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A SIMPLIFIED APPROACH TO THE COMPUTATION OF THE HYPERTHERMAL FREE-MOLECULAR FLOW OF A GAS-MIXTURE PAST A CONVEX BODY

by

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SUMMARY

The free-molecular hyperthermal flow of a rarefied gas-mixture past a convex solid, not necessarily cold or cooled, body of constant geometry is considered. It is assumed that the constituent incident gases, which may not be monatomic, are in equilibrium at infinity and that the boundary conditions at the surface of the body are given in terms of the accommodation coefficients, the same for each gas and independent of velocities and angle of attack. It is shown that, unlike in the case of subsonic or transonic free-molecular flows, the introduction of a certain hypothetical (singlet) velocity distribution function for the gas-mixture as a whole yields results sufficiently exact for various practical applications if an error margin of 1% - 1.5%, in comparison with the corresponding results of the "exact" theory, is permissible. Such a simplified approach can also be used (i) to compute hypersonic and supersonic free-molecular flows of similar type if an error margin of 3.5% is permissible; (ii) to evaluate very accurately the aerodynamic characteristics of a convex solid body in high altitude (upper atmosphere) flight.

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I. Main aim and contents of this paper

The main aim of this paper is to show when and how, for the regime as described below in Section II, sufficiently exact for various practical purposes, results regarding the aerodynamic quantities of interest can be obtained, on the basis of a certain (one-particle) velocity distribution function which is defined for the gas as a whole (see Section IV). [This is done for an arbitrary angle of attack.] To make the comparison between the corresponding results of the usual "exact" theory and the approximate simplified theory given here more convenient for the reader, appropriate formulae of the exact hyperthermal* free-molecular theory are given in Section III. A specification of the ranges of applicability of the theory proposed as well as the "exact" theory leads to a description of some limitations of the theories. This is given mainly in Sections II and V. The latter section also describes some related work to appear later.

II. The regime and introductory remarks

Consider the hyperthermal free-molecular flow of a rarefied non-reacting gas-mixture past a convex solid (not necessarily cold) obstacle, assuming that the boundary conditions at its surface are given in terms of accommodation coefficients (a.c.'s, for brevity) and that the constituent incident gases are each in equilibrium at infinity. More precisely, the following basic assumptions and conditions are to be satisfied:

(i) [equilibrium at infinity]
The velocity distribution function of the i^{th} gas constituent at infinity in the system of coordinates fixed in the body, $f_i(\underset{\sim}{x} \to \infty, \underset{\sim}{\xi}_i) \equiv f_{i\infty}(\underset{\sim}{\xi}_i) \equiv f_{i\infty}$, is given by

$$f_{i\infty} = \frac{n_{i\infty}}{(\pi^{\frac{1}{2}} h_{i\infty})^3} \exp\left(- \frac{(\underset{\sim}{\xi}_i - \underset{\sim}{u}_\infty)^2}{h_{i\infty}^2} \right) \qquad (i = 1,2,\ldots,K; \quad K \geq 2). \tag{1}$$

Here $n_{i\infty}$, $h_{i\infty} \equiv (2kT_\infty/m_i)^{\frac{1}{2}}$ are respectively the number density and the so-called most probable random speed of the molecules of the i^{th} gas at infinity; k is the Boltzmann constant; m_i and $\underset{\sim}{\xi}_i$ are respectively the constant mass of a molecule of the i^{th} gas and its molecular velocity; T_∞ and $\underset{\sim}{u}_\infty$ are respectively the absolute temperature and the mean uniform velocity of the gases at infinity, each being the same for every constituent.

(ii) [hyperthermality]
This is usually understood to mean the fulfilment of the condition

$$S_{i\infty} \gg 1 \quad (\forall i), \qquad S_{i\infty} \equiv u_\infty/h_{i\infty}, \qquad u_\infty \equiv |\underset{\sim}{u}_\infty|. \tag{2}$$

As is well-known $S_\infty \sim M_\infty$, where M_∞ is the Mach number in the free stream. Thus $M_{i\infty} \gg 1$ ($\forall i$) i.e. the flow is hypersonic. [Note that, e.g. for a satellite regime, the $S_{i\infty}$ are quite high (for example, see Cook [3], [4]), which means that the condition (2) can be practically satisfied.] It should be pointed out that for slender bodies (2) have to be replaced by a stronger and more precise condition ([5], p. 404)

$$S_{i\infty} \cos\theta \gg 1 \quad (\forall i), \tag{3}$$

where θ is the angle between the outward normal to the surface of the body [see (vi) below] and the free stream vector $\underset{\sim}{u}_\infty$. It is assumed in the following that this latter condition is satisfied, whenever applicable.

