The interpretation is that we transmit the waveform $w(t)$, it bounces off a stationary object located at the radar, hence we receive $w(t)$. We correlate $w(t)$ with a time and frequency shifted version of $v(t)$, forming $A_{w,v}(t,f)$.
- If $|A_{w,w}(t,f)|$ is a sharp peak centred at the origin, then $w(t)$ is a good waveform.

The Ambiguity Function
Questions

- What does the set of all $A_{w,w}$ look like, where $w \in L^2(\mathbb{R})$?
- How close can $A_{w,w}$ be made to a sharp peak?
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- What if the set of possible w is restricted to being "implementable in practice"?
- To attempt to answer these questions, we can ask:
  - Are any symmetries present? (Given that $A(t, f)$ is an ambiguity function, for what functions $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is it true that $A(\alpha(t, f))$ is also an ambiguity function?)
  - Is there an orthonormal basis $w_1, w_2, \cdots$ of $L^2(\mathbb{R})$ such that A_{w_i,w_i} are "nice"?

Outline

A Suggestive Expression for the Ambiguity Function

- Let $\langle \cdot, \cdot \rangle$ denote the inner product on $L^2(\mathbb{R})$:

$$\langle w, v \rangle = \int_{-\infty}^{\infty} w(t)\overline{v(t)}\, dt.$$

- Let $\rho_{(t,f)} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the unitary operator $\rho_{(t,f)}(v)(\tau) = e^{-2\pi i f \tau} v(\tau - t)$.
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- Then $A_{w,v}(t, f) = \langle w, \rho_{(t,f)} v \rangle$.
- If ρ were a group representation of $\mathbb{R}^2$ then A would be a special function on $\mathbb{R}^2$.

Representations

- We consider $\mathbb{R}^2$ in $\rho : \mathbb{R}^2 \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$ as a group (the additive group).
- For $g \in \mathbb{R}^2$, ρ_g is a unitary operator on $L^2(\mathbb{R})$, i.e. it belongs to the unitary group.
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- However, $\rho_{(t+t',f+f')} = e^{-2\pi i f' t} \rho_{(t,f)} \rho_{(t',f')}$, so it fails to be a representation only in that there is the scalar $e^{2\pi i f' t}$ present.

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- In fact, it is a *multiplier representation*.

Multiplier Representations

- A multiplier is a Borel map $\sigma : G \times G \to T$ (T is group of complex numbers with magnitude 1) satisfying the cocycle condition $\sigma(g_1, g_2)\sigma(g_1 g_2, g_3) = \sigma(g_1, g_2 g_3)\sigma(g_2, g_3)$ and the normalisation $\sigma(1, 1) = 1$.
- A multiplier representation is a Borel-measurable map ρ from G to the unitary group of some Hilbert space that satisfies $\rho_{g_1 g_2} = \sigma(g_1, g_2)\rho_{g_1}\rho_{g_2}$ for some multiplier σ.
- Note the cocycle condition is natural: $\rho_{(g_1 g_2)g_3} = \rho_{g_1(g_2 g_3)}$ hence $\sigma(g_1 g_2, g_3)\sigma(g_1, g_2)\rho_{g_1}\rho_{g_2}\rho_{g_3} =$ $\sigma(g_1, g_2 g_3)\sigma(g_2, g_3)\rho_{g_1}\rho_{g_2}\rho_{g_3}$.

Multiplier Representations Are Natural

- The theory of multiplier representations is essentially the theory of projective representations. (Cohomologous multipliers have essentially the same representation theory.)
- If one uses "Mackey analysis" to construct ordinary representations of a (locally compact) group G, one needs to consider projective representations of subgroups of G.
- If we extend the Mackey analysis to construct projective representations, then we still only have to consider projective representations of subgroups of G.
- Hence, projective representations (and thus multiplier representations) form a natural completion of the class of ordinary representations. Besides, radar theory shows they should exist!

The Heisenberg Group

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- For the ρ in radar, $\tilde{G}$ is the Heisenberg group.
- There is an exact correspondence between the σ-representation theory of G and the ordinary representation theory of those representations of $\tilde{G}$ that restrict on the central subgroup T to be the homomorphism $[g, z] \to \overline{z}I$.

Outline

Two Key Theorems

Stone-von Neumann Theorem

The σ-representation ρ is irreducible, and it is the unique irreducible σ-representation of $\mathbb{R}^2$ up to equivalence.

