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Calculation of Potential Flows using Surface Distributions of Vorticity

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1 INTRODUCTION

The work reported in this paper represents an extension of studies reported at an earlier conference in the series (Joynt, 1971) where an axisymmetric vorticity distribution on the surface of a series of contiguous frustums of cones was used to design contractions, etc., lying everywhere internal to this singularity array, having a surface velocity distribution substantially at the disposition of the designer. The present extension inverts this process so that the velocity field associated with a given boundary shape can be determined; both two-dimensional and three-dimensional axisymmetric cases are covered.

The option of surface vorticity distributions was chosen because the fundamental theory showed that it conferred some advantages, compared with the source distributions preferred by some other workers (e.g. Hess, 1975). These advantages included the identification of the magnitude of the vorticity with the surface flow speed, the capacity to deal with cyclic and lifting flows without the introduction of new concepts, and the ability to extend the work more logically into sheared and rotational flows (some of this latter work is currently under development). These benefits, and an inferred - although not proven - facilitation of the numerical work, were judged to outweigh the complication of using a solenoidal vector, rather than a scalar, singularity distribution.

Publication was felt to be warranted partly because of the use of this less usual approach but also to illustrate the way in which the evaluation of the singular integrals was accomplished and the unusual, but successful, way of meeting the boundary conditions that was applied.

2 THEORY

The representation of flows by means of surface distributions of singularities is treated in the literature (Lamb, 1962, Art. 57) where it is shown that, if the chosen boundary is a stream surface, double-sources, with their axes normal to the surface, provide one suitable formulation from the infinity of choices available. Further, with this choice, the strength of the double-source is equal to the velocity potential at that point, so that, as the magnitude of the vorticity vector is equal to the gradient of the double-source strength, it is also equal to the magnitude of the velocity vector.

This representation thus produces a relationship between the variables that is particularly easy to interpret; on the stream surface the vorticity vector and the velocity vector are mutually orthogonal, equal in magnitude and both lie in the surface. As the vorticity distribution must produce a discontinuity in the tangential component of velocity, equal to its own strength, and cannot produce a discontinuity in the normal component of velocity, it follows that the velocity is zero just the other side of the stream surface (the same result is proved in a different way in the original reference).

The condition of zero velocity just outside the stream surface is used to set up a boundary condition that is more convenient, in the present work, than the usual one in which the normal component of velocity is set equal to zero. It is found to be sufficient to calculate the vorticity distribution that produces zero tangential component of velocity just outside the stream surface. At first sight it might appear that this could be a necessary but not sufficient boundary condition, but this would require there to be a flow in which the streamlines were everywhere normal to the stream surface. For the cases of practical interest this can be shown to be impossible. In practice the use of this boundary condition is efficient as an element of vorticity makes a strong contribution to the tangential velocity at points adjacent to it, leading to well-conditioned matrices in the numerical solution.

With the above selection of singularity distribution and boundary condition it is now possible to write down the equations to be solved. The velocity $\bar{q}_p$ at the general point P is given by the Biot Savart law:

$$\bar{q}_p = \frac{1}{4\pi}\int_S \frac{\bar{K} \times \overline{QP}}{|\overline{QP}|^3}\, dS$$

where $\bar{K}$ is the local vorticity vector at the point Q of the stream surface S. This is converted into an integral equation for $\bar{K}$ by evaluating the tangential component of q_p at a general point just outside S and equating to zero.

The cases considered were all duct flows in either two dimensions or in three dimensions with axial symmetry. The ducts were considered in terms of sections where the shape was specified and the velocity remained to be calculated, and entry and exit sections having a simple form where, remote from the shaped section, the velocity field was known. In the shaped region the profile was discretised into a piecewise linear representation and on each of the elements so formed it was assumed that the vorticity had constant, but as yet unknown, strength K_r (r = 1, 2, ... N). Typical duct shapes are illustrated on Figure 1 for axisymmetric geometries (two-dimensional geometries are analogous although bends can also be accommodated in this case).