(iii) [free-molecularity]
We assume that the obstacle is a convex body of constant (rather smooth) shape** having a characteristic dimension L such that

$$\lambda_{i\infty}/L \equiv Kn_{i\infty} \gg 1 \quad (\forall i), \tag{4}$$

where $\lambda_{i\infty}$ is the mean free path of the i^{th} gas at infinity, taken with respect to the system of coordinates fixed with the gas. (For example, at an altitude of 160 km above the Earth $\lambda_\infty \sim 50$ m, L of a very big satellite ~ 25 m (King-Hele [7] p. 14); for information on the regimes of satellite, in particular, for values of the mean free path, see Cook [3], [4].) It should be stressed that whilst condition (4), originated by Tsien [8], was until the beginning of sixties used as a universal criterion for a rarefied gas flow to be free or almost free-molecular (e.g. see [9], [10], [11]), it is not in fact sufficient in the case of deviations

* The term "hyperthermal" was introduced by R. Schamberg [1], [2].

** It seems useful to point out that for a non-stationary shape of the body, the convexity of the body is sufficient to guarantee that collisions of the gas molecules with the body occur only once ([6], footnote on p. 402).

from moderate Mach's numbers M_∞* ([3], [6], [12], [13]). For example, under the assumption of close to perfectly diffuse boundary conditions (in the meaning as given in [5], pp. 396-398), the flow satisfying (4) is free-molecular, or almost free-molecular, if simultaneously

$$Kn_{i\infty}/S_{ir} \gg 1 \qquad (\forall i). \tag{5}$$

Here $S_{ir} \equiv u_\infty/h_{ir}$, where $h_{ir} \equiv (2kT_r/m_i)^{1/2}$, and T_r is the temperature of the re-emitted gases, the same for every constituent. (More on this latter condition, especially on values of S_{ir} in satellite flight, may be found in [3] and [4].) Practically, T_r has to be bounded from above to avoid the so-called "real gas effects", e.g. dissociation and ionization, [5] p. 15 [see also (v) below].

(iv) [over-all boundary (wall conditions)]
It is assumed that the gas-surface interaction is given in terms of three a.c.'s, namely the tangential momentum a.c. α_τ, the normal momentum a.c. α_n and the thermal a.c. α_e. The coefficients are assumed to be the same for each gas and independent of the angle of attack, speeds of incident and reflected molecules, and temperatures (including the surface temperature T_w). Note that generally the a.c.'s should not be fixed independently [5] pp. 396-398, [6] p. 102. (For very thorough and detailed considerations and discussions of these and other, often much more general and precise, a.c.'s (including their applications) see Kogan [6] § 2.10, § 6.1 and references contained there. See also reviews in [3], [4], [5] and [14].) The surface is not necessarily cold, i.e. even the situations for which $S_{iw} \le 1$ can be included, where

$$S_{iw} \equiv u_\infty/h_{iw}, \qquad h_{iw} \equiv (2kT_w/m_i)^{1/2}. \tag{6}$$

(Clearly $S_{iw} \gg 1$ represents a highly cooled body.) It is assumed that the gas molecules may have classical internal degrees of freedom. Thus the heat flux may contain contributions due to these degrees of freedom but it is assumed that there is no transfer of the translational energy into the internal energy even at impacts with the wall.

(v) [chemical neutrality]
It is assumed that the gases, and the surface of the body, are chemically neutral. Thus we omit such real gas-surface interaction effects as chemisorption at moderate temperatures and/or hypersonic speeds of incident flow ([3], especially Section 3.4).

(vi) [sufficiently smooth surface]
For mathematical convenience, to guarantee the existence of the normal to a unit surface element, we suppose that each element of the body surface is sufficiently smooth (with the exception perhaps of only a negligible set of points). This assumption, although commonly used (very often implicitly), in fact can be properly applied only if surface irregularities satisfy the condition $2\ell \cos\theta < \lambda$, where ℓ is the average height of the irregularities of the surface and λ is the de Broglie wavelength of the moving molecule (de Boer [15], Ch. III. § 19).

