Extreme Point Results for Robust Stability of Nested Polynomial Families with Real Coefficients

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Abstract

A family of polynomials is robustly stable if each member of the family is stable, meaning all its zeros lie in the open left half plane. Consider a one dimensional affine family of polynomials, visualised as a line segment. For some such families, it is known the family is robustly stable if and only if the two end points are stable. Indeed, a sufficient condition is for the line segment to be parallel to what is called a convex direction. This paper extends this result to nested polynomial families. Specifically, a set of directions is found such that, if the line segment is parallel to one of these directions, then any nested family obtained by composing a fixed polynomial with the line segment is robustly stable if and only if its end points are stable. Moreover, it is proved this set of directions is the largest possible. An extension to higher dimensional families is also derived.

Key words: robust stability, extreme point results, nonlinear uncertainty structure, uncertain uniform system, parametric uncertainty, polygon uncertainty bounds.

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1 Introduction

A polynomial p over either the real or the complex field is said to be *Hurwitz stable*, or simply Hurwitz or stable, if all its zeros lie in the open left half plane. By convention, a polynomial of degree zero is not Hurwitz stable. A family $\mathcal{P}$ of polynomials is *robustly stable* if every polynomial in $\mathcal{P}$ is stable. Given a family $\mathcal{P}$ and a polynomial ϕ, the associated *nested polynomial family* is

$$\mathcal{N} = \phi(\mathcal{P}) = \{\phi \circ p : \ p \in \mathcal{P}\} \tag{1}$$

where $\circ$ denotes composition. This paper restricts attention to nested polynomial families of the form (1), where the polynomials in the base family $\mathcal{P}$ have real coefficients. The fixed polynomial ϕ can have complex coefficients.

Proving an arbitrary nested family $\mathcal{N}$ is robustly stable in general necessitates verifying each and every polynomial in $\mathcal{N}$ is stable. Of interest then are classes of families for which the whole family can be proved to be robustly stable simply by verifying a finite number of strategically chosen elements of the family are stable [3, 5, 12]. To this end, this paper proves the following two results. For clarity, Corollary 1 is stated instead of Theorem 13.

Corollary 1 *Let p_0 and p_1 be two polynomials with real coefficients. It is assumed*

$$\frac{d\theta}{d\omega} \leq \begin{cases} \frac{\sin(2\theta)}{2\omega} & \text{if } \sin(2\theta) \geq 0, \\ \frac{-\sin(2\theta)}{4\omega} & \text{otherwise} \end{cases} \tag{2}$$

holds for all $\omega > 0$ for which $\theta(\omega) \arg\{p_1(\jmath\omega) - p_0(\jmath\omega)\}$ is defined, that is, for which $p_1(\jmath\omega) \neq p_0(\jmath\omega)$, and the degree of every element of the line segment

$$\mathcal{P} = \{p : \ p = \lambda p_1 + (1-\lambda)p_0, \ \lambda \in [0,1]\} \tag{3}$$

is the same and greater than zero. Let ϕ be an arbitrary polynomial with possibly complex coefficients. Then the one dimensional nested family $\mathcal{N} = \phi(\mathcal{P})$ is robustly stable if and only if both $\phi(p_0)$ and $\phi(p_1)$ are stable.

Corollary 1 extends to the following class of higher dimensional families. The class is larger than the ones considered in [13, Theorem 2] and [18, Theorem 3]. First, for $i = 1, \cdots, n$, define the interval polynomial families

$$\mathcal{A}_i = \left\{ p : \ p(s) = \sum_{j=0}^{d_i} a_j^{(i)} s^j, \ \ a_j^{(i)} \in [a_j^{(i-)}, a_j^{(i+)}] \subset \mathbb{R} \right\} \tag{4}$$

where it is assumed $d_i > 0$ and $0 \notin [a_{d_i}^{(i-)}, a_{d_i}^{(i+)}]$, so the degree of every element of $\mathcal{A}_i$ is d_i. Then, for fixed polynomials $q_1, \cdots, q_n$ with real coefficients, define the family

$$\mathcal{P} = \{p : \ p(s) = q_1(s)a_1(s) + \cdots + q_n(s)a_n(s), \ a_i(s) \in \mathcal{A}_i, \ i = 1, \cdots, n\} \tag{5}$$

of linear combinations of the interval polynomials. Finally, define the nested family $\mathcal{N} = \phi(\mathcal{P})$ where ϕ is an arbitrary polynomial with possibly complex coefficients.

Theorem 2 *Consider the nested family $\mathcal{N}$ defined above. Define $\theta_i(\omega) = \arg\{q_i(\jmath\omega)\}$. It is assumed, for all $i = 1, \cdots, n$ and for all $\omega > 0$ for which $\theta_i(\omega)$ is defined,*

$$\frac{d\theta_i}{d\omega} \leq \left| \frac{\sin(2\theta_i)}{4\omega} \right|. \tag{6}$$

It is also assumed the degree of every polynomial in $\mathcal{P}$ is the same and greater than zero. Let $K_i^1, \cdots, K_i^4$ denote the Kharitonov polynomials [16] associated with the family $\mathcal{A}_i$, namely

$$K_i^1(s) = a_0^{(i-)} + a_1^{(i-)}s + a_2^{(i+)}s^2 + a_3^{(i+)}s^3 + a_4^{(i-)}s^4 + a_5^{(i-)}s^5 + a_6^{(i+)}s^6 + a_7^{(i+)}s^7 + \cdots \tag{7}$$

$$K_i^2(s) = a_0^{(i-)} + a_1^{(i+)}s + a_2^{(i+)}s^2 + a_3^{(i-)}s^3 + a_4^{(i-)}s^4 + a_5^{(i+)}s^5 + a_6^{(i+)}s^6 + a_7^{(i-)}s^7 + \cdots \tag{8}$$

$$K_i^3(s) = a_0^{(i+)} + a_1^{(i-)}s + a_2^{(i-)}s^2 + a_3^{(i+)}s^3 + a_4^{(i+)}s^4 + a_5^{(i-)}s^5 + a_6^{(i-)}s^6 + a_7^{(i+)}s^7 + \cdots \tag{9}$$

$$K_i^4(s) = a_0^{(i+)} + a_1^{(i+)}s + a_2^{(i-)}s^2 + a_3^{(i-)}s^3 + a_4^{(i+)}s^4 + a_5^{(i+)}s^5 + a_6^{(i-)}s^6 + a_7^{(i-)}s^7 + \cdots \tag{10}$$

and define the finite family

$$\tilde{\mathcal{P}} = \left\{p : \ p(s) = q_1(s)a_1(s) + \cdots + q_n(s)a_n(s), \ a_i(s) \in \left\{K_i^1, \cdots, K_i^4\right\}, \ i = 1, \cdots, n\right\}. \tag{11}$$

Then $\mathcal{N}$ is robustly stable if and only if $\tilde{\mathcal{N}} = \phi(\tilde{\mathcal{P}})$ is robustly stable.

Motivation for considering these robust stability problems is now given. The set of stable polynomials forms a subset of the set of polynomials. The shape or geometry of this subset depends on the particular parameterisation (or coordinate system) used. If, for example, monic polynomials are parameterised by their roots, so that $(\lambda_1, \cdots, \lambda_p)$ represents the polynomial $p(s) = (s - \lambda_1)\cdots(s - \lambda_p)$, then the set of stable polynomials is simply the space $\{(\lambda_1, \cdots, \lambda_p) : \ \Re\lambda_i < 0, \ i = 1, \cdots, p\}$. In this geometry, stability results for polytopes of polynomials are trivial; the whole family is robustly stable if and only if every extreme point of the family is stable. Unfortunately, many practical problems do not result in families forming polytopes under this geometry.

Another natural parameterisation of polynomials is via their coefficients, so that $(a_0, \cdots, a_p)$ represents the polynomial $p(s) = a_p s^p + \cdots + a_0$. In this geometry, the straight line segment connecting the polynomials p_0 and p_1 is given by $\{p : \ p = \lambda p_1 + (1 - \lambda)p_0, \ \lambda \in [0, 1]\}$. Although the shape of the set of stable polynomials in this geometry is hard to visualise, and hence robust stability results are often non-trivial, it is known [19, Theorem 2] that if $p_1 - p_0$ satisfies a certain phase growth condition similar to (2) then the line segment joining p_0 and p_1 is fully contained in the set of stable polynomials if and only if its end points, p_0 and p_1, are stable. Directions $p_1 - p_0$ for which this is true are called *convex directions* [3, 5].

The usefulness of convex direction like results is, given a polytopic family of polynomials (under the coefficient geometry described above), if every edge is a convex direction then the whole family is robustly stable if and only if its extreme points are stable [19]. Such families have been found to arise in practice.

Several pointers to the literature are now provided. For an overview, see [3, 5, 16, 19]. Kharitonov's Theorem [12] states the family of interval polynomials A_i in (4) is robustly stable if and only if the four Kharitonov polynomials (7)–(10) are stable. The graphical significance of the Kharitonov polynomials was discovered by Dasgupta [9] and used in [16] to give an elementary proof of Kharitonov's theorem. The theorem remains true in the presence of a degree drop [22]. Similar extreme point results to Kharitonov's, but for polytopes of polynomials and under necessarily restrictive conditions (see the counter-example in [1]), are presented in [19], which build on the work of [4] and others. Multilinear families are considered in [2, 17, 21]. At the other extreme, very general structures are studied in [23]. Nested families of interval polynomials, which arise in stability analysis of uncertain uniform systems [18], are considered in [13]. Very recently, robust stability of multivariate polynomials has been investigated [14]. Frequency response arcs of Hurwitz polynomials are known to be convex [11, 15, 10].

The remainder of this paper is organised as follows. Section 2 states preliminary results while Section 3 presents the main technical result. Corollary 1 and Theorem 2 are proved in Section 4. Several applications are mentioned in Section 5 while Section 6 concludes the paper.

Notation: If $z \in \mathbb{C}$ then $\Re z$, $\Im z$ and $\bar{z}$ denote the real part, imaginary part and complex conjugate respectively. The symbol $\jmath$ denotes $\sqrt{-1}$. Strictly positive real numbers are denoted by $\mathbb{R}^+$. A non-constant polynomial is one whose degree is greater than zero, a real polynomial is one whose coefficients are real valued, and similarly for a complex polynomial. A prime always denotes differentiation with respect to s, even if the argument is $\jmath\omega$. For example, if $p(s) = s^2 + 5s$ then $p'(s) = 2s + 5$ and $p'(\jmath\omega) = 2\jmath\omega + 5$.

2 Preliminaries

2.1 Hurwitz Stability

If $p(s)$ is a polynomial, or indeed any complex analytic function, the number of zeros $p(s)$ has in a region Ω containing the origin equals the winding number of $p(s)$ along the boundary of Ω. If Ω is the open left half plane then, for the purposes of computing the winding number, its boundary can be taken to be the imaginary axis

$(-\jmath\infty, \jmath\infty)$ and the semi-circle $re^{\jmath\theta}$ where $\theta \in [\pi/2, 3\pi/2]$ and $r \to \infty$. If $p(s)$ is a polynomial of degree d then the contribution to the winding number along the semi-circle is $d/2$. Thus, $p(s)$ has all d zeros in the open left half plane if and only if $p(\jmath\omega) \neq 0$ for all $\omega \in (-\infty, \infty)$ and $\frac{1}{2\pi} \arg\{p(\jmath\omega)\}$ increases by $d/2$ as ω goes from $-\infty$ to ∞. A direct proof of this from first principles appears in [16].

