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Numerical Investigations about the Predictions of Free Surface Shallow Water Motions

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SUMMARY This paper shows how numerical techniques developed for predicting compressible flows described by hyperbolic partial differential equations, may be used for solving the corresponding problems of shallow water motions. The usual analogy between these two classes of flows is extended to the computational area with reference to the integration of the interior points, computation at the boundaries and treatment of discontinuities such as shock waves and hyperbolic jumps or bores. Examples of numerical experiments on shallow waters motions are presented for different kinds of flows. These results have been obtained by adding only minor modifications to existing and experimented codes, developed for solving typical problems of gasdynamics.

1 INTRODUCTION

Many steps have been done in the past twenty years on the solution of problems in fluid dynamics by means of numerical computations. The most investigated fields have been the ones related to aeronautical and aerospace sciences. In particular remarkable progress have been achieved in the numerical solution of gasdynamical problems described by hyperbolic partial differential equations and representing steady supersonic or unsteady compressible flows. The main difficulties in solving these problems are related to the correct computation at the boundaries and to the treatment of discontinuities in the flow fields such as shock waves or contact discontinuities. Beside these two main points, it has been proved that the proper choice of variables and coordinates systems is a very important requirement. While many efforts have been made in solving these problems with reference to aeronautical applications, little attention has been paid to the investigation of problems very similar but related to different physical areas, such as the prediction of free surface water motions in the case of shallow water. The purpose of this paper is the extension of the very well known analogy between compressible flows and shallow water motions, beyond the usual comparison which is generally confined to comments on the similarity of the equations which describe these two classes of physical phenomena. Namely we try to show here how the broad experience achieved in the numerical computation of hyperbolic compressible flows can help easily in predicting the corresponding hyperbolic shallow water flows. We have developed a certain number of numerical experiments which represent different physical problems in the area of shallow waters. We may divide these in four classes:

- unsteady one-dimensional flow in channels
- two-dimensional supercritical flow in channels
- unsteady two-dimensional flow
- two-dimensional sub and supercritical flow.

These problems are often complicated by the formation of discontinuities, bores or hydraulic jumps, which represent the counterpart of shock waves in gasdynamics. In the following a brief review is given on the main features on the numerical procedure, and examples of computations are presented for different flows.

2 THE EQUATIONS OF MOTION ANALOGY

We recall quite briefly the analogy between the equations of motion which describe the compressible flow and the shallow water motion in the case of unsteady one-dimensional flow. The basic equations in gasdynamics are:

$$P_t + u P_x + \gamma u_x + \gamma u \alpha = 0$$
$$u_t + u u_x + T P_x = 0$$
$$S_t + u S_x = 0$$

where $P = \ln p$, $T = \exp\left(\frac{\gamma-1}{\gamma}P + \frac{S}{\gamma}\right)$ and $\alpha = A_x/A$ (A = cross area of the duct). All the variables have been here normalized with respect to reference values (length l_∞, pressure p_∞, temperature T_∞, velocity $c_\infty = \sqrt{p_\infty/\rho_\infty}$ and time $t_\infty = l_\infty/c_\infty$).
The basic equations for constant depth shallow water are:

$$H_t + u H_x + 2 u_x + 2 \alpha u = 0$$
$$u_t + u u_x + h H_x = 0$$

where $H = 2\ln h$ (h = water level) and the reference velocity is $c_\infty = \sqrt{g l_\infty/2}$.
Both Eq (1) and (2) are hyperbolic partial differential equations and the comparison between these reveals the well known analogy (Ref.1).
A very important feature of these equations is represented by the possible generation in the flow field of discontinuities known as shock waves in gas dynamics and bores or hydraulic jumps in hydraulics. The Rankine-Hugoniot equations relate the flow properties on the two sides of the discontinuity (1,2) which propagates with the velocity U_s:

$$\frac{p_1 w_1}{T_1} = \frac{p_2 w_2}{T_2}$$

$$p_1\left(1+\frac{w_1^2}{T_1}\right)=p_2\left(1+\frac{w_2^2}{T_2}\right)$$

$$T_1\left(1+\frac{\gamma-1}{2\gamma}\frac{w_1^2}{T_1}\right)=T_2\left(1+\frac{\gamma-1}{2\gamma}\frac{w_2^2}{T_2}\right)$$

where w represents the gas velocity relative to the shock ($w = u - U_s$).

Similar equations hold for the bore:

$$h_1 w_1 = h_2 w_2$$

$$h_1^2\left(1+\frac{w_1^2}{h_1}\right)=h_2^2\left(1+\frac{w_2^2}{h_2}\right)$$

The same analogy between Eq.(1,3) and Eq.(2,4) may be found for different and more complicated flows (two-dimensional steady or unsteady, two-dimensional steady sub and supercritical flows).