(vii) [absence of external forces]
It is assumed that the field of external forces is of negligible importance. In such a way we neglect in particular gravity.

III. The exact hyperthermal approximation

If the assumption (3) is satisfied, then the incident streams can be treated to a very good approximation as uniform beams of molecules moving with the same velocity $\underset{\sim}{u}_\infty$. [I.e. mathematically the Maxwellian distributions (1) are replaced by the generalized functions $n_{i\infty}\delta(\underset{\sim}{\xi}_i - \underset{\sim}{u}_\infty)$, where $\delta(\underset{\sim}{z} - \underset{\sim}{z}_0)$ is the 3-dimensional Dirac delta-function.] For the regime described in Section II, the number and mass of incident molecules per unit area per unit time striking the surface, the surface pressure, the surface shear and heat transfer rate per unit area per unit time, are given in the "exact" hyperthermal approximation, respectively by (e.g. see [5], [6] or [14])**:

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* It is generally agreed on the basis of experiments that for moderate Mach numbers M_∞ the condition $Kn_\infty \ge 3$ is usually sufficient to treat such a rarefied gas flow as free-molecular.

** The expressions (7a-7e) and (8a-8e) are particular but generalized to mixtures, and in some cases slightly modified (e.g. to suit the notation used), versions or implications of some formulae from the references quoted. Note that the formulae for the total heat flux are derived in [5] and [14] without stressing that they are correct in so far as the assumption given in the last statement of (iv) [see above] is satisfied. As has been pointed out by Kogan [6], § 6.1 this assumption represents a significant restriction in the case of hypersonic flows.

$$\left.\begin{array}{l} N^+ \\ M^+ \\ p \\ \tau \\ Q \end{array}\right\} = \cos\theta \times \begin{cases} \Sigma n_{i\infty}\, u_\infty, & \text{(7a)} \\ \Sigma \rho_{i\infty}\, u_\infty, & \text{(7b)} \\ \Sigma \rho_{i\infty}\, u_\infty^2\, [(2 - \alpha_n)\cos\theta + \tfrac{1}{2}\alpha_n \pi^{\frac{1}{2}} S_{iw}^{-1}], & \text{(7c)} \\ \Sigma \alpha_\tau \rho_{i\infty}\, u_\infty^2 \sin\theta, & \text{(7d)} \\ \Sigma \tfrac{1}{2}\alpha_e \rho_{i\infty}\, u_\infty^3\, (1 - \varepsilon_i S_{iw}^{-2}), & \text{(7e)} \end{cases}$$

where

$$\rho_{i\infty} \equiv m_i n_{i\infty}, \qquad \varepsilon_i \equiv \tfrac{1}{2}(\gamma_i + 1)/(\gamma_i - 1).$$

Hereafter $\Sigma \equiv \Sigma_{i=1}^{K}$; $\rho_{i\infty}$ denotes the ordinary density of the i^{th} gas at infinity, γ_i is the i^{th} gas isentropic constant (also often called "specific heat ratio in the free stream"), i.e. $\tfrac{1}{2}/\varepsilon_i$ represents the limiting density ratio for strong shocks in the perfect i^{th} gas; $\varepsilon_i = 2$ for monatomic gases, and $\varepsilon_i = 3$ for diatomic gases at moderate temperatures; S_{iw} is defined by (6).

It is easily seen that the set of the quantities given by (7a-7e) depends as a whole on the surface temperature T_w, but all of the quantities are independent of T_∞. (Of course this is a result connected with the abovementioned delta-function representation.) Hence, if for all i the conditions $\tilde{S}_{i\infty} \cos\theta \gg 1$ are satisfied, where $\tilde{S}_{i\infty} \equiv u_\infty/\tilde{h}_{i\infty}$, $\tilde{h}_{i\infty} \equiv (2kT_{i\infty}/m_i)^{\frac{1}{2}}$, then in the expressions for each $f_{i\infty}$ [given by (1)] T_∞ could be replaced by these, not necessarily equal, $T_{i\infty}$ without producing any change in (7a-7e).