Showing $\frac{1}{2\pi} \arg\{p(\jmath\omega)\}$ increases by $d/2$ is not always easy. An alternative is to investigate the values of ω when $p(\jmath\omega)$ crosses the real axis and the imaginary axis; as long as these values strictly interlace, $p(\jmath\omega)$ must encircle the origin the required number of times. This classical result is now stated formally; see [5] for a proof. The condition $a_{d-1}a_d + b_{d-1}b_d > 0$ ensures $p(\jmath\omega)$ encircles the origin in the correct direction.

Lemma 3 (Hermite-Biehler Theorem) *Let $p(s) = \sum_{i=0}^{d}(a_i + \jmath b_i)s^i$ with $a_i, b_i \in \mathbb{R}$ be a complex polynomial of degree $d \geq 1$. Define the polynomials*

$$u(\omega) = \Re p(\jmath\omega) = a_0 - b_1\omega - a_2\omega^2 + b_3\omega^3 + \cdots, \tag{12}$$

$$v(\omega) = \Im p(\jmath\omega) = b_0 + a_1\omega - b_2\omega^2 - a_3\omega^3 + \cdots. \tag{13}$$

Then $p(s)$ is stable if and only if $a_{d-1}a_d + b_{d-1}b_d > 0$ and the zeros of $u(\omega)$ and $v(\omega)$ are all simple and real and interlace as ω runs from $-\infty$ to ∞.

Corollary 4 is Lemma 3 specialised to real polynomials.

Corollary 4 *Let $p(s)$ be a real non-constant polynomial. Decompose $p(s)$ uniquely as $p(s) = u(s^2) + sv(s^2)$. Then $p(s)$ is Hurwitz if and only if its leading two coefficients are non-zero and have the same sign, and the roots of $u(t)$ and $v(t)$, denoted by x_i and y_i respectively, are simple, real, negative and interlace in the following way:*

$$0 > x_1 > y_1 > x_2 > y_2 > \cdots. \tag{14}$$

Lemma 5 is a simple extension of [16, Property 2]. Lemma 6 follows from Lemma 3.

Lemma 5 *If $p(s)$ is a complex Hurwitz polynomial then $\arg\{p(\jmath\omega)\}$ is smooth and strictly increasing in ω.*

Lemma 6 *If $p(s)$ is a real Hurwitz polynomial of degree at least two then $\Omega = \{\omega > 0 : \Re p(\jmath\omega) = 0\}$ is non-empty and finite.*

The stability of a nested polynomial depends in a straightforward way on the roots of the outer polynomial.

Lemma 7 *If $\phi(z) = \prod_{i=1}^{n}(z - z_i)$ then $\phi(p(s))$ is stable if and only if $p(s) - z_i$ is stable for $i = 1, \cdots, n$.*

PROOF. The lemma, which appears in [13], follows from the factorisation $\phi(p(s)) = \prod_{i=1}^{n}(p(s) - z_i)$. □

2.2 Convex Directions

Let $p_0(s)$ and $\delta(s)$ be polynomials and define S to be the set of $\lambda \in \mathbb{R}$ for which $p_0(s) + \lambda\delta(s)$ is stable. In general, little can be said about S. For certain $\delta(s)$ though, known as convex directions, S is an interval. Two convex direction results are stated. The first corrects an erroneous statement and proof in [5].

Lemma 8 *Let $\delta(s)$ be a real polynomial and let $d \neq 2$ be any positive integer greater than or equal to the degree of $\delta(s)$. The following two statements are equivalent.*

1. *There exists an $\omega_0 \in \mathbb{R}^+$ such that $\delta(\jmath\omega_0) \neq 0$ and*

$$\left.\frac{d \arg\{\delta(\jmath\omega)\}}{d\omega}\right|_{\omega=\omega_0} > \left|\frac{\sin(2\arg\{\delta(\jmath\omega_0)\})}{2\omega_0}\right|. \tag{15}$$

2. *There exists a real polynomial p_0 such that $p_0(s) + \lambda\delta(s)$ has degree d for $\lambda \in [0, 1]$ and $p_0(s) + \lambda\delta(s)$ is Hurwitz for $\lambda \in \{0, 1\}$ but not for all $\lambda \in [0, 1]$.*

Furthermore, if $d \leq 2$ then Statement 2 is false.

Proof. The equivalence of the statements is proved in [19] when $d \geq 4$. It was erroneously extended to $d \geq 1$ in [5]. However, the proof in [5] that Statement 2 implies Statement 1 for $d \geq 1$ is correct, and also follows from the results in [19]. The remaining cases are proved in Appendix A. □

If $\delta(s)$ is such that Statement 2 of Lemma 8 is false then $\delta(s)$ is called a convex direction for real Hurwitz polynomials of degree d. The analogous result for complex Hurwitz polynomials is Lemma 9.

Lemma 9 *Let $\delta(s)$ be a complex polynomial and let $d \geq 1$ be any integer greater than or equal to the degree of $\delta(s)$. The following two statements are equivalent.*

1. *There exists an $\omega_0 \in \mathbb{R}$ such that $\delta(\jmath\omega_0) \neq 0$ and*

$$\left.\frac{d \arg\{\delta(\jmath\omega)\}}{d\omega}\right|_{\omega=\omega_0} > 0. \tag{16}$$

2. *There exists a complex polynomial p_0 such that $p_0(s) + \lambda\delta(s)$ has degree d for $\lambda \in [0, 1]$ and $p_0(s) + \lambda\delta(s)$ is Hurwitz for $\lambda \in \{0, 1\}$ but not for all $\lambda \in [0, 1]$.*

Proof. This is proved in [19] for $d \geq 2$. The case $d = 1$ is proved in Appendix A. □

2.3 Rate of Change of the Phase

A formula for the frequently arising function $\frac{d}{d\omega}\arg\{p(\jmath\omega)\}$ is derived. Note that since $p'(s)$ is the derivative with respect to s then

$$\frac{dp(\jmath\omega)}{d\omega} = \jmath p'(\jmath\omega). \tag{17}$$

By using the identity $\arg p(\jmath\omega) = \Im \log p(\jmath\omega)$ or otherwise, it follows that

$$\frac{d}{d\omega}\arg\{p(\jmath\omega)\}\bigg|_{\omega=\omega_0} = \Im\left\{\frac{\frac{dp(\jmath\omega)}{d\omega}\Big|_{\omega=\omega_0}}{p(\jmath\omega_0)}\right\} \tag{18}$$

$$= \Re\left\{\frac{p'(\jmath\omega_0)}{p(\jmath\omega_0)}\right\}. \tag{19}$$

Lemma 10 gives a lower bound on (19) and follows from the proof of [19, Lemma 11].

Lemma 10 *If $p(s)$ is a real Hurwitz polynomial then*

$$\frac{d}{d\omega}\arg\{p(\jmath\omega)\}\bigg|_{\omega=\omega_0} \geq \left|\frac{\sin\left(2\arg\{p(\jmath\omega_0)\}\right)}{2\omega_0}\right| \tag{20}$$

for all $\omega_0 \in \mathbb{R}^+$. Moreover, equality is obtained if and only if $p(s)$ has degree 1.

2.4 Root Location and Continuity

Lemma 11 is usually called Lucas' theorem [6]. It implies that if $p(s)$ is Hurwitz then so too is $p'(s)$.

Lemma 11 *Let $p(s)$ be a complex polynomial. All the roots of $p'(s)$ are contained in the closed convex hull of the set of roots of $p(s)$.*

Part B of Lemma 12 is the Zero Exclusion Principle [19, Proposition 1] and follows from Part A.

Lemma 12 *Let $p(s;\lambda)$ be a polynomial in s whose coefficients depend continuously on $\lambda \in \mathbb{R}^m$, $m \geq 1$. For $\Lambda \subset \mathbb{R}^m$, define $\mathcal{P} = \{p:\ p(s) = p(s;\lambda),\ \lambda \in \Lambda\}$. Assume every $p \in \mathcal{P}$ has the same degree. A) If Λ is open and $p(z;\bar{\lambda}) = 0$ for some $z \in \mathbb{C}$ and $\bar{\lambda} \in \Lambda$, then for any $\delta > 0$ there exists an $\epsilon > 0$ such that a root of $p(s;\lambda)$ is within δ of z if $|\lambda - \bar{\lambda}| < \epsilon$. B) If Λ is connected and there exist $p_1, p_2 \in \mathcal{P}$ with p_1 Hurwitz but p_2 not, then there exist $p_3 \in \mathcal{P}$ and $\omega \in \mathbb{R}$ such that $p_3(\jmath\omega) = 0$.*

3 A Convex Direction Result for Nested Polynomials

The key technical result is Theorem 13.

Theorem 13 *Let $\delta(s)$ be a real polynomial and $d \geq 1$ any integer greater than or equal to the degree of $\delta(s)$. The following three statements are equivalent.*

1. *There exists an $\omega_0 \in \mathbb{R}^+$ such that $\delta(\jmath\omega_0) \neq 0$ and*

$$\left.\frac{d \arg\{\delta(\jmath\omega)\}}{d\omega}\right|_{\omega=\omega_0} > \begin{cases} \frac{\sin(2\arg\{\delta(\jmath\omega_0)\})}{2\omega_0} & \text{if } \sin(2\arg\{\delta(\jmath\omega_0)\}) \geq 0, \\ \frac{-\sin(2\arg\{\delta(\jmath\omega_0)\})}{4\omega_0} & \text{otherwise.} \end{cases} \tag{21}$$

2. *There exist non-constant real polynomials p_0 and ϕ such that $p_0(s) + \lambda\delta(s)$ has degree d for $\lambda \in [0, 1]$ and $\phi\big(p_0(s) + \lambda\delta(s)\big)$ is Hurwitz for $\lambda \in \{0, 1\}$ but not for all $\lambda \in [0, 1]$.*

3. *There exists a real polynomial p_0 and a non-constant complex polynomial ϕ such that $p_0(s)+\lambda\delta(s)$ has degree d for $\lambda \in [0, 1]$ and $\phi\big(p_0(s) + \lambda\delta(s)\big)$ is Hurwitz for $\lambda \in \{0, 1\}$ but not for all $\lambda \in [0, 1]$.*

In view of the equivalence of Statements 2 and 3 of Theorem 13, it is not necessary to distinguish between ϕ being a real or a complex polynomial. Therefore, a real polynomial $\delta(s)$ for which Statement 1 of Theorem 13 is false is simply called a convex direction for nested polynomials.

The proof of Theorem 13 is broken into several parts. Lemma 15 proves the equivalence of Statements 2 and 3.