3 THE NUMERICAL SOLUTION

The broad experience achieved in gasdynamics on the solution of hyperbolic partial differential equations, has suggested the finite difference method as a powerful tool for the numerical prediction of this kind of flow fields. The following steps are done. The physical region is generally mapped into a computational domain. This step is done by using simple or very sophisticated transformations of coordinate, depending on the complexity of the boundaries. One may use just only a transformation for normalizing some length, or complicated conformal mappings in case of very irregular shape of the physical region. Even if the problem is very different from those which occur in hydraulics, Ref.2 shows how powerful is the use of sophisticated transformations. The equations of motion are then written in the new system of coordinates of the computational domain. The new set of equations are then discretized, having fixed a constant intervals grid in the computational region. Among the several schemes of integration which are available, we use the one suggested by Mac Cormack (Ref.3). This scheme is very simple for coding purposes, even in case of multidimensional flows and has been experimented successfully in a very large number of cases. It is a predictor-corrector scheme which is expected to give second-order-accuracy. The integration of the equations at the interior points of the computational region does not show difficulties, provided that no discontinuities are generated in the flow field. On the other hand, good care must be paid to computing the flow at the boundaries. We may have many kinds of boundaries. Solid walls are the ones which occur more often. However we may have other different boundaries. The bow discontinuity in front of a blunted body in a supercritical stream represents a permeable boundary (Ref.4). Special surfaces for having finite computational domains (Ref.5). Contact surfaces separating two different flow regions, and others. In all these cases, we follow the ideas and procedures suggested by G. Moretti in many papers on gasdynamical problems. An auxilary frame of reference is assumed, according to the nature of the boundary . For example, in the case of a solid wall in a two dimensional problem, the frame will be with axis oriented as normal and tangent to the wall, so that the velocity can be written in terms of the normal and tangential components. If the boundary moves, the auxilary frame will move with it. The equations of motions are then written by introducing the new frame of reference. By combining continuity and momentum equations, one gets compatibility equations along characteristic lines. On the other hand the conditions which hold at the boundary are differentiated at the boundary itself. For example, in case of 2D unsteady flow at solid walls, one puts zero the components of the velocity and acceleration normal to the wall. The compatibility equations and the boundary equations are then combined and the resulting differential equation allows the integration of the unknown variable at the boundary. The integration scheme follows the same used at the interior points. The detailed description of this procedure is reported in Ref. 5. The nature of the hyperbolic equations for compressible flow and shallow water allows the formation of discontinuity inside the flow field. In gasdynamics, the real gas effects do not generate a real discontinuity and the physical shock wave has a finite thickness. However this thickness is so small with respect to reference lengths that the shock may be regarded as a discontinuity. On the contrary the region perturbed in the hydraulic jump is quite large (Ref.1) and, depending on the physical problem, its thickness may be too large with respect to the reference length. In this case the jump can not be modeled as a discontinuity, as Eqs.(2) would require. It follows that while in gasdynamics it is almost always correct to consider the shock as a discontinuity, one can do the same in hydraulics only in the case of reference length quite larger than the width of the actual jump. From a numerical point of view there are two distinct approaches in dealing with the discontinuity. One is known as the "shock capturing technique" where the flow equations are written in conservation form and the discontinuity is not treated explicitely, but comes out from the computation as steep gradients in the flow spread over few meshes. On the other hand the discontinuity may be treated explicitely as a double value point in the flow field. The procedure for computing the evolution of discontinuities has been indicated by G. Moretti and is well known as "shock fitting technique". It consists in the matching of the compatibility equations in the two continous flow regions separated by the discontinuity with the algebraic equations across it /Eq.(3,4)/ differentiated along the discontinuity itself. Details on this procedure may be found in Ref. 8 for the simple one dimensional case or in many other papers of the same author for multidimensional flows. In computing the hydraulic jump evolution we follow this discontinuity explicit treatment.

4 SOME EXAMPLES

We report now some examples of prediction of shallow water motions. The results have been achieved by using codes developed for the corresponding gasdynamical problems and by adding just minor changes for the hydraulics application. In the following figures the flow pattern is indicated with constant level (h) lines.

i) Unsteady one-dimensional flow in channels.

We have performed several examples of this flow, involving the formation of hydraulic jumps. (Ref.6,7).