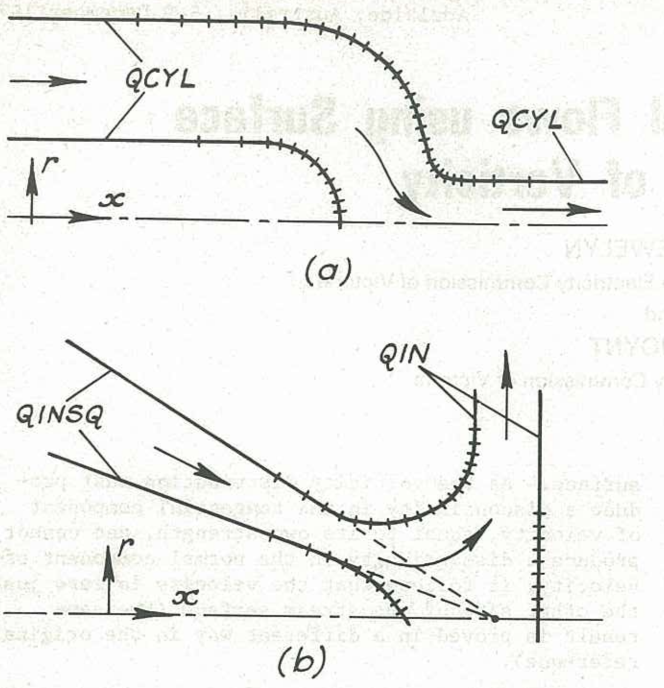

Figure 1 Specimen axisymmetric duct configurations

In the axisymmetric case the end elements are seen to be able to accommodate cylindrical or constant cross-section annular inflows and outflows, the corresponding purely radial flows as well as those occurring between a pair of cones with a common apex. In all cases the requirement of continuity of volume flow must be complied with in the overall sense and applying it to each end flow permits the variation of q with x and/or r to be determined. Thus the Biot-Savart integrals can be defined.

The midpoints of the short straight elements introduced on those parts of the duct profile where the velocity is varying in an unknown manner, are used as nodes. It is necessary to calculate the tangential component $b_i (i = 1, 2, \ldots N)$ of the velocity induced by all the end elements at each node, leading to non-singular integrals. The integral along the length of the element is performed analytically (see Appendix) and the remaining integration around the circumference, in the case of the axisymmetric elements, must in any event be performed numerically as it is an elliptic integral. The only other point to be watched in these evaluations is that the two-dimensional end elements must be evaluated in parallel pairs with vorticity of opposite sign, otherwise unwanted infinities appear in the expressions. The appropriate computer subprograms for the axisymmetric case were named QCYL, QIN and QINSQ as indicated on Figure 1(a) and (b).

It remains to calculate the tangential velocity component induced at the centre of the i^{th} piecewis linear element, by unit vorticity strength distributed on the j^{th} such element, as a_{ij}. Then solution of the integral equation is approximated by the solution of the following set of simultaneous linear algebraic equations:

$$\sum_{j=1}^{N} a_{ij} K_j + b_i = 0 , \qquad i = 1, 2, \ldots N.$$

For the cases $i \neq j$ the a_{ij} integral is non-singular and, although somewhat more tedious, was evaluated by similar methods to those described for the end elements. A computer subprogram, QCAX, was written for this case as, although the one written for the singular integral, QAXVOR, would handle the non-singular evaluation, it was considerably slower.

In two dimensions evaluating a_{ii} involves a straightforward Cauchy singularity that poses no particular difficulty. In the axisymmetric case the fact that the integral is in any event elliptic, and its algebraically complicated form, posed a significant problem. Essentially it was solved by expressing $\overline{QP}$ as the sum of three orthogonal vectors so that the vector product could be easily evaluated, performing the integration along the generators of the frustum of the cone analytically and then either reducing the remaining singular integrals to standard forms or absorbing the singularity into the differential term so that the transformed integral was non-singular and could be accurately evaluated using a quadrature. The broad details of this, the most interesting integral, are given in the Appendix.

The well-conditioned matrix equation involving a_{ij} lends itself to accurate solution by either inversion or iterative techniques giving a vector K_j which equates to the surface speed distribution in the duct. Internal velocities can also be determined but a set of coefficients similar to a_{ij} and b_j $(i = 1, 2, \ldots N)$ must be calculated for each such evaluation.