Lemma 14 *If $p(s)$ is a real polynomial and $\beta \in \mathbb{R}$ then $p(s) + \jmath\beta$ is Hurwitz if and only if $p(s) - \jmath\beta$ is Hurwitz.*

Proof. Since $p(s)$ has real coefficients, $\overline{p(s)} = p(\bar{s})$. If $p(s) + \jmath\beta = \alpha \prod_{i=1}^{n}(s - \gamma_i)$, where $\alpha \in \mathbb{R}$ and $\gamma_i \in \mathbb{C}$, then $p(s) - \jmath\beta = \overline{p(\bar{s}) + \jmath\beta} = \alpha \prod_{i=1}^{n} \overline{(\bar{s} - \gamma_i)} = \alpha \prod_{i=1}^{n}(s - \bar{\gamma}_i)$, proving the roots of $p(s) - \jmath\beta$ are the complex conjugates of the roots of $p(s) + \jmath\beta$. The lemma follows. □

Lemma 15 *Statements 2 and 3 of Theorem 13 and the following statement are all equivalent.*

4. *There exists a real polynomial p_0 and a $\beta \in \mathbb{R}$ such that $p_0(s) + \lambda\delta(s)$ has degree d for $\lambda \in [0, 1]$ and $p_0(s) + \lambda\delta(s) + \jmath\beta$ is Hurwitz for $\lambda \in \{0, 1\}$ but not for all $\lambda \in [0, 1]$.*

Proof. Statement 2 clearly implies Statement 3. If $\phi(s) = c \prod_{i=1}^{n}(s + \alpha_i + \jmath\beta_i)$, where $c \in \mathbb{C}$ and $\alpha_i, \beta_i \in \mathbb{R}$, then $\phi\big(p(s)\big) c \prod_{i=1}^{n} \big(p(s) + \alpha_i + \jmath\beta_i\big)$ is Hurwitz if and only if $\big(p(s) + \alpha_i\big) + \jmath\beta_i$ is Hurwitz for all i. Thus, if Statement 3 is true, there exists an i such that $\big(p_0(s) + \alpha_i\big) + \lambda\delta(s) + \jmath\beta_i$ is Hurwitz for $\lambda \in \{0, 1\}$ but not for all $\lambda \in [0, 1]$, implying Statement 4. Finally, assume Statement 4 is true and define $\phi(s) = s^2 + \beta^2$. Then $\phi\big(p_0(s) + \lambda\delta(s)\big) = \big(p_0(s) + \lambda\delta(s) + \jmath\beta\big)\big(p_0(s) + \lambda\delta(s) - \jmath\beta\big)$. Lemma 14 implies this $\phi(s)$ satisfies Statement 2. □

Lemma 16 proves Theorem 13 for $d = 1$. If $d \geq 3$, Lemma 17 proves the top condition in (21) implies Statement 4 above. The other cases, being more involved, are proved in Sections 3.1 to 3.4.

Lemma 16 *If $d = 1$ then Statement 1 of Theorem 13 and Statement 4 of Lemma 15 are false.*

PROOF. Either $\delta(s)$ is a constant or $\delta(s) = \gamma(s + \alpha)$. The right side of (21) is never less than zero. If $\delta(s)$ is a constant then the left side of (21) is zero. If $\delta(s) = \gamma(s + \alpha)$, where $\gamma \neq 0$, then the left side of (21) equals $\frac{\alpha}{\alpha^2+\omega_0^2}$. If $\alpha < 0$ then this is negative, while if $\alpha \geq 0$ then the right side of (21) also equals $\frac{\alpha}{\alpha^2+\omega_0^2}$. In all cases, Statement 1 cannot hold. In Statement 4, since $p_0(s) + \lambda\delta(s) = \alpha_1 s + \alpha_0 + \jmath\beta + \lambda(\eta_1 s + \eta_0)$ has constant degree, as λ increases, the sign of $\alpha_1 + \lambda\eta_1$ is constant while the sign of $\alpha_0 + \lambda\eta_0$ can change at most once. Thus, the root of $p_0(s) + \lambda\delta(s)$ can cross the imaginary axis at most once, so Statement 4 is false. □

Lemma 17 *In Theorem 13, if the top inequality in (21) holds and $d \geq 3$ then Statement 4 of Lemma 15 holds.*

PROOF. From Lemma 8, there exists a p_0 satisfying Statement 4 with $\beta = 0$. □

3.1 Local Examples

Let $\mathcal{H}$ denote the set of Hurwitz polynomials of the form $p(s) + \jmath\beta$ where $p(s)$ has real coefficients and $\beta \in \mathbb{R}$. Addition of polynomials endows $\mathcal{H}$ with an affine structure. Statement 4 of Lemma 15 asserts there exists, in the direction $\delta(s)$, a line segment with its endpoints in $\mathcal{H}$ but which is not wholly contained in $\mathcal{H}$. Whether or not this can occur is a global consideration in that it depends on the overall shape of the boundary of $\mathcal{H}$. If the boundary of $\mathcal{H}$ is well behaved though, it can be anticipated the property of being a convex direction is only a local consideration, in a sense explained below. Rantzer expresses this fact in [19, Figure 1], but for a different set of polynomials, by saying convex directions are identical to inner tangent directions.

Consider a non-Hurwitz polynomial $p(s) + \jmath\beta$ lying on the boundary of $\mathcal{H}$, with a simple root at $\jmath\omega_0$ and the others in the open left half plane. Note $p'(\jmath\omega_0) \neq 0$. Let $\delta(s)$ be a polynomial whose degree is less than or equal to that of $p(s)$. By the implicit function theorem, there exists a smooth function $s(\mu)$ satisfying $s(0) = \jmath\omega_0$ and $p\big(s(\mu)\big) + \jmath\beta + \mu\delta\big(s(\mu)\big) = 0$ for $|\mu|$ sufficiently small. If $\delta(\jmath\omega_0) = 0$ then $s(\mu) = \jmath\omega_0$ and is uninteresting. Otherwise,

$$\left.\frac{ds}{d\mu}\right|_{\mu=0} = -\frac{\delta(\jmath\omega_0)}{p'(\jmath\omega_0)}, \tag{22}$$

$$\left.\frac{d^2 s}{d\mu^2}\right|_{\mu=0} = \left(\frac{\delta(\jmath\omega_0)}{p'(\jmath\omega_0)}\right)^2 \left(\frac{2\delta'(\jmath\omega_0)}{\delta(\jmath\omega_0)} - \frac{p''(\jmath\omega_0)}{p'(\jmath\omega_0)}\right). \tag{23}$$

A sufficient condition for there to exist an $\epsilon > 0$ such that $p(s) + \jmath\beta + \mu\delta(s)$ is Hurwitz in the punctured neighbourhood $0 < |\mu| < \epsilon$ of the origin is for the real part of (22) to equal zero and the real part of (23) to be less than zero. In this case, $p(s) + \jmath\beta$ is called a *local example* of $\delta(s)$ not being a convex direction. Note $p_0 = \frac{1}{2\epsilon}p(s) - \frac{1}{2}\delta(s)$ will then satisfy Statement 4. This motivates Lemma 18.

Lemma 18 *For a given* $d \geq 1$*, let* $\delta(s)$ *be a real polynomial of degree at most* d *and let* $\omega_0 \in \mathbb{R}$ *be a scalar satisfying* $\delta(\jmath\omega_0) \neq 0$*. If there exist a real polynomial* $p(s)$ *of degree* d *and a* $\beta \in \mathbb{R}$ *such that i)* $p(s) + \jmath\beta$ *has a simple root at* $\jmath\omega_0$ *and the others in the open left half plane, ii)* $\sin(2\arg\{p'(\jmath\omega_0)\}) = -\sin(2\arg\{\delta(\jmath\omega_0)\})$*, and iii)* $\frac{d}{d\omega}\arg\{p'(\jmath\omega)\}\big|_{\omega=\omega_0} < 2\,\frac{d}{d\omega}\arg\{\delta(\jmath\omega)\}\big|_{\omega=\omega_0}$*, then Statement 4 of Lemma 15 holds.*

Proof. The condition that the real part of (22) equals zero is equivalent to $\arg\{\delta(\jmath\omega_0)\} = \arg\{p'(\jmath\omega_0)\} + \frac{\pi}{2} + n\pi$ for some integer n. This is also equivalent to condition (ii). Since the real part of (22) is zero, the condition that the real part of (23) is less than zero is equivalent to $\Re\left\{\frac{2\delta'(\jmath\omega_0)}{\delta(\jmath\omega_0)} - \frac{p''(\jmath\omega_0)}{p'(\jmath\omega_0)}\right\} > 0$. Using (19), this is equivalent to condition (iii). Therefore, conditions (i) to (iii) imply $p(s) + \jmath\beta$ is a local example of $\delta(s)$ not being a convex direction and Statement 4 holds. □

These ideas in reverse lead to the following lower bound on $\frac{d}{d\omega}\arg\{\delta(\jmath\omega)\}$.

Lemma 19 *If the real polynomials* $p(s), \delta(s)$ *and the scalars* $\gamma, \omega_0 \in \mathbb{R}$ *satisfy i)* $\delta(\jmath\omega_0) \neq 0$*, ii)* $p(s) + \jmath\gamma$ *has a simple root at* $\jmath\omega_0$*, and iii) there exists an* $\epsilon > 0$ *such that the roots of* $p(s)+\jmath\gamma+\mu\delta(s)$ *are in the closed left half plane if* $|\mu| < \epsilon$*, then* $\sin(2\arg\{p'(\jmath\omega_0)\}) = -\sin(2\arg\{\delta(\jmath\omega_0)\})$ *and* $\frac{d}{d\omega}\arg\{\delta(\jmath\omega)\}\big|_{\omega=\omega_0} \geq \frac{1}{2}\,\frac{d}{d\omega}\arg\{p'(\jmath\omega)\}\big|_{\omega=\omega_0}$.

Proof. The conditions imply the real part of (22) is zero and the real part of (23) is less than or equal to zero. An argument analogous to the proof of Lemma 18 now applies. □

3.2 The Forward Implication

Lemma 18 is used to prove the bottom condition in (21) implies Statement 4 of Lemma 15. This is accomplished by choosing $p(s)$ to make $\frac{d}{d\omega}\arg\{p'(\jmath\omega)\}$ as small as possible while enforcing conditions (i) and (ii) in Lemma 18. Because condition (i) implies $p'(\jmath\omega_0) \neq 0$, it follows from Lemma 11 that $p'(s)$ is Hurwitz, hence Lemma 10 lower bounds $\frac{d}{d\omega}\arg\{p'(\jmath\omega)\}$. Lemma 20 proves this bound is tight under certain conditions. Its proof is deferred.

Lemma 20 *For any* $\theta_0 \in (0, \pi/2)$*,* $\omega_0 \in \mathbb{R}^+$ *and* $d \geq 3$ *there exist a sequence of real polynomials* $p_k(s)$ *of degree* d *and scalars* $\beta_k \in \mathbb{R}$ *satisfying i)* $p_k(s) + \jmath\beta_k$ *has a simple root at* $\jmath\omega_0$ *and all others in the open left half plane, ii)* $\arg\{p_k'(\jmath\omega_0)\} = \theta_0$*, and iii)* $\lim_{k\to\infty}\frac{d}{d\omega}\arg\{p_k'(\jmath\omega)\}\big|_{\omega=\omega_0} = \frac{\sin(2\theta_0)}{2\omega_0}$.

Lemma 21 *In Theorem 13, assume there exists an* $\omega_0 \in \mathbb{R}^+$ *such that* $\delta(\jmath\omega_0) \neq 0$ *and*

$$\left.\frac{d\arg\{\delta(\jmath\omega)\}}{d\omega}\right|_{\omega=\omega_0} > \frac{-\sin\left(2\arg\{\delta(\jmath\omega_0)\}\right)}{4\omega_0} > 0. \tag{24}$$

If $d \geq 3$ *then Statement 4 of Lemma 15 holds.*

PROOF. Define $\phi_0 = \arg\{\delta(\jmath\omega_0)\}$. Since (24) implies $\sin(2\phi_0) < 0$, there exists a $\theta_0 \in (0, \pi/2)$ such that $\sin(2\theta_0) = -\sin(2\phi_0)$. Therefore, from (24), $2\,\frac{d}{d\omega}\arg\{\delta(\jmath\omega)\}\big|_{\omega=\omega_0} > \frac{\sin(2\theta_0)}{2\omega_0}$. Thus, from Lemma 20, there exists a $p(s) + \jmath\beta$ satisfying all three conditions in Lemma 18. □

Lemma 20 is proved by starting with a sequence of polynomials $r_k(s)$ satisfying (ii) and (iii) with r_k replacing p_k, then constructing sequences $\beta_k, c_k \in \mathbb{R}$ so $p_k(s)r_k(s) + c_k$ and β_k satisfy (i). To make this viable, r_k is chosen so $|r_k(\jmath\omega)|$ increases for $\omega \geq 0$. Lemmas 22 and 23 prove the required existence results.