The one reported in Fig. 1 is quite significant. A constant depth channel is placed between two infinite capacity reservoirs with different water levels. The channel width is shaped in a convergent-divergent fashion along the abscissa x. The steady flow through the channel is subcritical in the convergent portion of the channel till the throat; then, it becomes supercritical, if the level in the downstream reservoir is low enough. However for a particular range of values of this level, a hydraulic jump is expected to take place in the divergent portion of the channel; the stream becomes then subcritical till it matches, at the end of the channel, the water level in the discharge reservoir. This steady flow can be obtained as the asymptotic result of the following transient (time-dependent technique). Assume that at the time $t = 0$ the channel is closed by a wall at $x = 1$ and open at $x = 0$, so that the level is every-where equal to the one of the left reservoir and the water is at rest. Suddenly the end wall is removed and a depression wave propagates upstream. During this transient a shock is generated by the coalescence of characteristics going from right to left. First it moves upstream and then downstream until it gets stabilized in the equilibrium location typical of the steady state flow. The description of this transient is given in the x,t plot of Fig. 1. In order to get some idea on the capability of this numerical methodology to deal with very complicated one-dimensional unsteady motions, the reader may refer to Reg. 8.

ii) Two-dimensional steady supercritical flow in channels.

The example reported in Fig. 2 (Ref.6) shows a steady supercritical stream (Froude = 3) in a channel. The computation is done by marching along the channel. Because of the walls shape, the rising waves generated on the lower wall form an oblique jump moving towards the upper wall. Once the jump reaches the wall, it is reflected on the opposite direction. Even for these flows the reader may refer to Ref.9, where similar, but more complicated cases are presented, in case of compressible flows.

iii) Two-dimensional unsteady flow.

The example we present here, is reported in extended form in Ref.10. A region of water at rest (h=1.) with variable depth, is bounded by three solid walls and on the fourth side (AD) by a diaphragm wall, which separates the region from an outer capacity with water at a lower level (h =.9). When the diaphragm wall is removed, water is flowing out (from right to left) and depression waves travel inside the region. The boundary AD, which separates the region from the outer capacity is a special surface for simulating this external reservoir; details are given in Ref.10. Friction effects on the bottom are here taken into account by adding the proper terms in the momentum equations. The sequence of Fig. 3 shows the evolution in time of the water waves and the velocity profiles on AD describing the flow going in and out of the investigated region. After many cycles of waves moving across the region, the water level will be stabilized at the outer capacity level with water at rest.

iiii) Two-dimensional steady sub and supercritical flow.

We refer here to those flows where the steady flow equations are both elliptic and hyperbolic. The example shown here (Fig. 4), represent the counter part of the "blunt body" problem in gasdynamics. A supercritical stream flows against a blunted obstacle; a curved jump is then wrapping the obstacle and a pocket of subcritical flow is generated just in front of it. The steady flow configuration is achieved through a "time dependent technique" (Ref.4). The equations of motion are written for 2D unsteady flow; an initial flow configuration is guessed "a priori" in the region between the obstacle and the bow jump. The region is confined on the two sides by lines where supercritical flow is expected. Because the initial guessed configuration is not the correct one, the unsteady flow equations will exibit derivatives in time of the flow properties different from zero. Wave are then generated which will move the bow jump in the right position and will create the balanced steady flow configuration. Once the flow pattern is obtained in the front of the obstacle, a marching technique for 2D steady supercritical flow will generate the flow on the two sides of the obstacle. The example reported in Fig. 4 (Ref.11) refers to a Froude number = 3 and an incidence of 10°.

5 CONCLUSION

The examples reported in this paper show that the analogy between compressible flow and shallow water motion may be extended well behind the usual comparison of equations as reported in text books. Many of the features of the numerical methods developed for solving gasdynamical hyperbolic problems may be utilized for the corresponding hydraulic problems. Only minor changes should be added to existing and reliable codes for compressible flows in order to get solutions of shallow waters motions. May be that the examples shown here are just academic exercises. However we think that the use of these numerical methods may give a good contribution to the understanding of practical problems in the area of water resources management.

