21st Australasian Fluid Mechanics Conference
Adelaide, Australia
10-13 December 2018

Nonlinear Exact Coherent Structures in power-law fluids in pipe geometry*

Ozge Ozcakir

School of Mathematical Sciences
Monash University, Clayton, Victoria, 3800, Australia

Abstract

Coherent structures are believed to play an important role in transition to turbulence. Most work done in investigation of coherent structures in pipe geometry has been limited to numerical calculation of solutions for Newtonian fluids at moderate Reynolds numbers [18, 17, 9].

In the present work, we develop a code to solve Navier-Stokes equation for non-Newtonian fluids to find nonlinear exact coherent structures in the form of three-dimensional travelling wave solutions. We present our preliminary findings at low Reynolds numbers only for few viscosity parameters using a simple Power-law fluids model.

Themes: :Non-Newtonian flows, Pipe flows

Introduction

Research on exact coherent structures in flows in pipes, channels and boundary-layer flows has attracted much recent interest. There have been several investigations of travelling wave (TW) solutions for Newtonian fluids. Such solutions have been computed by [8], [14], [15], [16], [4] and [1] in channel flow. [3], [6], [17], [7], [9], [13] have given related results for pipe flow with different degrees of rotational symmetry. There is evidence from, for example, [12] to suggest that travelling wave (TW) states can be part of the edge state that characterizes the boundary of the basin of attraction of the linearly stable Plane Couette or pipe Pouisseuille flow. These nonlinear TW states are also of potential technological importance if suitable controls can be inserted to stabilise a coherent state with a significantly smaller drag than an uncontrolled turbulent flow. The question of the influence of a non-Newtonian rheology on these traveling waves is of fundamental interest, since these waves are believed to play important role in transition to turbulence.

The physical mechanism to sustain such steady states in Newtonian fluids for large Reynolds number is now well-understood. In general for pipe flow, three dimensional traveling wave solutions to Navier-Stokes equations can be written, in cylindrical coordinates (r,θ,z), of the form

$$\mathbf{u} = \mathbf{v}_B(r) + \mathbf{U}(r,\theta) + \mathbf{v}_w(r,\theta,z-ct) \tag{1}$$

where $\mathbf{v}_B(r)$ is the base flow ($(1-r^2)\hat{\mathbf{z}}$ for Newtonian fluids) , $\mathbf{U}(r,\theta)$ is streamwise independent part of the flow and $\mathbf{v}_w = (u,v,w)$ is 2π periodic in both θ and in $\tilde{z} := \alpha(z-ct)$, with zero axial average over a period, denoted by $\langle \mathbf{v}_w \rangle = 0$. It is assumed that cylinder axis is aligned along z. If we write $\mathbf{U}(r,\theta) = (U(r,\theta),V(r,\theta),0) + (0,0,W(r,\theta))$, the first term contains radial and azimuthal components of streamwise independent velocity $\mathbf{U}(r,\theta)$ which represents streamwise vortices and is referred to as the *roll* part of the flow; the latter is termed the *streak* and represents streamwise-independent axial velocity. The last term in (1) represents three dimensional wave part of the flow. Three way interaction between rolls, streaks and waves is essential to sustain these states.

In Ozcakir *et al.*[11], previous numerically calculated states with two-fold azimuthal periodicity in pipe geometry were roughly identified as finite R realizations of a Vortex Wave Interaction (VWI) states with an asymptotic structure similar to the ones in channel flows studied earlier by Hall & Sherwin [5]. In the context of non-Newtonian fluids, there hasn't been much work done that focuses on the study of travelling waves. Recently [19] investigated the effect of shear thinning on three-fold rotationally symetric travelling waves using Carreu Model.

The aim of this short paper is to briefly describe the methodology used in computing nonlinear travelling waves in non-Newtonian fluids in general and to present our initial findings for non-Newtonian counterparts of VWI states described in [11].