Lemma 22 *Let $p(s)$ be a real Hurwitz polynomial of degree at least two. Define $\Omega = \{\omega > 0 : \Re p(\jmath\omega) = 0\}$, $\omega_0 = \min\Omega$ and $\beta = -\Im p(\jmath\omega_0)$. If there exists a $c \in \mathbb{R}$ such that $|p(\jmath\omega) - c|$ strictly increases on the interval $[0, \infty)$, then $p(s) + \jmath\beta$ has a simple root at $\jmath\omega_0$ and all other roots in the open left half plane.*

PROOF. Lemma 6 ensures ω_0 is well defined. Since $|p(\jmath\omega) - c|$ is strictly increasing, and $\Im\{p(\jmath\omega) - c\} = \Im p(\jmath\omega)$, if $\omega \in \Omega$ and $\omega > \omega_0$ then

$$|\Im p(\jmath\omega)| = |\Im p(-\jmath\omega)| > |\beta|. \tag{25}$$

Note $\beta \neq 0$ because $p(s)$ is Hurwitz. It is claimed the only solution of $p(\jmath\omega) + \lambda\jmath\beta = 0$ with $\lambda \in [0, 1]$ and $\omega \in \mathbb{R}$ is $\omega = \omega_0$ and $\lambda = 1$. Indeed, any solution must satisfy $|\omega| \in \Omega$ and $\Im p(\jmath\omega) = -\lambda\beta$. If $\omega = -\omega_0$ then, since $\beta \neq 0$, λ must equal -1. If $|\omega| > \omega_0$ then, from (25), $|\lambda|$ must exceed unity. This proves the claim. By Lemma 12, as λ increases from 0 to 1, the roots of $p(s) + \lambda\jmath\beta$ remain in the open left half plane except for a root at $\jmath\omega_0$ appearing when $\lambda = 1$. This root at $\jmath\omega_0$ is simple because the roots of $p'(s)$ are in the open left half plane by Lemma 11. □

Lemma 23 *Let $q(s) = c(s+\alpha)^n(s+\gamma)$ be a real Hurwitz polynomial with $n \geq 1$ and $c \neq 0$. Let ω_* be the smallest $\omega_* > 0$ satisfying $\Re q(\jmath\omega_*) = 0$. Then, for any $\omega_0 \in (0, \omega_*)$, there exist a real Hurwitz polynomial $p(s)$ and a $\beta \in \mathbb{R}$ such that $p'(s) = q(s)$ and the roots of $p(s) + \jmath\beta$ are in the open left half plane except for a simple root at $\jmath\omega_0$.*

PROOF. Lemma 6 ensures ω_* is finite. Without loss of generality, it is henceforth assumed $c = n + 2$. Define $r(s)(s+\alpha)^{n+1}(s - \frac{1}{n+1}\alpha + \frac{n+2}{n+1}\gamma)$, so $r'(s) = q(s)$. Note $\alpha > 0$ and $\gamma > 0$ because $q(s)$ is Hurwitz. Since $p(s)$ can differ from $r(s)$ by at most an additive real constant, and since $p(\jmath\omega_0) + \jmath\beta 0$, the only candidates for $p(s)$ and β are $p(s) = r(s) - \Re r(\jmath\omega_0)$ and $\beta = -\Im p(\jmath\omega_0)$. Define the polynomials u and v so that $r(s) = u(s^2) + sv(s^2)$. Note

$$u(-\omega^2) = \Re r(\jmath\omega), \qquad \omega v(-\omega^2) = \Im r(\jmath\omega). \tag{26}$$

Therefore, $p(s) = \bar{u}(s^2) + sv(s^2)$ where $\bar{u}(t) = u(t) - u(t_0)$ and $t_0 = -\omega_0^2$. Corollary 4 is ultimately used to prove $p(s)$ is Hurwitz. First, properties of $u(t)$ and $v(t)$ on the interval $[t_*, 0]$ are determined, where $t_* = -\omega_*^2$. Since

$\arg\{q(0)\} = 0$, Lemma 5 implies $\arg\{q(\jmath\omega)\}$ monotonically increases from 0 to $\pi/2$ as ω increases from 0 to ω_*. Therefore, using (17),

$$\arg\left\{\frac{dr(\jmath\omega)}{d\omega}\right\} = \arg\{\jmath r'(\jmath\omega)\} \tag{27}$$

$$= \arg\{q(\jmath\omega)\} + \pi/2 \tag{28}$$

monotonically increases from $\pi/2$ to π. Thus, $\Im r(\jmath\omega)$ monotonically increases from zero and $\Re r(\jmath\omega)$ monotonically decreases. It follows from (26) that on the interval $[t_*, 0]$, $u(t)$ is strictly increasing and, since it is proved shortly that $v(0) > 0$, $v(t)$ is non-zero.

First assume n is odd. Set $m = (n+1)/2$. Define u_1 and v_1 so $(s+\alpha)^{n+1} = u_1(s^2) + sv_1(s^2)$. Note $u_1(t) = t^m + \cdots + \alpha^{n+1}$ and $v_1(t) = (n+1)\alpha t^{m-1} + \cdots + (n+1)\alpha^n$. Since $(s+\alpha)^{n+1}$ is Hurwitz, Corollary 4 implies the roots of $u_1(t)$ and $v_1(t)$, denoted by μ_i and ν_i respectively, are real, negative and interlace:

$$\mu_m < \nu_{m-1} < \mu_{m-1} < \cdots < \nu_1 < \mu_1 < 0. \tag{29}$$

Set $\xi = -\frac{1}{n+1}\alpha + \frac{n+2}{n+1}\gamma$. Information about the roots of $u(t)$ and $v(t)$ is gained through the relations

$$u(t) = \xi u_1(t) + tv_1(t), \qquad v(t) = u_1(t) + \xi v_1(t). \tag{30}$$

Consider $v(t)$ first. It is a degree m polynomial with leading term t^m. Let $y_1, \cdots, y_m$ denote its roots. At the origin, $v(0)u_1(0) + \xi v_1(0) = (n+2)\alpha^n\gamma > 0$. Moreover, $v(\nu_i)u_1(\nu_i)$, and since $u_1(0) = \alpha^{n+1} > 0$ and the roots of u_1 and v_1 interlace, $v(\nu_i) < 0$ if i is odd and $v(\nu_i) > 0$ if i is even. Therefore, precisely one simple root of $v(t)$ is located in each of the m intervals $(-\infty, \nu_{m-1})$, (ν_{m-1}, ν_{m-2}), $\cdots$, $(\nu_1, 0)$. Order the y_i so $y_i \in (\nu_i, \nu_{i-1})$, where $\nu_0 = 0$ and $\nu_m = -\infty$. Note $y_i < t_* < t_0 < 0$ because $v(t)$ is non-zero for $t \in [t_*, 0]$.

Now consider $u(t)$. It is a degree m polynomial with leading term $(\xi + (n+1)\alpha)t^m$. Note $\xi + (n+1)\alpha > 0$. From (30), $u(t) = \xi v(t) + (t - \xi^2)v_1(t)$. Thus, for $t < 0$, the signs of $u(y_i)$ and $v_1(y_i)$ are opposite. Since $\nu_i < y_i < \nu_{i-1}$ and $v_1(0)(n+1)\alpha^n > 0$, it follows that $u(y_i) > 0$ if i is even and $u(y_i) < 0$ if i is odd.

Finally, consider $\bar{u}(t)$. Since $u(t)$ is strictly increasing on $[t_*, 0]$, $\bar{u}(0) = u(0) - u(t_0) > 0$. The roots of $r(s)$ lie on the real axis, hence $|r(\jmath\omega)|$ strictly increases on the interval $\omega \in [0, \infty)$. Because $y_i < t_0$, $\sqrt{-y_i} > \omega_0$ and $|u(t_0)| \leq |r(\jmath\omega_0)| < |r(\jmath\sqrt{-y_i})| = |u(y_i)|$. Thus, $\bar{u}(y_i)$ has the same sign as $u(y_i)$, that is, $\bar{u}(y_i) > 0$ if i is even and $\bar{u}(y_i) < 0$ if i is odd. Since $\bar{u}(t)$ has degree m and $\bar{u}(0) > 0$, its roots $x_1, \cdots, x_m$ satisfy $x_i \in (y_i, y_{i-1})$ where $y_0 = 0$. That is, the roots of $\bar{u}(t)$ are simple, real, negative and interlace with the roots of $v(t)$ according to (14). Direct expansion of $r(s)$ shows its leading coefficients are 1 and $(n+1)\alpha + \xi > 0$, so $p(s)$ is Hurwitz by Corollary 4.

Define $\Omega = \{\omega > 0 : \Re p(\jmath\omega) = 0\}$. Then $\Omega = \{\sqrt{-x_1}, \cdots, \sqrt{-x_m}\}$. Note $x_1 = t_0$ because $y_1 < t_0 < 0$ and $\bar{u}(t_0) = 0$. Therefore, $\omega_0 \min\Omega$ and Lemma 22 completes the proof, for n odd.

If n is even, set $m = n/2$. Only minor changes to the above proof are required. Although the leading terms of $u_1(t)$, $v_1(t)$, $u(t)$ and $v(t)$ change, the required inequalities still hold. The m roots of $v(t)$ still satisfy $y_i \in (\nu_i, \nu_{i-1})$ where $\nu_0 = 0$. Note ν_m is finite because v_1 has degree m. The $m+1$ roots of $\bar{u}(t)$ still satisfy $x_i \in (y_i, y_{i-1})$ where $y_0 = 0$ and $y_{m+1} = -\infty$. This proves $p(s)$ is Hurwitz and Lemma 22 completes the proof, for n even. □

Proof of Lemma 20. For $k = 1, 2, \cdots$, define $q_k(s) = \left(\tan(\theta_0)s + \omega_0 + \frac{1}{k}\right)(\xi_k s + 1)^{d-2}$ where $\xi_k > 0$ is chosen so that $\arg\{q_k(\jmath\omega_0)\}\,\theta_0$. Since $\xi_k \to 0$, $\lim_{k\to\infty} \frac{d}{d\omega} \arg\{q_k(\jmath\omega)\}\big|_{\omega=\omega_0} = \frac{\sin(2\theta_0)}{2\omega_0}$. Thus, by Lemma 23, there exist p_k and β_k with all three properties. □

3.3 The Reverse Implication

Lemma 19 is used to prove Statement 4 of Lemma 15 implies Statement 1 of Theorem 13. First, Lemma 24 proves Statement 4 implies the existence of a $p(s) + \jmath\gamma$ satisfying the conditions in Lemma 19. Importantly, $p(s)$ is chosen to be Hurwitz, allowing bounds on $\frac{d}{d\omega} \arg\{p'(\jmath\omega)\}$ to be derived. Proofs are deferred.