3 EXAMPLES AND COMPARISONS

Figure 2 shows results for a two-dimensional 90° bend with a 3:1 contraction ratio. The bend shape was determined by a conformal transformation method which has been shown to give good agreement with experimental measurements. The results obtained by approximating the boundary with 48 and 120 elements are shown.

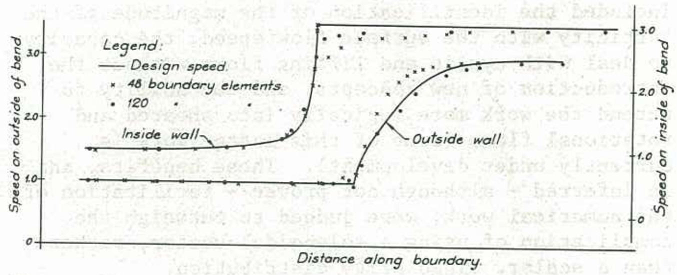

Figure 2 Accuracy of two-dimensional calculations

Axisymmetric results for the flow around a unit sphere are presented in Figure 3. Two stream surfaces, with radii 1.0 and 2.0 at infinity, have been taken as the boundaries of an annular region. The speeds calculated by approximating the boundary with conical elements are compared with the known potential flow around a sphere.

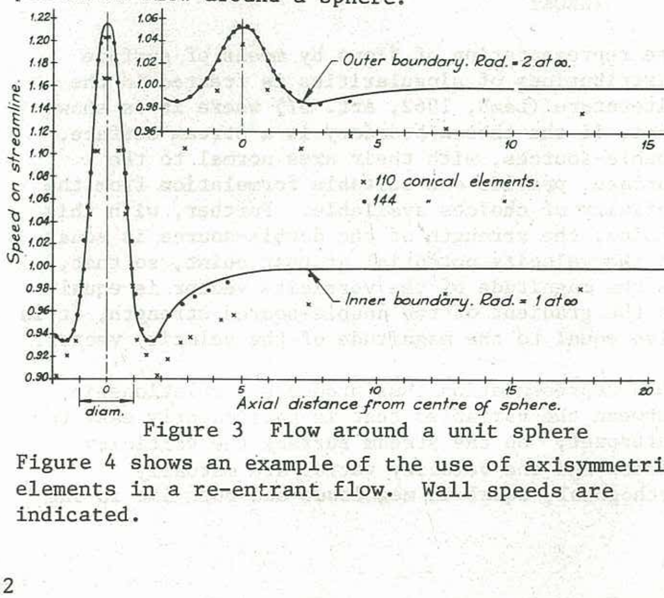

Figure 3 Flow around a unit sphere

Figure 4 shows an example of the use of axisymmetric elements in a re-entrant flow. Wall speeds are indicated.