Description of the problem

We consider a flow driven by a constant pressure gradient along a pipe of radius a where the velocity field $\mathbf{u}$ and pressure p satisfy

$$\rho[\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}] = -\nabla p + \nabla\cdot\left[\eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathbf{T}})\right],\ \nabla\cdot\mathbf{u} = \mathbf{0}. \tag{2}$$

As mentioned in the introduction, TW states correspond to solutions of the Navier-Stokes of the form $\mathbf{u} = \mathbf{v}_B(r,\theta) + \mathbf{v}(r,\theta,z-ct)$ where the first term $\mathbf{v}_B$ is the base state and $\mathbf{v}$ is the perturbation velocity which travels with real phase speed c. It is clear that $\mathbf{v}$ satisfies

$$\begin{aligned} c\frac{\partial\mathbf{v}}{\partial z} - \mathbf{v}\cdot\nabla\mathbf{v} - \mathbf{v_B}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{v_B} - \nabla\Delta^{-1}\mathcal{N}[\mathbf{v}] = \\ -\nabla\cdot\left[\nu(\mathbf{u})(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathbf{T}})\right] \\ -\nabla\cdot\left[\nu(\mathbf{u})(\nabla\mathbf{v_B} + (\nabla\mathbf{v_B})^{\mathbf{T}})\right] \\ +\nabla\cdot\left[\nu(\mathbf{v_B})(\nabla\mathbf{v_B} + (\nabla\mathbf{v_B})^{\mathbf{T}})\right], \end{aligned} \tag{3}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{4}$$

where $\eta(\mathbf{u}) = \rho\nu(\mathbf{u})$ and the perturbed pressure $q = \Delta^{-1}\mathcal{N}[\mathbf{v}]$ is determined by solving $\Delta q = \mathcal{N}[\mathbf{v}]$ with Neumann boundary condition $\frac{\partial q}{\partial n} = \mathcal{N}_b[\mathbf{v}]$ at the pipe wall. Here

$$\begin{aligned} \mathcal{N}[\mathbf{v}] := -\nabla\cdot\left[(\mathbf{v}\cdot\nabla)\mathbf{v} + \mathbf{v_B}\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v_B}\right] \\ -\nabla\cdot\nabla\cdot\left[\nu(\mathbf{u})(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathbf{T}})\right] \\ -\nabla\cdot\nabla\cdot\left[\nu(\mathbf{u})(\nabla\mathbf{v_B} + (\nabla\mathbf{v_B})^{\mathbf{T}})\right], \end{aligned} \tag{5}$$

$$\begin{aligned} \mathcal{N}_b[\mathbf{v}] := -\nabla\cdot\left[\nu(\mathbf{u})(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathbf{T}})\right] \\ +\nabla\cdot\left[\nu(\mathbf{v_B})(\nabla\mathbf{v_B} + (\nabla\mathbf{v_B})^{\mathbf{T}})\right] \\ -\nabla\cdot\left[\nu(\mathbf{u})(\nabla\mathbf{v_B} + (\nabla\mathbf{v_B})^{\mathbf{T}})\right], \end{aligned} \tag{6}$$

In our computations, we find it more efficient to eliminate the pressure through a Poisson equation rather than by enforcing the divergence condition directly. For this reason, instead of using $-\nabla q$ in (3), we replace it using the notation $-\nabla\Delta^{-1}\mathcal{N}[\mathbf{v}]$ merely to emphasize that pressure modes do not appear in the

Newton iteration scheme for traveling waves or stability calculations since are eliminated in terms of velocity modes.

Base flow using Power-Law

For power-law fluids, viscosity is described as

$$\eta = K\dot{\gamma}^{n-1} \tag{7}$$

with shear rate $\dot{\gamma}$, consistency constant K and flow index n where $n < 1$ describes shear thinning, $n = 1$ newtonian and $n > 1$ shear thickening fluids.

If we assume that the base flow is of the form $v_B(r) = (0, 0, w_B(r))$, shear rate is calculated as $\dot{\gamma}_B(r) = |w'_B(r)|$. Using Navier-Stokes equations (2), we get

$$\eta_B w'_B(r) = \frac{r}{2}\frac{\partial p}{\partial z} \tag{8}$$

where $\frac{\partial p}{\partial z}$ is assumed constant pressure gradient and η_B is the viscosity for base flow defined as

$$\eta_B = K\dot{\gamma}_B{}^{n-1} = K(-w'_B(r))^{n-1}.$$

Hence, (8) becomes

$$(-1)^{n-1}Kw'_B(r)^n = \frac{r}{2}\frac{\partial p}{\partial z}$$

which has a solution

$$w_B(r) = U_c\left(1 - \left(\frac{r}{a}\right)^{\frac{n+1}{n}}\right)$$

where $U_c = a^{\frac{n+1}{n}}\dfrac{n}{n+1}\left(-\dfrac{1}{2K}\dfrac{\partial p}{\partial z}\right)^{\frac{1}{n}}$ is the centerline velocity. (Note that K $\frac{\partial p}{\partial z} < 0$.)