Lemma 24 *For a given $\delta(s)$ and $d \geq 1$, assume $p_0(s)$ and β satisfy Statement 4 of Lemma 15. If $\delta(s)$ is a convex direction for real Hurwitz polynomials of degree d (see Lemma 8) then there exist $\omega_0 \in \mathbb{R}^+$, $\gamma \in \mathbb{R}$ and a real Hurwitz polynomial $p(s)$ of degree d satisfying conditions (i) to (iii) of Lemma 19.*

Lemma 25 *Let $p(s)$ be a real Hurwitz polynomial of degree at least two. Define $\theta(\omega) = \arg\{p'(\jmath\omega)\}$ and $\Omega = \{\omega > 0 : \Re p(\jmath\omega) = 0\}$. Then $\theta(\omega)$ is well defined and smooth. Furthermore, if $\omega \in \Omega$ then*

$$\frac{d\theta}{d\omega} > \frac{-\sin(2\theta)}{\omega}. \tag{31}$$

Lemma 26 *If $d \geq 3$ then Statement 4 of Lemma 15 implies Statement 1 of Theorem 13.*

Proof. Assume Statement 4 holds. If $\delta(s)$ is not a convex direction for real Hurwitz polynomials of degree d then Lemma 8 implies Statement 1 of Theorem 13 holds. Otherwise, apply Lemma 24 to obtain $p(s)$, γ and ω_0. Since $p(s)$ is Hurwitz, so too is $p'(s)$ by Lemma 11, and Lemma 10 implies $\frac{d}{d\omega} \arg\{p'(\jmath\omega)\}\big|_{\omega=\omega_0} > \frac{\sin(2\arg\{p'(\jmath\omega_0)\})}{2\omega_0}$. Furthermore, since $p(\jmath\omega_0) + \jmath\gamma = 0$, Lemma 25 is applicable, and $\frac{d}{d\omega} \arg\{p'(\jmath\omega)\}\big|_{\omega=\omega_0} > \frac{-\sin(2\arg\{p'(\jmath\omega_0)\})}{\omega_0}$ too. Therefore, Lemma 19 implies Statement 1. □

Lemma 27 *If $p(s)$ is a non-constant real polynomial then there exists a $\bar{\beta} \in [0, \infty]$ such that $p(s) + \jmath\beta$ with $\beta \in \mathbb{R}$ is Hurwitz if and only if $|\beta| < \bar{\beta}$. Moreover, if $0 < \bar{\beta} < \infty$ then both $p(s) + \jmath\bar{\beta}$ and $p(s) - \jmath\bar{\beta}$ have at least one root on the imaginary axis and all their roots in the closed left half plane.*

Proof. Lemma 9 implies $\delta(s) = \jmath\beta$ is a convex direction, so the set of values of $\beta \in \mathbb{R}$ for which $p(s) + \jmath\beta$ is Hurwitz is a possibly empty interval. Lemma 14 implies the interval is symmetric about the origin. Part A of Lemma 12 implies the interval is open and Parts A and B complete the proof. □

Proof of Lemma 24. Since $p_0(s) + \jmath\beta$ and $p_0(s) + \jmath\beta + \delta(s)$ are Hurwitz, so too are $p_0(s)$ and $p_0(s) + \delta(s)$ by Lemma 27. This implies $p_0(s) + \lambda\delta(s)$ is Hurwitz for any $\lambda \in [0, 1]$ because $\delta(s)$ is a convex direction for real Hurwitz polynomials of degree d. Referring to Lemma 27, define $f(\lambda)$ so that $p_0(s) + \lambda\delta(s) + \jmath\gamma$ with $\gamma \in \mathbb{R}$ is Hurwitz if and only if $|\gamma| < f(\lambda)$. Define $\gamma = \inf_{\lambda\in[0,1]} f(\lambda)$. Clearly, $0 < \gamma \leq \beta$. Note Lemma 27 implies the roots of both $p_0(s) + \jmath\gamma + \lambda\delta(s)$ and $p_0(s) - \jmath\gamma + \lambda\delta(s)$ are in the closed left half plane for any $\lambda \in [0, 1]$.

Since $[0, 1]$ is compact, there exists a $\lambda_* \in [0, 1]$ such that $f(\lambda_*) = \gamma$. In fact, $\lambda_* \in (0, 1)$ because $f(0) > \beta$ and $f(1) > \beta$. Define $p(s) = p_0(s) + \lambda_*\delta(s)$ and note it is Hurwitz. By Lemma 27, there exists an $\omega_0 \in \mathbb{R}$ such that $p(\jmath\omega_0) + \jmath\gamma = 0$. Note $\omega_0 \neq 0$ because $p(0) \in \mathbb{R}$ and $\gamma \neq 0$. Since $p(-\jmath\omega_0) - \jmath\gamma = 0$, if $\omega_0 < 0$ then replace ω_0 with $-\omega_0$ and γ with $-\gamma$ so that $\omega_0 > 0$. It is shown conditions (i) to (iii) of Lemma 19 are satisfied. If (i) does not hold then $p_0(\jmath\omega_0) + \jmath\gamma = p(\jmath\omega_0) + \jmath\gamma 0$, implying $p_0(s) + \jmath\gamma$ is not Hurwitz. This contradicts $f(0) > \beta$. Since $p(s)$ is Hurwitz, so too is $p'(s)$ by Lemma 11, hence $p'(\jmath\omega_0) \neq 0$ and (ii) holds. Finally, (iii) holds with $\epsilon = (1/2)\min\{\lambda_*, 1 - \lambda_*\}$. □

Proof of Lemma 25. Lemma 11 implies $p'(s)$ is Hurwitz, hence $\theta(\omega)$ is well defined since $p'(\jmath\omega)$ cannot be zero. Lemma 5 ensures it is smooth. Define the polynomials $u(t)$ and $v(t)$ so that $p(s)u(s^2) + sv(s^2)$. Then $p'(s) = \tilde{u}(s^2) + s\tilde{v}(s^2)$ where $\tilde{u}(t) = v(t) + 2t\dot{v}(t)$, $\tilde{v}(t) = 2\dot{u}(t)$ and the dot denotes differentiation with respect to t. Since $p'(s)$ is Hurwitz, Corollary 4 implies $\tilde{u}(t)$ is not the zero polynomial. Therefore, $X(\omega) = \omega\tilde{v}(-\omega^2)/\tilde{u}(-\omega^2)$ exists and is smooth at all but a finite number of ω. For $\omega > 0$ such that $X(\omega)$ exists,

$$\frac{d\theta}{d\omega} = \frac{d\arctan X}{d\omega} \tag{32}$$

$$= \frac{1}{1 + X^2}\frac{dX}{d\omega}. \tag{33}$$

Therefore, (31) holds if and only if

$$\frac{dX}{d\omega} > (1 + X^2)\frac{-\sin(2\theta)}{\omega} \tag{34}$$

$$= \frac{-2X}{\omega}, \tag{35}$$

or equivalently, if and only if

$$\tilde{u}(-\omega^2)\tilde{v}(-\omega^2) + 2\omega^2\left[\dot{\tilde{u}}(-\omega^2)\tilde{v}(-\omega^2) - \tilde{u}(-\omega^2)\dot{\tilde{v}}(-\omega^2)\right] > -2\tilde{u}(-\omega^2)\tilde{v}(-\omega^2). \tag{36}$$

Define $t = -\omega^2$. Substituting for $\tilde{u}$ and $\tilde{v}$ yields

$$[3\dot{u}(t) + 2t\ddot{u}(t)]\, v(t) + 4t^2 \left[\ddot{u}(t)\dot{v}(t) - \dot{u}(t)\ddot{v}(t)\right] > 0. \tag{37}$$

Although the derivation assumed ω is such that $X(\omega)$ exists, the continuity of all relevant terms implies (31) holds at a point $\omega > 0$ if and only if (37) holds at the point $t = -\omega^2$. Note that if $\tilde{u}(t)\tilde{v}(t) > 0$ then (31) holds trivially because X, and hence $\sin(2\theta)$, are positive, while Lemma 10 implies the left side of (31) is non-negative. Since $\omega \in \Omega$ implies $u(-\omega^2) = 0$, it suffices to prove (37) holds for all t such that $u(t) = 0$ and $\tilde{u}(t)\tilde{v}(t) \leq 0$.

Since (37) is linear in both $u(t)$ and $v(t)$, without loss of generality $u(t)$ and $v(t)$ are assumed to be of the form $u(t) = \prod_{i=1}^{n}(t + \mu_i)$ and $v(t) = \prod_{i=1}^{m}(t + \nu_i)$. Corollary 4 implies $0 < \mu_1 < \nu_1 < \cdots$ and either $m = n - 1$ or $m = n$. If $n = 1$ and $m = 0$, so that $u(t) = t + \mu_1$ and $v(t) = 1$, direct substitution proves (37) holds for all t.

Induction is now used. First assume $n = m$. Define $v_k(t) = \prod_{i=1}^{k}(t + \nu_i)\prod_{i=k+2}^{n}(t + \mu_i)$ so that

$$v(t) = u(t) + \sum_{k=0}^{n-1}(\nu_{k+1} - \mu_{k+1})v_k(t). \tag{38}$$

If t satisfies $u(t) = 0$ and $v(t)$ in (37) is replaced by $u(t)$ then the left side of (37) is zero. By the induction hypothesis, if $v_{n-1}(t)$ replaces $v(t)$ then (37) holds. By the induction hypothesis and continuity, if $v_k(t)$ replaces $v(t)$ then (37) holds but with possible equality. Since the coefficients $\nu_{k+1} - \mu_{k+1}$ are positive and (37) is linear in $v(t)$, it follows that (37) holds, as required.

Now assume $m = n - 1$. Define $\bar{u}(t)$ so that $u(t)(t + \mu_n)\bar{u}(t)$. Note $\tilde{u} = v + 2t\dot{v}$, $\tilde{v}2\bar{u} + 2(t + \mu_n)\dot{\bar{u}}$ and

$$[3\dot{u} + 2t\ddot{u}]\, v + 4t^2 \left[\ddot{u}\dot{v} - \dot{u}\ddot{v}\right] = (t + \mu_n)\left\{[3\dot{\bar{u}} + 2t\ddot{\bar{u}}]\, v + 4t^2 \left[\ddot{\bar{u}}\dot{v} - \dot{\bar{u}}\ddot{v}\right]\right\} + \left\{\bar{u}\left(3v - 4t^2\ddot{v}\right) + 4t\dot{\bar{u}}\,(v + 2t\dot{v})\right\} \tag{39}$$

where the argument t is dropped. If t satisfies $\bar{u}(t) = 0$ and $\tilde{u}(t)\tilde{v}(t) \leq 0$, the first term in braces is positive by the induction hypothesis while the second term is $4t\dot{\bar{u}}\,(v + 2t\dot{v}) = \frac{2t}{t+\mu_n}\tilde{u}\tilde{v} \geq 0$ because $-\mu_n < t \leq -\mu_1 < 0$. Therefore, (37) holds. If $t = -\mu_n$ then $\tilde{u}\tilde{v}2\bar{u}\,(v - 2\mu_n\dot{v})$. It is clear from their definitions that $\bar{u}(-\mu_n)$ and $v(-\mu_n)$ are non-zero and have the same sign, and $\dot{v}(-\mu_n)$ has the opposite sign, so $\tilde{u}(t)\tilde{v}(t) > 0$, completing the proof. $\square$

3.4 The Degree Two Case

This section completes the proof of Theorem 13.