Of course for $\cos\theta = 0$ (i.e. for zero angle of attack to the incident stream) the conditions (3) can no longer be satisfied. In this case, the <u>exact</u> free-molecular theory yields the following results (again see [5], [6] or [14]):

$$\left.\begin{array}{l} N^+ \\ M^+ \\ p \\ \tau \\ Q \end{array}\right\} = \tfrac{1}{2}\pi^{-\frac{1}{2}} \times \begin{cases} \Sigma n_{i\infty}\, h_{i\infty}, & \text{(8a)} \\ \Sigma \rho_{i\infty}\, h_{i\infty}, & \text{(8b)} \\ \Sigma \pi^{\frac{1}{2}} \rho_{i\infty}\, u_\infty^2\, [\tfrac{1}{2}\alpha_n S_{iw}^{-1} S_{i\infty}^{-1} + (1 - \tfrac{1}{2}\alpha_n) S_{i\infty}^{-2}], & \text{(8c)} \\ \Sigma \alpha_\tau \rho_{i\infty}\, u_\infty^2\, S_{i\infty}^{-1}, & \text{(8d)} \\ \Sigma \tfrac{1}{2}\alpha_e \rho_{i\infty}\, u_\infty^2\, h_{i\infty}(1 - \varepsilon_i S_{iw}^{-2} + \varepsilon_i S_{i\infty}^{-2}). & \text{(8e)} \end{cases}$$

In contrast to the previous case all the expressions (8a-8e) depend on T_∞ as could be expected. It can be also noted that N^+, M^+ and p are independent of u_∞, but τ and Q are strictly increasing functions of u_∞. Of course, in view of the hyperthermality conditions these terms which contain $S_{i\infty}^{-2}$ as a factor could be rejected. Nevertheless as it will be seen in Section IV it is not really necessary, and since it could be of interest to retain the terms, we present them explicitly. (It seems useful to point out that for a not necessarily cold body other terms cannot be rejected even for the abovementioned hyperthermal conditions.)

IV. A simplified approach

Let us compare the formulae (7a-7e) and (8a-8e) of Section III with the corresponding formal results (denoted in what follows by $\hat{N}^+$, $\hat{M}^+$, $\hat{p}$, $\hat{\tau}$ and $\hat{Q}$ respectively) obtained if one replaces the set of distribution functions as given by (1) by a certain hypothetical distribution function:

$$\hat{f}(\underset{\sim}{x}\to\infty, \underset{\sim}{\xi}) \equiv \hat{f}_\infty(\underset{\sim}{\xi}) = \frac{n_\infty}{(\pi^{\frac{1}{2}}\bar{h}_\infty)^3} \exp\left[-\frac{(\underset{\sim}{\xi} - \underset{\sim}{u}_\infty)^2}{\bar{h}_\infty^2}\right], \tag{9}$$

where

$$n_\infty \equiv \Sigma n_{i\infty}, \quad \bar{h}_\infty \equiv (2kT_\infty/\bar{m})^{\frac{1}{2}}, \quad \bar{m} \equiv \rho_\infty/n_\infty, \quad \rho_\infty \equiv \Sigma\rho_{i\infty}, \tag{10}$$

and uses the additional averaged quantity

$$\bar{\varepsilon} = \Sigma\varepsilon_i n_{i\infty}/n_\infty \tag{11}$$

as the isentropic constant for such a hypothetical gas.

For the formal hyperthermal approximation [more precisely, for $\bar{S}_\infty \cos\theta \gg 1$, where $\bar{S}_\infty \equiv u_\infty/\bar{h}_\infty$, together with the other similarly modified assumptions of Section I] we obtain from (7a-7e), using (9) as well as (10) and (11),

$$N^+ = \hat{N}^+, \qquad \text{where } \hat{N}^+ = n_\infty u_\infty \cos\theta; \qquad (12a)$$

$$M^+ = \hat{M}^+, \qquad \text{where } \hat{M}^+ = \rho_\infty u_\infty \cos\theta; \qquad (12b)$$

$$p = \hat{p} + \tfrac{1}{2}\alpha_n \pi^{\frac{1}{2}}(c_1 - 1)\rho_\infty u_\infty^2 (1/\bar{S}_w)\cos\theta, \qquad (12c)$$

$$\text{where } \hat{p} = \rho_\infty u_\infty^2[(2 - \alpha_n)\cos\theta + \tfrac{1}{2}\alpha_n \pi^{\frac{1}{2}}/\bar{S}_w]\cos\theta, \qquad (12c)_1$$