Corresponding pressure gradient in terms of velocity can be written as,

$$\frac{\partial p_B}{\partial z} = -\frac{2KU_c^n}{a^{n+1}}\left(\frac{n+1}{n}\right)^n$$

Nondimensionalisation

For our calculations of non-newtonian flows, we will choose viscosity scale as η_w (mean wall viscosity) where Reynolds number $Re = U_c a\rho/\eta_w$ for a pipe of radius a with centerline velocity of the base flow is U_c. Mean wall viscosity η_w can be calculated using axial momentum equation balance in terms of given pressure gradient. It is calculated from the mean wall shear stress, τ_w which is directly

$$\tau_w = -\frac{a}{2}\frac{\partial p}{\partial z} \tag{9}$$

For Power-Law fluids, it can be shown that mean wall viscosity is

$$\eta_w = K^{1/n}\tau_w^{1-1/n} \tag{10}$$

Therefore, Navier-Stokes equations (2) is written in the following non-dimensional form

$$[\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}] = -\nabla p + \frac{1}{Re}\nabla\cdot\left[\eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathbf{T}})\right],\ \nabla\cdot\mathbf{u} = \mathbf{0}. \tag{11}$$

with $w_B(r) = 1 - r^{\frac{n+1}{n}}$ and $\frac{\partial p_B}{\partial z} = -\frac{2K}{Re}(\frac{n+1}{n})^n$.

Computational Method

The efficient and accurate numerical calculation of the three dimensional travelling wave solutions of the Navier Stokes equations is challenging due to highly nonlinearity of the viscosity term. This requires a large number of modes which as a result leads to large matrix inversion problem for the associated Newton iteration scheme. Our computational method is based on a Galerkin truncation in the Fourier-modes in θ and $\tilde{z} = \alpha(z - ct)$ and collocation method that uses a Chebyshev representation in r with appropriate radial basis functions Φ_j, Ψ_j. The procedure is similar to that used by [17] and automatically accounts for the boundary condition. We use the following truncated basis representation for the velocity field

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \sum_{\substack{0\le j\le N \\ 0\le k\le M \\ 0\le l \text{ even}\le P}} \begin{pmatrix} (u^{(1)}_{jkl}\cos l\tilde{z} + u^{(2)}_{jkl}\sin l\tilde{z})\Phi_j(r;kk_0)\cos kk_0\theta \\ (v^{(1)}_{jkl}\cos l\tilde{z} + v^{(2)}_{jkl}\sin l\tilde{z})\Phi_j(r;kk_0)\sin kk_0\theta \\ (w^{(1)}_{jkl}\sin l\tilde{z} + w^{(2)}_{jkl}\cos l\tilde{z})\Psi_j(r;kk_0)\cos kk_0\theta \end{pmatrix}$$

$$+ \sum_{\substack{0\le j\le N \\ 0\le k\le M \\ 1\le l \text{ } odd\le P}} \begin{pmatrix} (u^{(1)}_{jkl}\cos l\tilde{z} + u^{(2)}_{jkl}\sin l\tilde{z})\Phi_j(r;kk_0)\sin kk_0\theta \\ (v^{(1)}_{jkl}\cos l\tilde{z} + v^{(2)}_{jkl}\sin l\tilde{z})\Phi_j(r;kk_0)\cos kk_0\theta \\ (w^{(1)}_{jkl}\sin l\tilde{z} + w^{(2)}_{jkl}\cos l\tilde{z})\Psi_j(r;kk_0)\sin kk_0\theta \end{pmatrix}. \tag{12}$$

which is suitable for k_0 fold azimuthally symmetric ($\mathbb{R}_{k_0}$) traveling wave states with shift-and-reflect ($\mathbb{S}$) symmetry, as discussed in [10]. This representation fixes the origin in θ, even in the case when basic state $\mathbf{v}_B = \mathbf{v}_P$ is rotationally symmetric θ. In addition, to fix the origin in z, we impose the phase condition