Lemma 28 *If $d = 2$ then Statement 4 of Lemma 15 implies Statement 1 of Theorem 13.*

Proof. If $\delta(s)$ satisfies Statement 1 then so does $-\delta(s)$, and similarly for Statement 4. Therefore, without loss of generality, assume $\delta(s) = \gamma_2 s^2 + \gamma_1 s + \gamma_0$ with $\gamma_2 \geq 0$. Note

$$\left.\frac{d \arg\{\delta(\jmath\omega)\}}{d\omega}\right|_{\omega=\omega_0} = \frac{\gamma_1(\gamma_0 + \gamma_2\omega^2)}{|\delta(\jmath\omega_0)|^2}, \tag{40}$$

$$\frac{\sin(2\arg\{\delta(\jmath\omega_0)\})}{2\omega_0} = \frac{\gamma_1(\gamma_0 - \gamma_2\omega^2)}{|\delta(\jmath\omega_0)|^2}. \tag{41}$$

It follows that if $\delta(s)$ does not satisfy Statement 1 then $\gamma_1 \leq 0$. If $\gamma_1 = 0$, or $\gamma_1 < 0$ and $\gamma_0 \geq 0$, then $\frac{d\arg\{\delta(\jmath\omega)\}}{d\omega} \leq 0$, so by Lemma 9, Statement 4 is false. Only the case $\gamma_0 < 0$ and $\gamma_1 < 0$ remains. If $p_0(s) = \alpha_2 s^2 + \alpha_1 s + \alpha_0$ then by Lemma 3, $p_0(s) + \jmath\beta + \lambda\delta(s)$ is Hurwitz if and only if: i) $(\alpha_1 + \lambda\gamma_1)(\alpha_2 + \lambda\gamma_2) > 0$, ii) $(\alpha_0 + \lambda\gamma_0)(\alpha_2 + \lambda\gamma_2) > 0$, and iii) $\beta^2 < \frac{\alpha_0 + \lambda\gamma_0}{\alpha_2 + \lambda\gamma_2}(\alpha_1 + \lambda\gamma_1)^2$. Assume to the contrary that Statement 4 holds. The degree constraint implies $\alpha_2 + \lambda\gamma_2$ is non-zero and does not change sign on the interval $\lambda \in [0, 1]$. Therefore, since (i) and (ii) hold for $\lambda \in \{0, 1\}$, they hold for all $\lambda \in [0, 1]$. Assume first $\alpha_2 > 0$. Then, $\alpha_0 > 0$ and $\alpha_1 > 0$ because (i) and (ii) hold for $\lambda = 0$. Since $\gamma_0 < 0$ and $\gamma_1 < 0$, the right side of (iii) is non-increasing on the interval $\lambda \in [0, 1]$. Since (iii) holds when $\lambda = 1$, this implies it holds for all $\lambda \in [0, 1]$, the required contradiction. If $\alpha_2 < 0$ then it can be shown analogously the right side of (iii) is non-decreasing, implying Statement 4 cannot hold. □

Lemma 29 *If $d = 2$ then Statement 1 of Theorem 13 implies Statement 4 of Lemma 15.*

Proof. As in the proof of Lemma 28, assume without loss of generality that $\delta(s) = \gamma_2 s^2 + \gamma_1 s + \gamma_0$ where $\gamma_2 \geq 0$. One way Statement 1 can hold is if there exists an $\omega_0 \in \mathbb{R}^+$ such that $\delta(\jmath\omega_0) \neq 0$ and $\left.\frac{d\arg\{\delta(\jmath\omega)\}}{d\omega}\right|_{\omega=\omega_0} > \frac{\sin(2\arg\{\delta(\jmath\omega_0)\})}{2\omega_0} \geq 0$. This is only possible if $\delta(s)$ is not the zero polynomial and, from (40) and (41), if $\gamma_0 > 0$, $\gamma_1 > 0$ and $\gamma_2 > 0$. Therefore, there exists an $\omega_0 > 0$ such that $\delta(\jmath\omega_0) \neq 0$ and $\gamma_0 < \gamma_2\omega_0^2$. It follows from (40) and (41) that this ω_0 satisfies $\left.\frac{d\arg\{\delta(\jmath\omega)\}}{d\omega}\right|_{\omega=\omega_0} > \frac{-\sin(2\arg\{\delta(\jmath\omega_0)\})}{4\omega_0} > 0$.

Therefore, if Statement 1 holds, there exists an $\omega_0 \in \mathbb{R}^+$ such that $\delta(\jmath\omega_0) \neq 0$ and $\left.\frac{d\arg\{\delta(\jmath\omega)\}}{d\omega}\right|_{\omega=\omega_0} > \frac{-\sin(2\phi)}{4\omega_0} > 0$, where $\phi_0 = \arg\{\delta(\jmath\omega_0)\}$. Let $\theta_0 \in (0, \pi/2)$ be such that $\sin(2\theta_0) = -\sin(2\phi_0)$. Define $\gamma = \frac{\omega_0}{\tan\theta_0}$, $p(s) = s^2 + 2\gamma s + \omega_0^2$ and $\beta = -2\gamma\omega_0$. It is claimed $p(s) + \jmath\beta$ satisfies conditions (i) to (iii) in Lemma 18, thus implying Statement 4. Indeed, (ii) holds because $\arg\{p'(\jmath\omega_0)\} = \theta_0$ and (iii) follows from $\left.\frac{d}{d\omega}\arg\{p'(\jmath\omega)\}\right|_{\omega=\omega_0} = \frac{\sin(2\theta_0)}{2\omega_0}$. Since $p(s)$ is Hurwitz and $r(s) = s^2 + 2\gamma s + \gamma^2$ is such that $|r(\jmath\omega)|$ strictly increases for $\omega \geq 0$, Lemma 22 implies (i) holds, proving the claim. □

Proof of Theorem 13. From Lemma 15, it suffices to prove Statement 1 is equivalent to Statement 4. If $d = 1$ then apply Lemma 16. If $d = 2$ then apply Lemmas 28 and 29. For $d \geq 3$, apply Lemmas 17, 21 and 26. □

4 Extreme Point Results

Before proving Corollary 1, it is remarked (2) is invariant to the ordering of p_0 and p_1, as it should be. Indeed, if $\tilde{\theta}(\omega) \arg\{p_0(\jmath\omega) - p_1(\jmath\omega)\}$ then $\tilde{\theta} = \theta + \pi$ and hence $\frac{d\tilde{\theta}}{d\omega} \frac{d\theta}{d\omega}$ and $\sin(2\tilde{\theta}) = \sin(2\theta)$.

Proof of Corollary 1. Define $\delta(s) = p_1(s) - p_0(s)$ and apply Theorem 13. □

Although Theorem 13 proves (2) is also a necessary condition for $\delta(s) = p_1(s) - p_0(s)$ to be a convex direction, (2) is not a necessary condition in the context of Corollary 1. For example, it can be shown that if $p_0(s) = s^3 + 2s^2 + 4s + 3$ and $p_1(s) = s^3 + 3s^2 + 6s + 4$ then, using the notation of Corollary 1, for any real polynomial ϕ, the family $\mathcal{N}$ is robustly stable if and only if both $\phi(p_0)$ and $\phi(p_1)$ are stable. However, $\theta(\omega) = \arg\{p_1(\jmath\omega) - p_0(\jmath\omega)\}$ does not satisfy condition (2).

The proof of Theorem 2 requires the following definition. If $\mathcal{P}$ is a family of polynomials then the value set $\mathcal{P}(\jmath\omega)$ is defined to be $\mathcal{P}(\jmath\omega) = \{p(\jmath\omega) : \ p \in \mathcal{P}\} \subset \mathbb{C}$.

Proof of Theorem 2. First, a set $\mathcal{P}_e$ satisfying i) $\mathcal{P}_e \subset \mathcal{P}$, and ii) for all $\omega \in \mathbb{R}$, the value set $\mathcal{P}_e(\jmath\omega)$ contains the boundary of the value set $\mathcal{P}(\jmath\omega)$, is constructed. By Lemma 12 and the fact $\xi \in \mathbb{R}$ merely shifts the value set of $\mathcal{P} - \xi$, it follows $\mathcal{P} - \xi$ is robustly stable if and only if $\mathcal{P}_e - \xi$ is. For any polynomial ϕ, this and Lemma 7 imply $\phi(\mathcal{P})$ is robustly stable if and only if $\phi(\mathcal{P}_e)$ is.

It is a standard result [9, 16] that the value set $A_i(\jmath\omega)$ is a rectangle with edges

$$E_i^1 = \left\{r : \ r(s) = \lambda K_i^1(s) + (1-\lambda) K_i^2(s), \ \lambda \in [0,1]\right\}, \tag{42}$$

$$E_i^2 = \left\{r : \ r(s) = \lambda K_i^2(s) + (1-\lambda) K_i^4(s), \ \lambda \in [0,1]\right\}, \tag{43}$$

$$E_i^3 = \left\{r : \ r(s) = \lambda K_i^4(s) + (1-\lambda) K_i^3(s), \ \lambda \in [0,1]\right\}, \tag{44}$$

$$E_i^4 = \left\{r : \ r(s) = \lambda K_i^3(s) + (1-\lambda) K_i^1(s), \ \lambda \in [0,1]\right\}. \tag{45}$$

Define $E_i = E_i^1 \cup E_i^2 \cup E_i^3 \cup E_i^4$. For any $\alpha \in \mathbb{C}$, αE_i is also a rectangle, hence

$$\mathcal{P}_e = \{p : \ p(s) = q_1(s) b_1(s) + \cdots + q_n(s) b_n(s), \ b_i(s) \in E_i, \ i = 1, \cdots, n\} \tag{46}$$

satisfies properties (i) and (ii) above. Thus, $\phi(\mathcal{P})$ is robustly stable if and only if $\phi(\mathcal{P}_e)$ is.

Define $\delta_i^1(s) = q_i(s)\left[K_i^1(s) - K_i^2(s)\right]$, $\delta_i^2(s) = q_i(s)\left[K_i^2(s) - K_i^4(s)\right]$, $\delta_i^3(s) = q_i(s)\left[K_i^4(s) - K_i^3(s)\right]$ and $\delta_i^4(s) = q_i(s)\left[K_i^3(s) - K_i^1(s)\right]$. It is claimed $\delta_i^j(s)$ is a convex direction for nested polynomials. Note $\arg\left\{\delta_i^1(\jmath\omega)\right\} = \theta_i + \arg\left\{K_i^1(\jmath\omega) - K_i^2(\jmath\omega)\right\}$ where $\theta_i \arg\{q_i(\jmath\omega)\}$. Moreover, $\arg\left\{K_i^1(\jmath\omega) - K_i^2(\jmath\omega)\right\}$ equals either $\pi/2$ or $-\pi/2$ because only odd powers of s occur in $K_i^1(s) - K_i^2(s)$. Thus, if $q_i(s)$ satisfies (6) then $\delta_i^1(s)$ satisfies (21), proving the claim for $j=1$. The cases $j=2,3,4$ are analogous. (If $j=2$ then note $\arg\left\{K_i^2(\jmath\omega)-K_i^4(\jmath\omega)\right\}$ equals either 0 or π because $K_i^2(s)-K_i^4(s)$ has only even powers of s.)

Since $\tilde{\mathcal{P}} \subset \mathcal{P}_e$, clearly $\phi(\tilde{\mathcal{P}})$ is robustly stable if $\phi(\mathcal{P}_e)$ is. To prove the converse, define

$$\mathcal{P}_k = \left\{p:\ p(s)=\sum_{i=1}^{n-k} q_i(s)a_i(s)+\sum_{i=n-k+1}^{n} q_i(s)b_i(s),\ a_i(s)\in\left\{K_i^1,\cdots,K_i^4\right\},\ b_i(s)\in E_i\right\}. \tag{47}$$

Note $\tilde{\mathcal{P}}=\mathcal{P}_0\subset\mathcal{P}_1\subset\cdots\subset\mathcal{P}_n=\mathcal{P}_e$. It is shown $\phi(\mathcal{P}_{k+1})$ is robustly stable if $\phi(\mathcal{P}_k)$ is, thus proving the theorem. For any $p(s)\in\mathcal{P}_{k+1}$, polynomials $p_0(s),p_1(s)\in\mathcal{P}_k$ can be found such that $p_1(s)-p_0(s)=\delta^j_{n-k+1}(s)$ for some $j\in\{1,2,3,4\}$ and $p(s)$ lies on the segment joining $p_0(s)$ and $p_1(s)$. Since $\delta^j_{n-k+1}(s)$ is a convex direction for nested polynomials, if $\phi(\mathcal{P}_k)$ is robustly stable then $\phi(p(s))$ is stable, as required. □

5 Applications

This section gives several illustrative applications of the main results of this paper.