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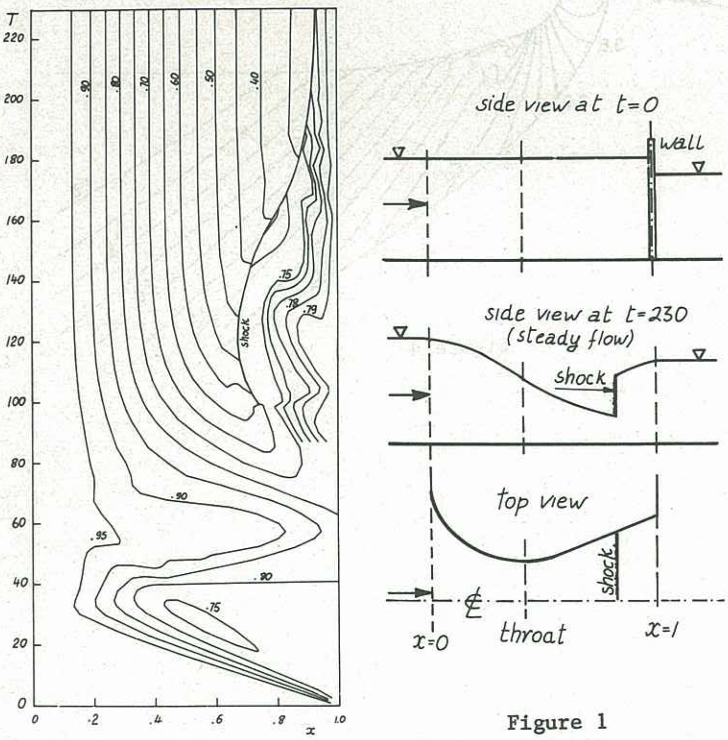