$$\sum_{j=0}^{N} u^{(1)}_{j,1,1} = 0 \tag{13}$$

We use representation (12) and equate coefficients of $\cos(kk_0\theta)\cos(l\tilde{z})$, $\sin(kk_0\theta)\cos(l\tilde{z})$, $\cos(kk_0\theta)\sin(l\tilde{z})$, and $\sin(kk_0\theta)\sin(l\tilde{z})$ for $0 \le k \le M$, $0 \le l \le P$ on both sides of (3) and evaluate resulting expressions at the given collocation points $\{r_{j'}\}$. This, together with scalar equation (13), results in a nonlinear system of algebraic equations for $(\mathbf{X}, c)$ in the form $\mathbf{G}(\mathbf{X}, c; \beta) = 0$ where $\mathbf{X} = \left\{u^i_{jkl}, v^i_{jkl}, w^i_{jkl}\right\}_{j,k,l,i}$ and $\beta = (\alpha, K, n)$ is the set of specified parameters. Details of our numerics and imposed symmetries can be found in [11].

The pressure elimination is efficacious in TW calculations since the inversions of relatively small matrices are involved for each Fourier mode in θ and z and that is more than compensated for by the reduction of the size of the Jacobian by a fourth in the Newton iteration scheme for the nonlinear system.

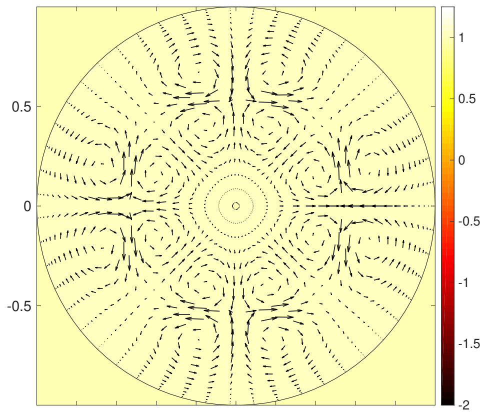
Numerical Results

Calculations described here are limited to $k_0 = 2$; i.e. two-fold azimuthally symmetric TW states. Table (1) shows list of viscosity parameters used in power-law fluid simulations of exact coherent structures. Here, we only present results with nondimensional constant K value while changing n to account for change in viscosity. In figures 1, 2 and 3 we display roll and wave components of these TW states in a plane perpendicular to the pipe axis at $Re = 5000$ when $\alpha = 1.55$ for n values given in Table (1). In each plot the rolls $\mathbf{U}(r,\theta) = (U, V, 0)$, radial and azimuthal waves $(u, v, 0)$ are depicted using arrows and axial waves w are represented in colors where the lighter color corresponds to positive values of w, while darker colors correspond to negative such values. Axial wave velocity $w(r,\theta,z_0)$ is shown

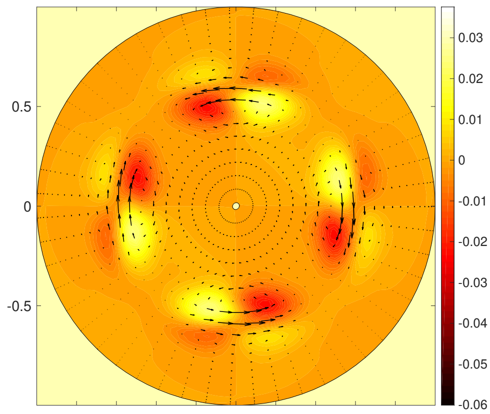
n	K	$\frac{\partial p}{\partial z}$
1	1	- 0.0008
.8	1	-0.00076525
.6	1	-0.00072051

Table 1: Parameters for simulations reported here for power-law fluids for a pipe of length $2\pi/\alpha$ with $\alpha = 1.55$ at Re=5000.

at a fixed $z_0 = 2\pi/\alpha$. The yellow background in figures 1a, 2a and 3a has no physical meaning; the color is chosen to make the arrows describing the roll behaviour more visible. Figure 1 shows a travelling wave for a Newtonian fluid at $Re = 5000$. Both rolls and waves are concentrated in a narrow region away from the center of the pipe and the wall. This is an example of a VWI state where the dominant flow behaviour is observed in a ciritical layer away from the origin ([5]). In the case of non-Newtonian fluids for $n = 0.8$ and 0.6 flow is still concentrated around a critical region as shown in figures 2 and 3. While there is no significant difference in the wave profiles, a clear change in the behaviour of the rolls is observed where counter-rotating vortices closer to the pipe center lose their intensity as flow becomes more shear thinning (fig. 2a & 3a) in comparison to figure 1a