5.1 Robust Stability of Uncertain Uniform Systems

In many practical problems, the regulation system can be regarded as consisting of identical elements, for instance, identical regulators or sensors or servos. Such systems are referred to as uniform systems [18]. Figure 1 shows an example of a single input single output (SISO) uniform system where $P(s)$ is the transfer function of the identical components and d_i and n_i for $i=1,2,\cdots,m$ denote arbitrary gains. If $P(s)=\frac{1}{p(s)}$ is an all pole plant then direct calculation shows the closed loop transfer function is

$$G(s) = \frac{n_0p^m+n_1p^{m-1}+\cdots+n_m}{p^m+d_1p^{m-1}+\cdots+d_m} \tag{48}$$

$$= \frac{N\big(p(s)\big)}{D\big(p(s)\big)} \tag{49}$$

where $D(z)=z^m+\sum_{i=1}^m d_iz^{m-i}$ and $N(z)=\sum_{i=0}^m n_iz^{m-i}$. By definition, the closed loop system is stable if and only if the polynomial $D\big(p(s)\big)$ is Hurwitz.

In practice, although $p(s)$ is usually not known exactly, it is often possible to determine a family $\mathcal{P}$ containing $p(s)$. It is then natural to ask if $D\big(p(s)\big)$, and hence the uniform system too, is stable for all $p(s)$ in $\mathcal{P}$. Several different families $\mathcal{P}$ are now considered in turn.

A polytope of real polynomials is any family of the form

$$\mathcal{P} = \left\{p:\ p(s)=p_0(s)+\sum_{k=1}^{n} c_kp_k(s),\ c_k\in\left[c_k^-,c_k^+\right]\right\} \tag{50}$$

where $p_0, \ldots, p_n$ are real polynomials. It is assumed the degree of every element of $\mathcal{P}$ is the same. It is shown in [13] that the robust stability of $D\big(p(s)\big)$ can be guaranteed by checking the stability of the exposed edges. Corollary 1 can help reduce the number of edges requiring checking since any edge satisfying the phase growth condition (2) is stable if and only if its two endpoints are stable.

If $p(s)$ belongs instead to the family in (5) then, by Theorem 2, only the extreme points defined in (11) need checking provided condition (6) holds.

Another polynomial family studied in the robust stability literature is the diamond of polynomials [20]. It can be viewed as a dual to the interval polynomials. Whereas the interval polynomial framework is associated with the l^∞ norm of the coefficient vector, Tempo [20] considers the l^1 norm and thus works with the family

$$\mathcal{P} = \left\{ p : \ p(s) = \sum_{i=0}^{n} a_i s^i, \ \sum_{i=0}^{n} |a_i - b_i| \leq r \right\} \tag{51}$$

where a_i, b_i and r are real valued. The polynomial $q(s) = \sum_{i=0}^{n} b_i s^i$ is called the nominal polynomial of $\mathcal{P}$. It is assumed the degree of every polynomial in $\mathcal{P}$ is the same. It is shown in [13] that for this family, robust stability is guaranteed if the $D\big(L_i(s)\big)$ are robustly stable, where $L_i(s)$ for $i = 1, \cdots, 8$ are the eight critical polynomial edges

$$L_1(s) = \{q(s) + \lambda r + (1-\lambda) rs : \ \lambda \in [0,1]\}, \quad L_5(s) = \left\{q(s) + \lambda r s^{n-1} + (1-\lambda) r s^n : \ \lambda \in [0,1]\right\}, \tag{52}$$

$$L_2(s) = \{q(s) + \lambda r - (1-\lambda) rs : \ \lambda \in [0,1]\}, \quad L_6(s) = \left\{q(s) + \lambda r s^{n-1} - (1-\lambda) r s^n : \ \lambda \in [0,1]\right\}, \tag{53}$$

$$L_3(s) = \{q(s) - \lambda r + (1-\lambda) rs : \ \lambda \in [0,1]\}, \quad L_7(s) = \left\{q(s) - \lambda r s^{n-1} + (1-\lambda) r s^n : \ \lambda \in [0,1]\right\}, \tag{54}$$

$$L_4(s) = \{q(s) - \lambda r - (1-\lambda) rs : \ \lambda \in [0,1]\}, \quad L_8(s) = \left\{q(s) - \lambda r s^{n-1} - (1-\lambda) r s^n : \ \lambda \in [0,1]\right\}. \tag{55}$$

Direct substitution proves $s+1$, $s-1$ and $s^{n-1}(s-1)$ satisfy condition (2) in Corollary 1, and so does $s^{n-1}(s+1)$ if n is odd. Thus, Corollary 1 implies that if the degree n is odd then $D\big(p(s)\big)$ is robustly stable if and only if $D\big(c_i(s)\big)$ is stable for $i = 1, 2, \cdots, 8$, where

$$c_1(s) = q(s) + r, \qquad c_2(s) = q(s) - r, \qquad c_3(s) = q(s) + rs, \qquad c_4(s) = q(s) - rs \tag{56}$$

$$c_5(s) = q(s) + rs^{n-1}, \quad c_6(s) = q(s) - rs^{n-1}, \quad c_7(s) = q(s) + rs^n, \quad c_8(s) = q(s) - rs^n. \tag{57}$$

Otherwise, if n is even, then $D\big(p(s)\big)$ is robustly stable if and only if $D\big(c_i(s)\big)$ is stable for $i = 1, 2, 3, 4$ and $D(L_6)$ and $D(L_7)$ are robustly stable.

Finally, consider the low order case when $p(s)$ belongs to the family

$$\mathcal{P} = \left\{p : \ p(s) = s^2 + a(\lambda_1, \cdots, \lambda_m) s + b(\lambda_1, \cdots, \lambda_m), \ \lambda_i \in [0,1], \ i = 1, \cdots, m\right\} \tag{58}$$

where $a(\lambda_1, \cdots, \lambda_m)$ and $b(\lambda_1, \cdots, \lambda_m)$ are affine in the λ_i. If $p_0(s)$ and $p_1(s)$ are two elements of this family then their difference $p_1(s) - p_0(s)$ is a degree 1 polynomial. Direct substitution shows any polynomial of degree 1 satisfies the condition (2) in Corollary 1. Therefore, the system is robustly stable if and only if $D\big(p(s)\big)$ is stable for the extreme points $p(s) = s^2 + a(\lambda_1, \cdots, \lambda_m)s + b(\lambda_1, \cdots, \lambda_m)$ where $\lambda_i \in \{0, 1\}$.

Note the above results extend to multiple input multiple output (MIMO) uniform systems considered in [18].

5.2 Robust Performance of Uncertain Uniform Systems

This section investigates the H_∞ performance and strict positive real (SPR) property of the uncertain uniform systems considered in Section 5.1. For the motivation and definitions of H_∞ norm and SPRness of a system, see [5, Chapter 9]. First, two lemmas from [7, 8] are combined and stated as:

Lemma 30 *Let $G(s) = \frac{B(s)}{A(s)}$ be a proper and strictly stable (real or complex) rational function. Let a_m and b_m be the coefficients of the s^m term in $A(s)$ and $B(s)$ respectively, where m is the degree of $A(s)$. Then $\|G(s)\|_\infty < 1$ if and only if 1) $|b_m| < |a_m|$, and 2) $A(s) + e^{\jmath\theta}B(s)$ is Hurwitz for all $\theta \in [0, 2\pi]$. Moreover, $G(s)$ is strictly positive real (SPR) if and only if i) $\Re\left\{\frac{B(0)}{A(0)}\right\} > 0$, and ii) $A(s) + \jmath\theta B(s)$ is Hurwitz for all $\theta \in \mathbb{R}$.*

Lemma 31 is a novel vertex lemma for the robust performance of uncertain uniform systems.

Lemma 31 *Define $D(\cdot)$ and $N(\cdot)$ as in Section 5.1. Let p_0 and p_1 be real polynomials and assume $D(p_0)$ and $D(p_1)$ are stable and the difference polynomial $p_1 - p_0$ satisfies condition (2) in Corollary 1. Then*

$$\max_{\lambda\in[0,1]} \left\| \frac{N\big(\lambda p_1 + (1-\lambda)p_0\big)}{D\big(\lambda p_1 + (1-\lambda)p_0\big)} \right\|_\infty = \max\left\{ \left\|\frac{N(p_0)}{D(p_0)}\right\|_\infty, \left\|\frac{N(p_1)}{D(p_1)}\right\|_\infty \right\} \tag{59}$$

and, for any $\alpha \in \mathbb{R}$, $\alpha + \frac{N\left(\lambda p_1+(1-\lambda)p_0\right)}{D\left(\lambda p_1+(1-\lambda)p_0\right)}$ is SPR for all $\lambda \in [0,1]$ if and only if $\alpha + \frac{N(p_0)}{D(p_0)}$ and $\alpha + \frac{N(p_1)}{D(p_1)}$ are SPR.

Proof. Note the leading coefficient of $N\big(\lambda p_1 + (1-\lambda)p_0\big)$ is that of $D\big(\lambda p_1 + (1-\lambda)p_0\big)$ multiplied by n_0. By Lemma 30, for any $\lambda \in [0, 1]$ and $\gamma > 0$, $\left\|\frac{N\left(\lambda p_1+(1-\lambda)p_0\right)}{D\left(\lambda p_1+(1-\lambda)p_0\right)}\right\|_\infty < \gamma$ if and only if $|n_0| < \gamma$ and $\phi\big(\lambda p_1 + (1-\lambda)p_0; \theta\big)$ is Hurwitz for all $\theta \in [0, 2\pi]$, where $\phi(z; \theta) = \gamma D(z) + e^{\jmath\theta}N(z)$. From Corollary 1, the latter condition holds if $\phi(p_1; \theta)$ and $\phi(p_0; \theta)$ are Hurwitz for all $\theta \in [0, 2\pi]$. It follows that if $\left\|\frac{N(p_0)}{D(p_0)}\right\|_\infty < \gamma$ and $\left\|\frac{N(p_1)}{D(p_1)}\right\|_\infty < \gamma$ then $\max_{\lambda\in[0,1]}\left\|\frac{N\left(\lambda p_1+(1-\lambda)p_0\right)}{D\left(\lambda p_1+(1-\lambda)p_0\right)}\right\|_\infty < \gamma$. This proves (59) because γ is arbitrary. The SPR result is proved similarly by defining $\phi(z; \theta) = D(z) + \jmath\theta(\alpha D(z) + N(z))$. □

Recall that extreme point results for robust stability were obtained in Section 5.1 for various families $\mathcal{P}$ of polynomials. Lemma 31 allows analogous extreme point results for worst case H_∞ performance and robust SPRness to be obtained for the same polynomial families.

5.3 Extreme Point Results for Polygonal Uncertainty Bounds

Consider the closed loop system in Figure 2. It is assumed $P(s) = \frac{1}{p(s)}$ where $p(s)$ is known to belong to one of the families $\mathcal{P}$ considered in Section 5.1. The feedback transfer function $\Delta(s)$ is arbitrary but it is assumed there exist a finite number of points $\delta_1, \cdots, \delta_k \in \mathbb{C}$ such that, for all $\omega \in \mathbb{R}$, $\Delta(\jmath\omega)$ lies in the convex hull of $\delta_1, \cdots, \delta_k$. Define Δ to be this convex hull. As in [11, Section V], the system is said to be robustly stable if and only if the analogous system having a constant gain δ in the feedback path is stable for every $\delta \in \Delta$.