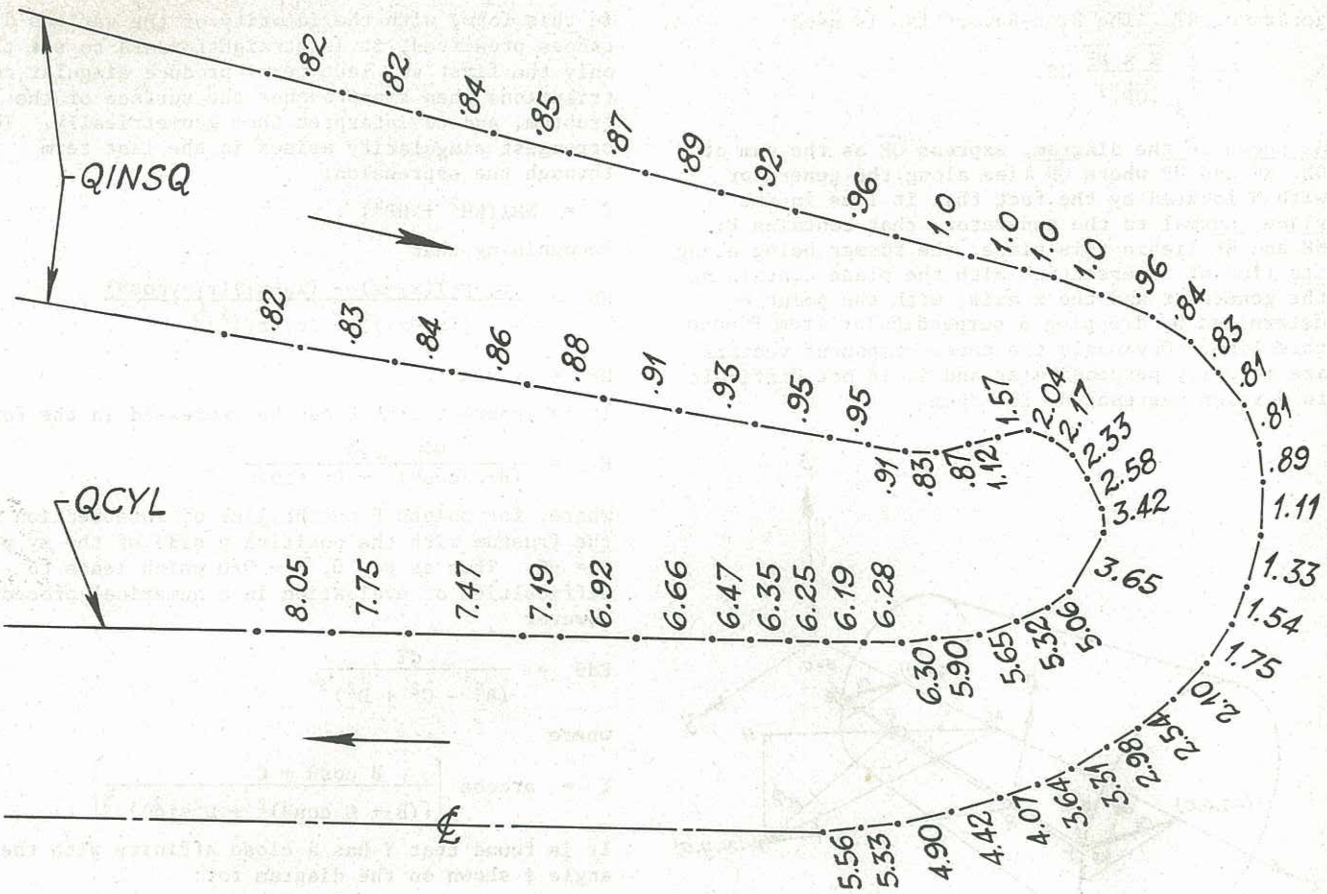

Figure 4 Axisymmetric re-entrant flow

4 DISCUSSION

The method presented herein has been found to be a useful engineering tool in both its two-dimensional and axisymmetric forms. Adequate accuracy has been obtained with a number of elements, and hence equations, that places no undue burden on the available computing resource. However there is little doubt that the use of more sophisticated elements would shorten computation times for a given accuracy. The literature suggests (Hess, 1975) that an upgrading to elements having an arc of a parabola as a profile should be accompanied by a change to a linearly varying vorticity strength. The resulting increase in the complexity of the integrals would require the allocation of a considerable amount of effort, which currently seems unlikely to eventuate.

The use of vorticity has conferred not only the benefit of a saving in computer time because surface velocity does not have to be separately calculated, but has been found to be beneficial in that engineers readily understand and put the methods to intelligent use. The concept of the vorticity as a very thin, mathematically idealised boundary layer provides a suitable bridge to physical concepts. In both of these respects and because it is essentially a more powerful tool, it would seem to be preferable to the use of source distributions.

One exploitation of this greater power that has recently been explored is the calculation of a flow in a duct containing a shear flow. Successful iteration to find the strength and position of an embedded vortex sheet was achieved in two dimensions and there would seem to be no reason why a similar result could not be obtained in the axisymmetric case. Currently some attention is being directed towards the use of elements having distributed vorticity to represent rotational flows.