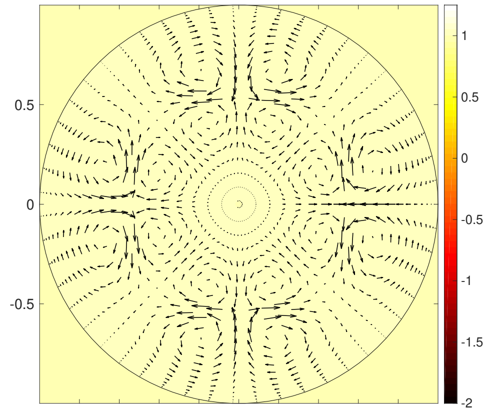


(a) $\mathbf{U}(r,\theta)$

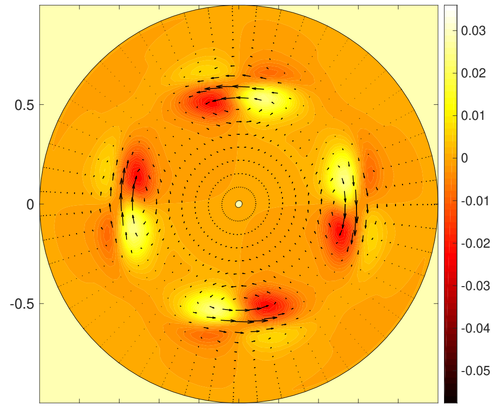


(b) $\mathbf{v}_w(r,\theta,z_0)$

Figure 1: (a)Roll $\mathbf{U}$ and (b) wave profiles $\mathbf{v}_w$ at $Re = 5000$, $n = 1$ at $\alpha = 1.55$. (b) 15 equispaced contour levels are plotted between minimum and maximum $w(r,\theta,z_0)$ where min/max taken over (r,θ).



(a) $\mathbf{U}(r,\theta)$



(b) $\mathbf{v}_w(r,\theta,z_0)$

Figure 2: (a)Roll $\mathbf{U}$ and (b) wave profiles $\mathbf{v}_w$ at $Re = 5000$, $n = 0.8$ at $\alpha = 1.55$. (b) 15 equispaced contour levels are plotted between minimum and maximum $w(r,\theta,z_0)$ where min/max taken over (r,θ).

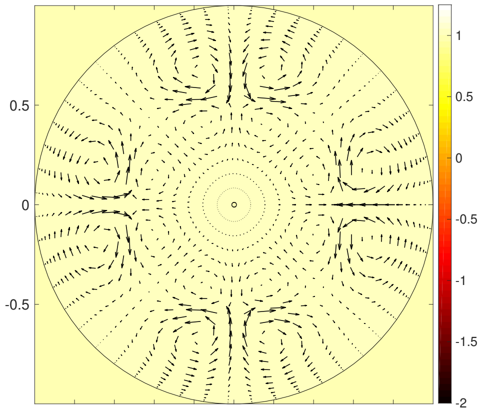
Conclusions

In this paper, we report our preliminary findings on non-linear travelling waves with two-fold rotational symmetry in pipe flows at low Reynolds numbers using power-law rheology model. Reported travelling waves are perturbations of off VWI states given in [10]. We present some features of velocity fields of these travelling waves and compare them to their Newtonian counterparts. Our calculations suggest that decreasing the value of n (making flow more shear thinning) while holding K constant, causes the rolls near to wall get stronger while rolls near the pipe center are dissapearing. On the other hand, the action of waves are still occurring inside a critical layer which is consistent with VWI theory.

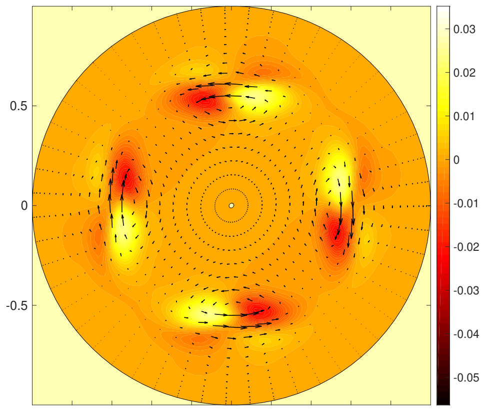
In an upcoming paper we will discuss stabilising effect of the shear-thinning rheology, which can be viewed as a part of the explanation of the delay in the transition to developed turbulence. Our calculations of travelling waves using different rheology models is a work in progress.