If $\delta \in \Delta$ is the actual feedback gain then the denominator of the transfer function of the system is $p(s) + \delta$. It follows from Lemma 9 that for a fixed $p(s)$, $p(s) + \delta$ is stable for all $\delta \in \Delta$ if and only if it is stable for $\delta \in \{\delta_1, \cdots, \delta_k\}$. Define $\phi_i(z) = z + \delta_i$. Thus, the system is robustly stable if and only if $\phi_i\big(p(s)\big)$ is stable for all i. It is now clear the results of Section 5.1 can be used to help verify the stability of $\phi_i\big(p(s)\big)$ for $p(s) \in \mathcal{P}$.

6 Conclusion

Theorem 13, the key technical result of this paper, extends the convex direction results of Rantzer [19] to nested polynomial families. This extension is the best possible in that the phase growth condition is both necessary and sufficient. Theorem 2 uses this result to generalise a theorem in [13] to a larger class of families. Several illustrative applications appear in Section 5.

A Miscellaneous Proofs

Proof of Lemma 8. Assume $d \leq 2$. A real polynomial of degree 1 or 2 is stable if and only if its coefficients are non-zero and have the same sign. The coefficients of $p(s) = p_0(s) + \lambda\delta(s)$ are affine functions of λ, hence they change sign at most once in the interval $\lambda \in [0, 1]$. The leading coefficient does not change sign because the degree of $p(s)$ is assumed constant. Therefore, Statement 2 cannot hold. Conversely, if $d = 1$ and $\delta(s) = \beta_1 s + \beta_0$ then

$$\left.\frac{d \arg\{\delta(\jmath\omega)\}}{d\omega}\right|_{\omega=\omega_0} = \frac{\beta_0\beta_1}{\beta_0^2 + \beta_1^2\omega_0^2} \tag{60}$$

$$= \frac{\sin\left(2\arg\{\delta(\jmath\omega_0)\}\right)}{2\omega_0}, \tag{61}$$

so Statement 1 is false. Trivially then, Statements 1 and 2 are equivalent if $d = 1$. (This is not true if $d = 2$.)

Assume $d = 3$ and Statement 1 holds. It is shown in [19, Proof of Theorem 2] that $p(s) = (s^2 + \omega_0^2)(s + b)$ is such that $p(s) + \mu\delta(s)$ is stable in a punctured neighbourhood $0 < |\mu| < \epsilon$ for some $\epsilon > 0$ if: i) $b > 0$, ii) $\Im\left\{\frac{\delta(\jmath\omega_0)}{\jmath\omega_0 + b}\right\} = 0$, and iii) $\left.\frac{d\arg\{\jmath\omega+b\}}{d\omega}\right|_{\omega=\omega_0} < \left.\frac{d\arg\{\delta(\jmath\omega)\}}{d\omega}\right|_{\omega=\omega_0}$. Thus, if a b satisfying these conditions exists then $\delta(s)$ is not a convex direction and Statement 2 holds. Define $\phi \arg\{\delta(\jmath\omega_0)\}$ and $\theta = \arg\{\jmath\omega_0 + b\}$. Note

$$\left.\frac{d\arg\{\jmath\omega+b\}}{d\omega}\right|_{\omega=\omega_0} = \frac{\sin(2\theta)}{2\omega_0}, \tag{62}$$

$$\left.\frac{d\arg\{\delta(\jmath\omega)\}}{d\omega}\right|_{\omega=\omega_0} > \frac{\sin(2\phi)}{2\omega_0}. \tag{63}$$

The second condition, $\Im\left\{\frac{\delta(\jmath\omega_0)}{\jmath\omega_0+b}\right\} = 0$, is equivalent to $\theta = \phi + n\pi$ for some integer n. Thus, if the first two conditions hold then so does the third because

$$\left.\frac{d\arg\{\delta(\jmath\omega)\}}{d\omega}\right|_{\omega=\omega_0} > \frac{\sin(2\phi)}{2\omega_0} \tag{64}$$

$$= \frac{\sin(2\theta)}{2\omega_0} \tag{65}$$

$$= \left.\frac{d\arg\{\jmath\omega+b\}}{d\omega}\right|_{\omega=\omega_0}. \tag{66}$$

Note from the definition of θ that there exists a $b > 0$ such that $\theta = \phi + n\pi$ if and only if $\phi + n\pi \in (0, \pi/2)$, or equivalently, $\sin(2\phi) > 0$. Summarising, if $\omega_0 > 0$ satisfies Statement 1 with $\sin(2\phi) > 0$ then Statement 2 holds.

Assume then that Statement 1 holds but with $\sin(2\phi) \leq 0$. It is claimed there exists another ω_0 satisfying Statement 1 with $\sin(2\phi) > 0$, thus proving the lemma. If $\delta(s) = \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$ then

$$\frac{\sin(2\phi)}{2\omega_0} = \frac{\left(\alpha_0 - \alpha_2\omega_0^2\right)\left(\alpha_1 - \alpha_3\omega_0^2\right)}{|\delta(\jmath\omega_0)|^2}, \tag{67}$$

$$\left.\frac{d\arg\{\delta(\jmath\omega)\}}{d\omega}\right|_{\omega=\omega_0} = \frac{\alpha_2\alpha_3\omega_0^4 + (\alpha_1\alpha_2 - 3\alpha_0\alpha_3)\omega_0^2 + \alpha_0\alpha_1}{|\delta(\jmath\omega_0)|^2} \tag{68}$$

$$= \frac{\sin(2\phi)}{2\omega_0} + \frac{2\omega_0^2(\alpha_1\alpha_2 - \alpha_0\alpha_3)}{|\delta(\jmath\omega_0)|^2} \tag{69}$$

$$= -\frac{\sin(2\phi)}{2\omega_0} + \frac{2(\alpha_2\alpha_3\omega_0^4 - 2\alpha_0\alpha_3\omega_0^2 + \alpha_0\alpha_1)}{|\delta(\jmath\omega_0)|^2}. \tag{70}$$

Since $\sin(2\phi) \leq 0$, (15) and (69) imply

$$\frac{2\omega_0^2(\alpha_1\alpha_2 - \alpha_0\alpha_3)}{|\delta(\jmath\omega_0)|^2} > -2\frac{\sin(2\phi)}{2\omega_0} \geq 0. \tag{71}$$

In particular, $\alpha_1\alpha_2 - \alpha_0\alpha_3 > 0$. Thus, from (69), any $\omega_0 > 0$ satisfying $\sin(2\phi) > 0$ and $\delta(\jmath\omega_0) \neq 0$ also satisfies Statement 1. Since $\delta(s)$ is not the zero polynomial, $\delta(s) = 0$ has a finite number of solutions, hence to prove the claim, it suffices to find an open sub-interval of $\omega_0 > 0$ on which $\sin(2\phi) > 0$. From (67), $\sin(2\phi) > 0$ if

$$\left(\alpha_0 - \alpha_2\omega_0^2\right)\left(\alpha_1 - \alpha_3\omega_0^2\right) > 0. \tag{72}$$

Provided $\alpha_2\alpha_3 \neq 0$, this is a quadratic in ω_0^2 with discriminant $(\alpha_1\alpha_2 - \alpha_0\alpha_3)^2$, which is strictly positive by (71), hence there exists an open sub-interval of $\omega_0 > 0$ on which (72) holds. If $\alpha_3 \neq 0$ but $\alpha_2 = 0$ then $\alpha_1\alpha_2 - \alpha_0\alpha_3 > 0$

implies $\alpha_0\alpha_3 < 0$, thus (72) holds if ω_0 is sufficiently large. If $\alpha_3 = 0$ then, since (15) holds with $\sin(2\phi) < 0$, (70) implies $\alpha_0\alpha_1 > 0$. Thus, for ω_0 sufficiently small, (72) holds. This proves the claim. □

Proof of Lemma 9. Assume $d = 1$ and $\delta(s) = \beta_1 s + \beta_0$. Note

$$\left.\frac{d\arg\{\delta(\jmath\omega)\}}{d\omega}\right|_{\omega=\omega_0} = \frac{\beta_0^r\beta_1^r + \beta_0^i\beta_1^i}{|\delta(\jmath\omega_0)|^2} \tag{73}$$

where superscripts r and i denote real and imaginary parts. If $p_0(s) = \alpha_1 s + \alpha_0$ then the real part of the root of $p_0(s) + \lambda\delta(s)$ is

$$\frac{-(\beta_0^r\beta_1^r + \beta_0^i\beta_1^i)\lambda^2 - (\alpha_0^r\beta_1^r + \alpha_1^r\beta_0^r + \alpha_0^i\beta_1^i + \alpha_1^i\beta_0^i)\lambda - (\alpha_0^r\alpha_1^r + \alpha_0^i\alpha_1^i)}{|\alpha_1 + \lambda\beta_1|^2}. \tag{74}$$

The numerator is a quadratic in λ with discriminant

$$\Delta = (\alpha_0^r\beta_1^r + \alpha_1^r\beta_0^r + \alpha_0^i\beta_1^i + \alpha_1^i\beta_0^i)^2 - 4(\beta_0^r\beta_1^r + \beta_0^i\beta_1^i)(\alpha_0^r\alpha_1^r + \alpha_0^i\alpha_1^i). \tag{75}$$

If Statement 1 is false, so either $\beta_0 = \beta_1 = 0$ or, from (73), $\beta_0^r\beta_1^r + \beta_0^i\beta_1^i \leq 0$, then the numerator of (74) is convex in λ, hence it is not possible for the sign of (74) to go from negative to positive back to negative again and Statement 2 is false. Conversely, if Statement 1 is true, so $\beta_0^r\beta_1^r + \beta_0^i\beta_1^i > 0$, then choose α_0 and α_1 so that $\alpha_0^r\alpha_1^r + \alpha_0^i\alpha_1^i > 0$ and α_1 is not a real-valued multiple of β_1. The latter condition ensures the denominator of (74) is never zero while the former condition ensures the numerator of (74) is negative if $\lambda = 0$ and, since (75) is positive, the numerator changes sign twice as λ goes from $-\infty$ to ∞. Therefore, the two zeros of the numerator can be brought to lie in $(0, 1)$ by jointly scaling α_0 and α_1 by a real number, thereby proving Statement 2. □

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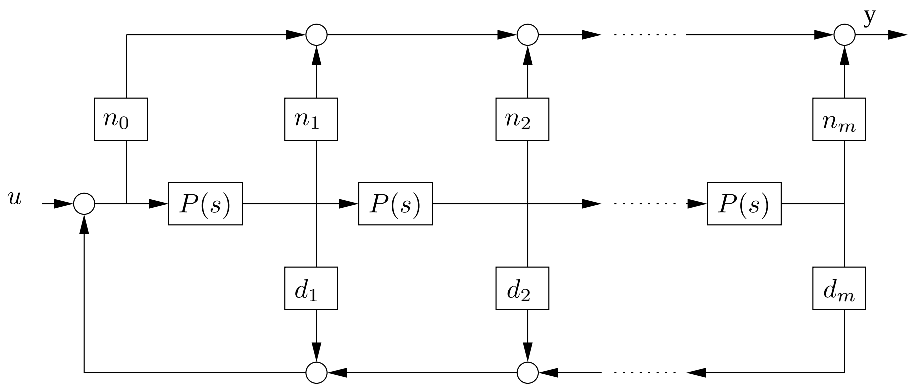

Figure 1: A Uniform System

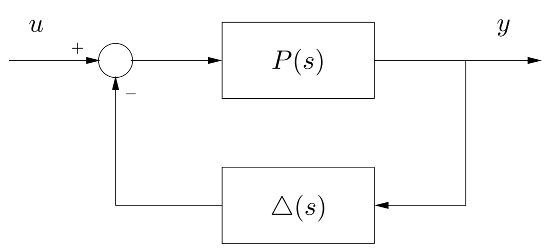

Figure 2: Standard Feedback Configuration