5 ACKNOWLEDGEMENT

This paper is published with the permission of the State Electricity Commission of Victoria.

6 REFERENCES

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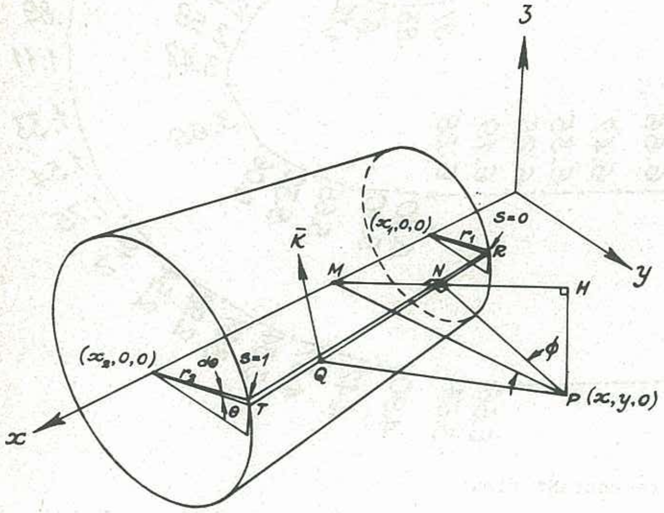
APPENDIX: EVALUATION OF THE SURFACE INTEGRALS

For each of the axisymmetric elements the integration along the generator of the cone can be performed analytically and the circumferential integration is done using Simpson's rule.

QAXVOR: The velocity $\bar{q}_p$ is to be calculated at the point P in the xy plane due to unit vorticity on the surface of the frustum with the vorticity vector tangential to the surface and normal to the generators. The general point Q is located on the generator RT. The Biot-Savart law is used:

$$\overline{q}_p = \frac{1}{4\pi}\int_S \frac{\overline{K}\times\overline{QP}}{|\overline{QP}|^3}\,dS.$$

As shown in the diagram, express $\overline{QP}$ as the sum of $\overline{QN}$, $\overline{NH}$ and $\overline{HP}$ where $\overline{QN}$ lies along the generator with N located by the fact that it lies in the plane, normal to the generator, that contains P; NH and HP lie in this plane, the former being along its line of intersection with the plane containing the generator and the x axis, with the point H determined by dropping a perpendicular from P onto this line. Obviously the three component vectors are mutually perpendicular and it is not difficult to develop expressions for them.



The vorticity has unit magnitude and is parallel to $\overline{PH}$, thus $\overline{K} = -\hat{HP}$. The elemental area can be expressed in the form:

$$dS = r_Q.d\theta.dQN, \text{ where } r_Q \text{ is the frustum radius at Q,}$$
$$= (r_1TQ + r_2RQ).d\theta.dQN/RT,$$
$$= \{r_1(TN - QN) + r_2(RN + QN)\}.d\theta.dQN/RT.$$

Thus, recognising the equality of the contributions from $0 \leqslant \theta \leqslant \pi$ and $\pi \leqslant \theta \leqslant 2\pi$, the expression for $\overline{q}_p$ becomes:

$$\overline{q}_p = \frac{-1}{2\pi RT}\int_0^{\pi} d\theta \int_{-RN}^{NT} \frac{\{r_1TN+r_2RN+QN(r_2-r_1)\}}{(QN^2+NH^2+HP^2)^{3/2}} \overline{HP}\times(\overline{QN}+\overline{NH}+\overline{HP}).dQN$$

Expressing $\overline{QN}$ as $QN.\hat{QN}$, and similarly for $\overline{NH}$ and $\overline{HP}$, the vector product becomes $QN.\hat{NH}-NH.\hat{QN}$. As, by symmetry, the $\underset{\sim}{k}$ component of $\underset{\sim}{q}$ is zero the corresponding components in $\hat{NH}$ and $\hat{QN}$ can be omitted from the numerical calculations from this point onwards. The integrals correspond to standard forms and, on evaluating and substituting the limits, give:

$$\overline{q}_p = \frac{-1}{2\pi.RT}\int_0^{\pi}\Bigg[(r_2-r_1)\,\hat{NH}\,\ln\frac{(NT^2+NH^2+HP^2)^{\frac{1}{2}}+NT}{(RN^2+NH^2+HP^2)^{\frac{1}{2}}-RN}$$
$$-\frac{(r_1TN + r_2RN)\,\hat{NH} - (r_2-r_1)(NH.\hat{QN} - NT.\hat{NH})}{(NT^2+NH^2+HP^2)^{\frac{1}{2}}}$$
$$+\frac{(r_1TN + r_2RN)\,NH - (r_2-r_1)(NH.QN + RN.NH)}{(RN^2+NH^2+HP^2)^{\frac{1}{2}}}$$
$$-\left(\frac{NT}{(NT^2+NH^2+HP^2)^{\frac{1}{2}}}+\frac{RN}{(RN^2+NH^2+HP^2)^{\frac{1}{2}}}\right)\frac{(r_1TN + r_2RN)\,NH.\hat{QN}}{NH^2+HP^2}\Bigg]\,d\theta\,.$$

In this form, with the identity of the various distances preserved, it is straightforward to see that only the first and last terms produce singular contributions when P approaches the surface of the frustum, and to interpret them geometrically. The strongest singularity arises in the last term through the expression:

$$E = NH/(NH^2 + HP^2)\,.$$

Recognising that

$$NH = \frac{(r_2-r_1)(x_1-x) - (x_2-x_1)(r_1-y\cos\theta)}{[(x_2-x_1)^2 + (r_2-r_1)^2]^{\frac{1}{2}}},$$

$$HP = y\sin\theta\,,$$

it is apparent that E can be expressed in the form:

$$E = \frac{B+C\cos\theta}{(B+C\cos\theta)^2 + D^2\sin^2\theta},$$

where, for points P on the line of intersection of the frustum with the positive y side of the xy plane, $B = -C$. Thus as $\theta \to 0$, $E \to 0/0$ which leads to difficulties of evaluation in a numerical procedure. However

$$E d\theta = \frac{dY}{(B^2 - C^2 + D^2)^{\frac{1}{2}}}$$

where

$$Y = \arccos\left[\frac{B\cos\theta + C}{\{(B + C\cos\theta)^2 + D^2\sin^2\theta\}^{\frac{1}{2}}}\right]$$

It is found that Y has a close affinity with the angle ϕ shown on the diagram for:

$$\cos Y = \cos\phi/\cos\alpha\,,$$

where

$$\cos\alpha = \frac{\{(x_2-x_1)^2 + (r_2-r_1)^2\cos^2\theta\}^{\frac{1}{2}}}{\{(x_2-x_1)^2 + (r_2-r_1)^2\}^{\frac{1}{2}}}$$

is another angle that can be shown on the diagram but has been omitted to avoid congestion. Thus for points P lying outside the frustum, Y rises from zero up to a maximum given by

$$Y_m = \arccos\{[(C^2-B^2)C^2/D^2]^{\frac{1}{2}}/C\}\,,$$

and then falls to zero again as θ varies from 0 to π. However for points P lying inside the frustum Y varies from π to 0 as θ varies from 0 to π. This sudden change in the range of Y produces the required discontinuity in the tangential component of $\overline{q}_p$. There is no difficulty in programming the integration of this term as the integral is now non-singular and dividing the range of Y into equal intervals, calculating θ from

$$\cos\theta = \frac{-BC(\cos^2Y-1)\pm\{(B^2-C^2+D^2)(B^2-C^2+D^2\cos^2Y)\cos^2Y\}^{\frac{1}{2}}}{(C^2-D^2)\cos^2Y - B^2}$$

and evaluating the remainder of the integrand, permits the use of a simple quadrature.

The other singular term is the first, which has a logarithmic singularity. In this case it is possible to isolate the essential singularity and integrate it as a standard form, leaving the remaining terms to be incorporated into the quadrature. However, there is insufficient space to illustrate this here.