$$c_1 = (\bar{m}^{\frac{1}{2}} n_\infty)^{-1} \Sigma m_i^{\frac{1}{2}} n_{i\infty} \equiv (\bar{m}^{\frac{1}{2}}/\rho_\infty)\Sigma\rho_{i\infty}/m_i^{\frac{1}{2}}, \qquad (12c)_2$$

$$\bar{S}_w \equiv u_\infty/\bar{h}_w, \quad \bar{h}_w \equiv (2kT_w/\bar{m})^{\frac{1}{2}}; \qquad (12c)_3$$

$$\tau = \hat{\tau}, \qquad \text{where } \hat{\tau} = \tfrac{1}{2}\alpha_\tau \rho_\infty u_\infty^2 \sin 2\theta; \qquad (12d)$$

$$Q = \hat{Q}, \qquad \text{where } \hat{Q} = \tfrac{1}{2}\alpha_e \rho_\infty u_\infty^3 (1 - \bar{\varepsilon}/\bar{S}_w^2)\cos\theta. \qquad (12e)$$

In a similar way, we obtain for $\cos\theta = 0$ [from (8a-8e)]

$$N^+ = c_2\hat{N}^+, \qquad \text{where } \hat{N}^+ = \tfrac{1}{2}\pi^{-\frac{1}{2}} n_\infty \bar{h}_\infty, \qquad (13a)$$

$$c_2 = (\bar{m}^{\frac{1}{2}}/n_\infty)\Sigma n_{i\infty}/m_i^{\frac{1}{2}}; \qquad (13a)_1$$

$$M^+ = c_1\hat{M}^+, \qquad \text{where } \hat{M}^+ = \tfrac{1}{2}\pi^{-\frac{1}{2}} \rho_\infty \bar{h}_\infty; \qquad (13b)$$

$$p = \hat{p}, \qquad \text{where } \hat{p} = \tfrac{1}{2}\rho_\infty u_\infty^2 [\tfrac{1}{2}\alpha_n/\bar{S}_w + (1 - \tfrac{1}{2}\alpha_n)/\bar{S}_\infty]/\bar{S}_\infty; \qquad (13c)$$

$$\tau = c_1\hat{\tau} \qquad \text{where } \hat{\tau} = \alpha_\tau \rho_\infty u_\infty^2/\bar{S}_\infty; \qquad (13d)$$

$$Q = \hat{Q} + \tfrac{1}{4}\pi^{-\frac{1}{2}} \alpha_e \rho_\infty \bar{h}_\infty [(c_1 - 1)u_\infty^2 - \bar{\varepsilon}(c_3 - 1)(\bar{h}_w^2 - \bar{h}_\infty^2)], \qquad (13e)$$

$$\text{where } \hat{Q} = \tfrac{1}{4}\pi^{-\frac{1}{2}} \alpha_e \rho_\infty u_\infty^2 \bar{h}_\infty(1 - \bar{\varepsilon}/\bar{S}_w^2 + \bar{\varepsilon}/\bar{S}_\infty^2), \qquad (13e)_1$$

$$c_3 = (\bar{m}^{\frac{1}{2}}/\bar{\varepsilon} n_\infty)\Sigma\varepsilon_i n_{i\infty}/m_i^{\frac{1}{2}}. \qquad (13e)_2$$

As we see, the simplified approach generally gives results completely equivalent to the results of the exact theory if and only if $c_j = 1$ $(j = 1,2,3)$. This precise but somewhat vague mathematical criterion can be replaced by a set of weaker statements which seems to be much more useful and convenient for practical and theoretical applications. First of all, it is clearly seen from (12a-12e) and (13a-13e) that the coefficients c_j are not equally important. In fact, in each case (i.e. either when $S_{i\infty} \cos\theta \gg 1$, or when $\cos\theta = 0$) a comparison of the order of the terms involved, taking into account the assumptions (ii) of Section II, shows the special importance of having an accurate value of the coefficient c_1, which appears as a factor in leading terms of the appropriate simplified expressions*. (In patricular, for a "cold" or "lukewarm" body this is the only essential coefficient. Of course, for a "highly cooled" body, the hyperthermal case $\bar{S}_\infty \cos\theta \gg 1$ does not depend on any coefficients at all, while the case $\cos\theta = 0$ depends only on c_1.) It is also clear that a very accurate value of the coefficient c_2 is of least importance from the aerodynamical point of view. Firstly, N^+ is a quantity of little importance from the aerodynamical point of view since, while dealing with gas-mixtures, one is (and should be) concerned with M^+ rather than with N^+. Secondly, from the same point of view the main quantities of interest are not M^+ or N^+ but p, τ and Q, since a direct integration of these over the surface of the body leads to evaluation of the aerodynamic characteristics of the body (namely the lift force, the drag force and the net thermal flow). While establishing the importance of the coefficients c_j in applications, it should be also realised that contributions from the grazing angle of incidence, being proportional to products of positive integer powers of $h_{i\infty}$ and h_{iw}, are usually very small compared with the corresponding contributions of the hyperthermal case. These small contributions normally produce even smaller contributions into the total aerodynamic characteristics, if a realistic solid flying object is considered, since for such an object the locus of the points of its surface defined by the condition $\cos\theta = 0$ constitutes only a small or even negligible portion of the surface. This latter remark is especially applicable to blunt, smoothly shaped bodies (e.g. for a sphere the locus is one of its great circles). Of course the situation can be drastically different for some (slender) bodies. A flat plate at zero angle of attack presents an extreme, although a not very realistic**, example supporting this claim. But for such specific shapes one may expect,