Acknowledgements

This research was supported is fully by the Australian Government through the Australian Research Council's Discovery Projects funding scheme (project DP170104703). Current project was undertaken with the assistance of resources and services from the National Computational Infrastructure (NCI), which is supported by the Australian Government. The author wishes to thank the referees for constructive comments on the first draft of this paper.



(a) $\mathbf{U}(r,\theta)$



(b) $\mathbf{v}_w(r,\theta,z_0)$

Figure 3: (a)Roll $\mathbf{U}$ and (b) wave profiles $\mathbf{v}_w$ at $Re = 5000$, $n = 0.6$ at $\alpha = 1.55$. (b) 15 equispaced contour levels are plotted between minimum and maximum $w(r,\theta,z_0)$ where min/max taken over (r,θ).

References

[1] Blackburn, H.M., Hall, P. & Sherwin, S.J., Lower branch equilibria in Couette flow: the emergence of canonical states for arbitrary shear flows. *J. Fluid Mech.*, **726**, 2013, **R2**, 12.

[2] Deguchi, K. & Hall, P., Free-stream coherent structures in parallel boundary-layer flows, *J. Fluid Mech.*, **752**, 2014, 602-625.

[3] Faisst, H. & Eckhardt, B. , Traveling waves in pipe flow, *Phys. Rev. Lett.*, **91**, 2003, 224502.

[4] Gibson, J.F., Halcrow, J. & Cvitanovic, P., Equilibrium and travelling-wave solutions of plane Couette flow., *J. Fl. Mech*, **638**,2009, 243-266.

[5] Hall, P. & Sherwin, S.,Streamwise vortices in shear flows: harbingers of transition and the skeleton of coherent structures, *J. Fluid Mech.*, **661**, 2010,178-205

[6] Hof, B.,van Doorne, C.,Westerweel, J.,Nieuwstadt, F., Faisst, H.,Eckhardt, B.,Wedin, H.,Kerswell, R., & Waleffe, F., Experimental observation of nonlinear traveling waves in the turbulent pipe flow, *Science* **305** (5690), 2004,1594-1598.

[7] Kerswell, R. & Tutty, O., Recurrence of traveling waves in transitional pipe flow, *J. Fluid Mech*, **584**, 2007, 69-102.

[8] Nagata, M., Three dimensional finite-amplitude solutions in plane Coutte flow: bifurcation from infinity, *J. Fluid Mech*, **217**, 1990, 519-527.

[9] Pringle, C.C.T & Kerswell, R.R., Asymmetric, helical and mirror-symmetric travelling waves in pipe flow, *Phys. Rev. Lett.*, **99**, 2007, 074502.

[10] Pringle, C.C.T., Duguet, Y., & Kerswell, R.R., Highly symmetric travelling waves in pipe flow, *Phil. Trans. R. Soc.*, A **367**(1888), 2009, 457-472.

[11] Ozcakir, O. , Tanveer, S. , Hall, P. & Overman E. A. ,Travelling waves in pipe flow, *J. Fluid Mech*, **791**, 2016, 284-328.

[12] Schneider, T. and Eckhardt, B. , Edge States intermediate between laminar and turbulent dynamics in pipe flow, *Phil. Trans. R. Soc.*, A **367**,2009, 577-587.

[13] Viswanath, D.,Recurrent motions within plane Couette turbulence, *J. Fluid Mech*, **580**, 2007, 339-358.

[14] Waleffe, F., Exact coherent structures in channel flow, *J. Fluid Mech*, **435**, 2001, 93-102.

[15] Waleffe, F., Homotopy of exact coherent structures in plane shear flows, *Phys. Fluids*, **15** (6), 2003,1517-1534.

[16] Wang, J., Gibson, J. & Waleffe, F., Lower branch coherent states in shear flows: transition and control, *Phys. Rev. Lett.*, **98** (20), 2007,204501.

[17] Wedin, H & Kerswell, R., Exact coherent structures in pipe flow: travelling wave solutions, *J. Fluid Mech*, **508**, 2004, 333-371.

[18] Faisst and Eckhardt *Phys. Rev. Lett.* **91**, 224502 (2003).

[19] E. Plaut, N. Roland & C. Nouar. Nonlinear waves with a threefold rotational symmetry in pipe flow: influence of a strongly shear-thinning rheology, *J. Fluid Mech*, **818**, 2017, 595-622.

[20] B.A. Toms. Some observations on the flow of linear polymer solutions through straight tubes at large Reynolds numbers. In *Proceedings of First International Congress on Rheology* North-Holland,1948.