Moyal's Identity

$$\langle A_{u,v}, A_{u',v'} \rangle_{L^2(\mathbb{R}^2)} = \langle u, u' \rangle \langle v, v' \rangle$$

which implies the "Heisenberg uncertainty principle"

$$\|A_{w,w}\|_{L^2(\mathbb{R}^2)} = \|w\|^2.$$

Indeed, $A_{w,w}(0,0) = \|w\|^2$ but this is also its L^2-norm, hence $A_{w,w}$ has considerable spread.

The Bargmann-Segal Representation

- Radar gives us a σ-representation of $\mathbb{R}^2$ on $L^2(\mathbb{R})$.
- Consider instead the (reproducing kernel) Hilbert space of entire functions $a(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfying $\sum_n n!|a_n|^2 < \infty$, with inner product $\langle a, b \rangle = \sum_n n! a_n \overline{b_n}$.
- An orthonormal basis is $z^k/\sqrt{k!}$ for $k = 0, 1, \cdots$.
- We can write down an explicit σ-representation of $\mathbb{R}^2$ on this new space. It is irreducible, hence equivalent to the σ-representation on $L^2(\mathbb{R})$.
- Moreover, the resulting isometry between the two Hilbert spaces maps the Hermite functions in $L^2(\mathbb{R})$ to $z^k/\sqrt{k!}$. Hence, this is a more convenient representation to use if Hermites are involved.

Outline

Ambiguity Functions of the Hermites

- Let v_n denote the nth Hermite function, which up to scale is $e^{-t^2/2}H_n(t)$ and $H_n(t)$ is the nth Hermite polynomial.
- We wish to calculate A_{v_n,v_m}. Very difficult to do from first principles.
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- By using the Bargmann-Segal Representation instead, can deduce that the A_{v_n,v_m} are the Laguerre polynomials.
- If we weren't thinking of A_{v_n,v_m} as $\langle v_n, \rho v_m\rangle$ then we wouldn't have thought to try the Bargmann-Segal representation to simplify the calculations.

Symmetries of Ambiguity Functions (I)

- (We change to an equivalent multiplier ν which only affects the phase of $A_{v,w}$.)
- We want to look for symmetries: Find functions α such that i) $A_{v,w}(\alpha(t,f))$ is a valid ambiguity function ($= A_{v',w'}(t,f)$), or ii) $A_{v,w}(\alpha(t,f)) = A_{v,w}(t,f)$.

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- $A_{v,w}(\alpha(t,f)) = \langle v, \rho_{\alpha(t,f)} w \rangle$. If there exists a unitary operator $U(\alpha)$ such that $\rho_{\alpha(t,f)} = U(\alpha)^{-1} \rho_{(t,f)} U(\alpha)$ then $A_{v,w}(\alpha(t,f)) = A_{U(\alpha)v, U(\alpha)w}(t,f)$, achieving (i). Moreover, (ii) is achieved if $U(\alpha)v = \lambda v$ and $U(\alpha)w = \bar{\lambda} w$ for some $\lambda \in T$.

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- The group of continuous automorphisms α which preserve ν, i.e. $\nu(\alpha(t,f),\alpha(t',f')) = \nu((t,f),(t',f'))$, is precisely $\mathrm{SL}(2,\mathbb{R})$.

Symmetries of Ambiguity Functions (II)

- Hence, if $\alpha \in \mathrm{SL}(2, \mathbb{R})$ and ρ is an irreducible ν-representation of $\mathbb{R}^2$, then $\rho \circ \alpha$ is also an irreducible ν-representation.

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- Moreover, the map $\alpha \to U(\alpha)$ is a projective representation of $\mathrm{SL}(2, \mathbb{R})$.
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- It lifts to a unitary representation of the double covering $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ of $\mathrm{SL}(2,\mathbb{R})$ called the *metaplectic representation*.
- This fact has not really been exploited in radar theory yet.

Outline

Summary

- A mathematical study of radar quickly leads to interesting and deep questions in group representation theory.
- The ambiguity function $A_{w,w}$ associated with a waveform $w(t)$ tells us how good the waveform is at locating an object.
- The connection is that $A_{w,w}(t,f) = \langle w, \rho_{(t,f)} w \rangle$ where ρ is a multiplier representation of $\mathbb{R}^2$.
- Although much has been done, it is likely very deep results in representation theory will lead to new results in radar theory.