* More precisely, the closer the coefficient c_1 is to unity, the better the agreement between the leading terms of both (original and simplified) approaches.

** In particular, some specific features of the so-called "leading edge problem" produce strong deviations from the free-molecular regime in front of this object. Also note that there exist essential contradictions within and between appropriate theoretical and experimental results. (For some interesting discussion and considerations see Kogan [6], § 6.6.)

on the basis of experiments and related theoretical results [1], [2], [16], [17], [18], that the values of* τ and $\hat{\tau}$ as well as Q and $\hat{Q}$ are rather small since in the regime under consideration the reflection pattern shows a strong tendency towards an almost fully specular reflection for which $\alpha_n \approx \alpha_\tau \approx \alpha_e \approx 0$. (For a precise definition of the fully specular reflection see [5], Section 10.3.) The somewhat general considerations and conclusions presented above can be complemented and made more precise and accurate by some rather elementary but lengthy estimates, computations and standard investigations connected with the study of behaviour of some multivariate functions. The additional analysis and computations show that the coefficients c_j are very close to unity with a deviation from unity ≤ 1% for various mixtures of practical importance, for example such as air at altitudes ≥ 150 km (see [19]). The coefficients also appear to be quite conservative and stable with respect to moderate variations of the actual components of a gas-mixture. These latter properties of the coefficients are connected with their particular forms, in which the smoothing action of the square root operators is combined with the specific semi-averaging procedure. (Strictly speaking, for the stability of c_3 an additional factor, namely, a sufficiently close range of the ε_i's involved, is of importance.) Not all of the work that has been done, and the results obtained, will be presented here. This is not so much motivated by the limitations on the size of paper in these Proceedings as by the strong conviction of this author that for various ranges of the parameters involved there exists an elegant and probably much shorter analytical way of obtaining precise estimates of c_j on the basis of some nontrivial manipulations involving the number theory.

This point of view can be well explained and illustrated by presenting some of the study connected with estimates for parameter c_1. First of all, by the Cauchy-Schwartz inequality,

$$(\Sigma m_i n_{i\infty})^{\frac{1}{2}}(\Sigma n_{i\infty})^{\frac{1}{2}} \geq \Sigma m_i^{\frac{1}{2}} n_{i\infty} \Rightarrow c_1 \leq 1. \tag{14}$$

Also it can be easily proved that c_1 is bounded from below by $\min_{\{i\}} m_i^{\frac{1}{2}}/\max_{\{i\}} m_i^{\frac{1}{2}}$. In fact

$$c_1 \geq \frac{\Sigma m_i n_{i\infty}/\max_{\{i\}} m_i^{\frac{1}{2}}}{(\Sigma m_i n_{i\infty})^{\frac{1}{2}}(\Sigma m_i n_{i\infty})^{\frac{1}{2}}/\min_{\{i\}} m_i^{\frac{1}{2}}} = \min_{\{i\}} m_i^{\frac{1}{2}}/\max_{\{i\}} m_i^{\frac{1}{2}} \equiv c. \tag{15}$$