Figure 1

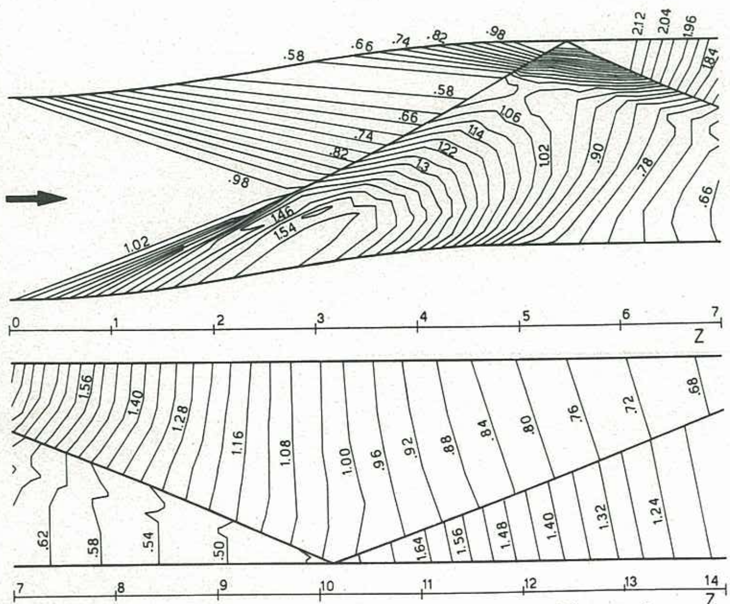

Figure 2

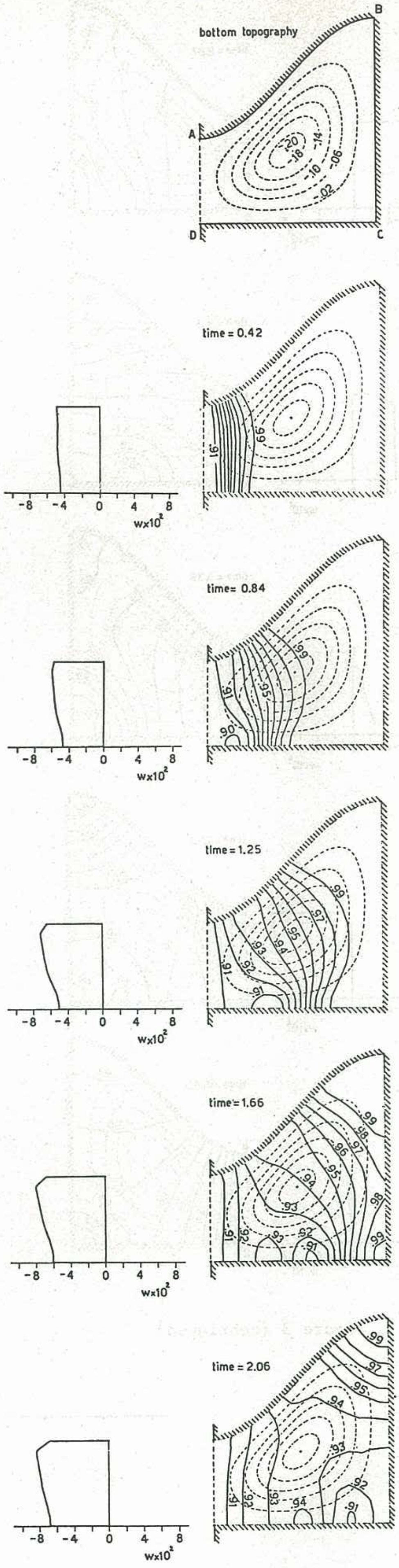

Figure 3

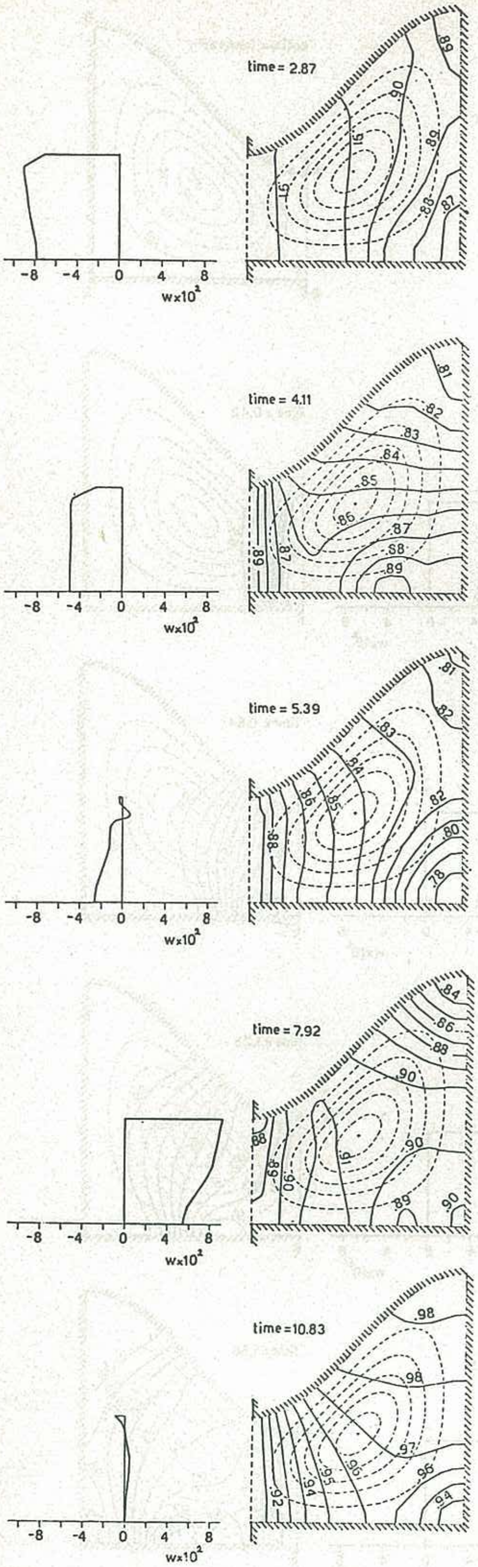

Figure 3 (continued)

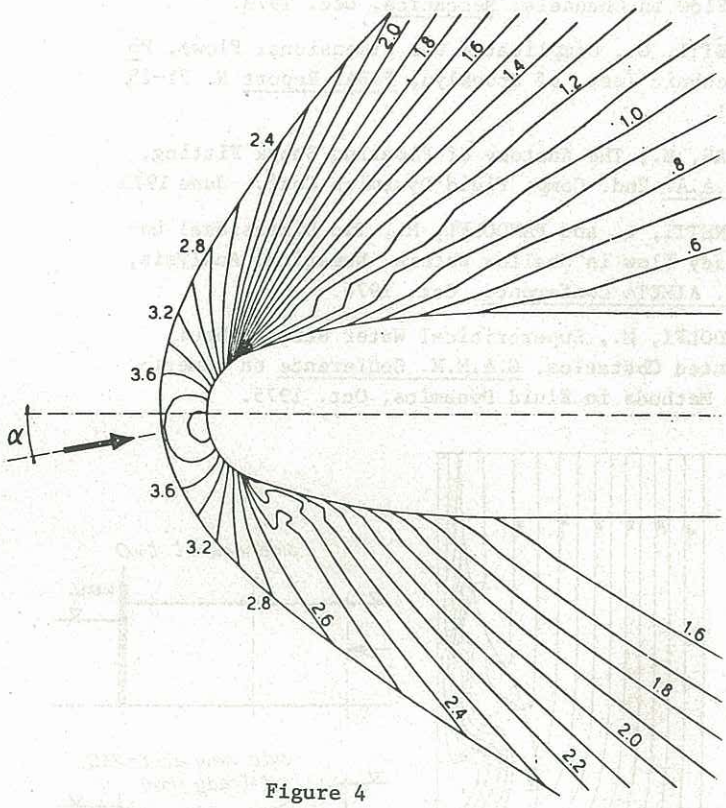

Figure 4