QCAX: If P is not on the cone the integration in terms of the usual basis vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ gives an expression of the form

$$\overline{q}_p = \int_0^{2\pi}\int_0^1 \frac{\overline{A}s^2 + \overline{B}s + \overline{C}}{(as^2 + bs + c)^{3/2}}\,ds\,d\theta\,,$$

which can be evaluated in half the time needed for the calculation by QAXVOR.

QCYL: Vorticity K is distributed uniformly on a cylinder of radius r extending from x_1 to $\pm\infty$.

$$\bar{q}_p = -\frac{Kr}{2\pi}\int_0^{\pi}\left(\frac{(r-y\cos\theta)\left(-1 \pm \frac{x_1-x}{D}\right)}{y^2 - 2yr\cos\theta + r^2}\,\underset{\sim}{i} \pm \frac{\cos\theta}{D}\,\underset{\sim}{j}\right)d\theta,$$

where $D = \{(x-x_1)^2 + (y-r\cos\theta)^2 + (r\sin\theta)^2\}^{\frac{1}{2}}$.

QIN: The vorticity at (x_1,r_1) is K and it decreases inversely with radius (Figure 1).

$$\bar{q}_p = \frac{Kr_1}{2\pi}\int_0^{\pi}\left(\underset{\sim}{i} + \frac{(D-r_1+y\cos\theta)(x-x_1)\cos\theta}{(x-x_1)^2 + y^2\sin^2\theta}\,\underset{\sim}{j}\right)\frac{d\theta}{D},$$

where $D = \{(x-x_1)^2 + (y-r_1\cos\theta)^2 + (r_1\sin\theta)^2\}^{1/2}$.

Smaller integration steps are needed when (x,y) is close to the surface, and particularly when it is close to (x_1,r_1). When $x = x_1$ the j component of q_p as given by this formula is zero, which is correct only for $|y|<r_1$. When $x = x_1$ and $|y|>r_1$ the limiting form of the integrand is used for the first or last step of the numerical θ integration. This step produces a contribution $\pm Kr_1 j/(2y)$ to the velocity, which is the expected discontinuity. At the central point, $\bar{q}(x_1, 0) = K/2\ i$.

QINSQ: The vorticity at (s, α) is K and it decreases inversely as the square of the distance from $(x_o, 0)$, the apex of the cone and origin of polar co-ordinates (Figure 1).

$$\bar{q}_p = \frac{K s^2 \sin\alpha}{2\pi}\int_0^{\pi}\Big\{\Big(\sin\alpha - \frac{yd}{\sigma^2}\cos\theta\Big)\underset{\sim}{i} + \Big((x-x_o)\frac{d}{\sigma^2}$$

$$-\cos\alpha\Big)\cos\theta\,\underset{\sim}{j}\Big\}\frac{D-s+d}{D(\sigma^2-d^2)} - \{-y\underset{\sim}{i} + (x-x_o)\,\underset{\sim}{j}\}$$

$$\frac{\cos\theta}{\sigma^2}\left\{\frac{1}{\sigma}\ln\left|\frac{s(\sigma-d)}{\sigma D + \sigma^2 - sd}\right| + \frac{1}{D}\right\}$$

where $\sigma = \{(x-x_o)^2 + y^2\}^{\frac{1}{2}}$,

$d = (x-x_o)\cos\alpha + y\sin\alpha\cos\theta$,

$D = (s^2 - 2sd + \sigma^2)^{\frac{1}{2}}$.

2 DIMENSIONS: Complex co-ordinates are used to represent the i and j components of position and velocity vectors. All vorticity is parallel to k. The velocity at z induced by vorticity K distributed on a straight line joining z_1 and z_2 is given by

$$\mathrm{conjg}\left(\frac{-K}{2\pi i}\,\frac{|z_2-z_1|}{z_2-z_1}\ln\frac{z-z_2}{z-z_1}\right).$$

A pair of parallel lines ending at z_l and z_r and extending to infinity in the direction α perpendicular to the line joining z_l and z_r (left and right subscripts determined looking in the direction α) and having vorticity K distributed on the left line and -K on the right line, induces a velocity at z of

$$\mathrm{conjg}\left(\frac{Ke^{-i\alpha}}{2\pi i}\ln\frac{z-z_l}{z-z_r}\right).$$