[Clearly this lower bound can be attained (e.g. if $(\forall i)\ m_i = m = \text{const.}$).]
It is easily seen from (15) that the closer the m_i's are to each other, the closer c_1 is to unity, as would be expected intuitively. On the other hand, despite the smoothing out due to the square roots in the numerator and denominator, the ratio c is too crude to provide a realistic bound for such cases when $\exists\, i,j\colon m_i \gg m_j$. It also follows from analytical and numerical computations that even for the case of two gases with quite close m_i's (say $m_1 = 2m_2$), where these computations given an error ≈ 1% [see the table below], the estimate (15) gives a significantly less accurate result. Unfortunately, despite some attempts by this author, an essential improvement on (15) is still lacking, except for some particular cases, e.g. a two-component gas-mixture. Results obtained for the latter case are of special significance since they can be used as a starting point for an easy procedure which can cover quite accurately the case of an arbitrary non-reacting gas-mixture of more than two components if the molecular masses are sufficiently close. The procedure consists of repeatedly reducing the number of species by one, replacing a pair of species by a single hypothetical gas, and evaluating an accumulative error. At each step the pairs of species with the closest m_i's should be always replaced first and, of course, the process can be always restricted to the replacement of only certain chosen pairs. In practice usually only a few components of a gas-mixture are really important. If this is so, the number of species can be reduced simply by rejecting those species for which $(\forall i \neq i')\ m_{i'} \ll m_i,\ n_{i'} \ll n_i$, since for each i^{th} gas M_i^{+}, p_i, τ_i and Q_i are proportional to $\rho_{i\infty}$. We now describe the two-species gas case in more detail. For two species, c_1 can be reduced to the following form:

$$c_1 = (1 + \nu\mu^{\frac{1}{2}})[(1 + \nu\mu)(1 + \nu)]^{-\frac{1}{2}}, \qquad \nu \equiv n_{2\infty}/n_{1\infty}, \qquad \mu \equiv m_2/m_1.$$

The behaviour of this c_1 as a continuous function of ν and μ has been studied, using some simple standard calculus and numerical computations, taking also into account natural physico-chemical restrictions on the range of μ. The main results will now be shortly described. (A more complete description, including appropriate graphs and numerical results, is contained in [20], where a more general problem is considered.) It was found that c_1 is strictly monotonic in μ for any fixed ν, namely, $c_1(\mu)$ is increasing for $\mu < 1$, decreasing for $\mu > 1$, and has its maximum at $\mu = 1$ [this latter result can also be easily derived from the estimate (15) combined with the implication in (14)]. More precisely, $\forall$ (fixed) $\nu_0 > 0$, $c_1(\mu;\nu_0)$ is quite rapidly (or even very rapidly for some ranges of ν_0) increasing if** $\mu \in (0, 0.4)$, then increasing at less

* Since values of p and $\hat{p}$ are identical if $\cos\theta = 0$, no discussion of these quantities is needed.

** Hereafter all numerical values except the number 1 are approximate with an error ≤ 1%.

than one-third that rate if $\mu \in (0.4, 1)$, and finally decreasing rather slowly for $\mu \in (1, 11)$ [for example, $\forall$(fixed) $\nu_0 > 0$, $c_1(11;\nu_0) \approx c_1(0.3;\nu_0)$]. For $\mu > 11$, c_1 is generally decreasing almost at the same rate as at $\mu = 11$.

Some useful numerical results for $c_1(\mu;\nu)$, restricted to $\nu \in (0.1, 10)$, are given in the following self-explanatory table:

μ	(0.3, 0.4)	(0.4, 0.6)	(0.6, 1.75)	(1.75, 2.0)	(2.0, 5.0)	(5.0, 11.0)
c_1	$\gtrsim 0.90$ $\lesssim 0.99$	$\gtrsim 0.95$ $\lesssim 1.0$	$\gtrsim 0.99$ $\lesssim 1$	$\gtrsim 0.98$ $\lesssim 1.0$	$\gtrsim 0.94$ $\lesssim 0.99$	$\gtrsim 0.91$ $\lesssim 0.98$

Clearly the optimal, almost stable, values of c_1 are obtained for $\mu \in (0.6, 2.0)$. It is also clear, that for some quite realistic values of μ_0 and ν_0, c_1 can be considerably less than 1, e.g. $c_1(0.04;10) \approx 0.76$. Since c_1 is an invariant under the nonsingular $[(\forall i) m_i,\ n_{i\infty} \in Z^+]$ transformation $(\mu,\nu) \to (\mu^{-1},\nu^{-1})$, the value for $c_1(\mu_0;\nu_0)$ is simultaneously the value for $c_1(\mu_0^{-1};\nu_0^{-1})$, a fact which has also been used in the above computations.

We have presented here some parts of our study on c_1. Rather similar, but by no means identical, considerations and computations lead to similar results for the essentially less important coefficients c_2 and c_3. As has been mentioned above, an exact extension of the considerations for a two-species gas mixture, to general gas mixtures of more than two species, is not simple.

V. Discussion, conclusions and continuing investigations

It has become a somewhat irresponsible custom that when describing and solving problems of motion of solid objects in a highly rarefied atmosphere, such an atmosphere, which is actually a gas-mixture, is treated simply as a monatomic gas of mean molecular weight. To the best knowledge of this author, neither a justification for such an approach, nor a precise way of replacing a gas-mixture by a single gas has been given so far. It is shown in this paper how to obtain a certain rather natural replacement by a single hypothetical gas for a general neutral gas-mixture in the regime described in Section II. It is also specified, using mainly mathematical manipulations and computations, when and where the replacement can be valid. It is clearly seen from Section IV that the problem in itself is far from being trivial and it seems certain that no simple replacement by a single monatomic gas can be valid for arbitrary gas-mixtures. The results obtained, which describe N^+, M^+, p, τ and Q in the terms of $\hat{N}^+$, $\hat{M}^+$, $\hat{p}$, $\hat{\tau}$, $\hat{Q}$ and c_1, c_2, c_3 [see (12a-12e) and (13a-13e)] can be used directly to evaluate the first set of quantities. The calculations are shorter than the original ones (since separate calculations of contributions due to each gas constituent are omitted) but still completely exact. One must add that in practice extreme precision in finding values of c_j seems to be useless if one considers the order of errors introduced by basic theoretical assumptions (cf. [3], [4]). In a particular case of high altitude (Earth's satellite) flights in conditions adequate to those of Section II (i.e. roughly speaking, at altitudes $\gtrsim$ 150 km and $\lesssim$ 800 km) one can check, using the available data (e.g. [3], [4]), that the simplified approach generally works with a very good accuracy, but not always, and only if a well-ordered selection of important components is taken successively (not simply by taking a mean molecular weight at an altitude as has been usually thought; e.g. cf. [3], [4]). A more detailed consideration, including (i) a discussion of "real gas effects", such as dissociation and ionization, (ii) a discussion of additional difficulty with the fulfilment of the basic condition $N^+ = N^-$ (where N^- is the outgoing number flux) due to physisorption and chemisorption, which is much less readily reversible than physisorption, and (iii) a discussion of the importance of the molecular ratio m/m_w (where m_w is the constant mass of a surface molecule), are also given in [19]. Some generalizations and extensions are of significant interest: (i) The above results, in terms of a.c.'s (independent of θ, T_w and u_∞, as well as re-emission parameters) suggest that such models (more flexible and realistic in many circumstances) as, for example, the Nocilla model* [16], [17], [18] containing diffuse and specular reflections as limiting cases, or Schamberg's model [1], [2], can be also included under some additional restrictions. (Note that any model other than the classical Maxwell's approximation yields a dependence of the a.c.'s on θ and T_w.) The Nocilla model is currently being considered [20] and it appears that this model can be included if the additional restrictions are imposed on parameters U_r and θ_r [18] connected with the macroscopic "drift" velocity; in particular, this velocity should not be too large. [Note that the behaviour of the model for the grazing angle of incidence exhibits a tendency to a specular reflection.] (ii) The hyperthermality requirements (2) or (3) can be weakened, to include ordinary hypersonic or even moderate supersonic flows of similar type to the above, if a greater error margin is permitted (e.g. $\lesssim$ 3.5% instead of $\lesssim$ 1%). A motivation for this is mainly based

* A very interesting departure from the Nocilla model to a class of Maxwellian type models involving more parameters has been proposed and investigated quite recently by T. Marshall (Jr.) [21].

on the similitude properties of the passage from $S_{i\infty} \gg 1$ to finite $S_{i\infty}$'s of order 3.5 or 4 ([5], Section 10.5), and the conclusion has been already strongly supported by some computations. (iii) A non-trivial generalization of the results can also be obtained by introducing some suitable $T_{i\infty}$ and T_{iw} and then averaging them in order to fit the formulae (7a-7e) and (8a-8e).

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