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Asymptotic Description of Localized Solutions of 2D Linear and Nonlinear Wave Equation With Degenerating Velocity and Application to the Tsunami Run-up Problem.

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Abstract

First, we consider the Cauchy problem with localized initial data and right-hand side for the two-dimensional wave equation with variable velocity $C(x_1;x_2)$ in a domain Ω, which describes long linear waves (tsunami waves) in the shallow water approximation. We assume that $C(x_1;x_2) = 0$ and $\operatorname{grad} C^2(x_1;x_2) \neq 0$ on the boundary of Ω (shore), which could be viewed as a special caustic. Using a modified semiclassical approximation (ray expansions), Maslov's canonical operator, and Fock quantization of canonical transformations, we construct an asymptotic solutions of this problem. Next, we consider the nonlinear shallow water equations. We use the Carrier–Greenspan transform and ideas due to Pelinovskii and Masova, based on this transform, about the calculation of 1D run-up parameters (e.g., the uprush magnitude) to derive approximate formulas for the run-up of long waves generated by a localized source.

Introduction

There are quite a few publications dealing with the generation, propagation and run-up of long waves in the ocean (e.g., see the monographs and papers [1, 3, 2, 4]). One important model is the system of shallow water equations, which, generally speaking, is defined in a basin of variable depth $D(x)$, $x = (x_1;x_2)$, in some region Ω on the sphere with boundary $\partial\Omega$ given by the equation $D = 0$. In most cases, the amplitude of tsunami waves in the open ocean is quite small, and far from the boundary (shore) we can restrict ourselves to the linearized system of shallow water equations. Further, if one considers only wave (rather than vortical) solutions, the linearized shallow water equations are reduced to the two-dimensional wave equation for the function $\eta(x,t)$ describing the free surface elevation. The description of tsunami waves in the framework of the piston model (corresponding to an instantaneous vertical motion of the bottom over the seismic source) is reduced to the solution of the Cauchy problem with a given initial data for this equation. If the source action is distributed in time, the initial conditions become zero, but the right-hand side appears, and the wave equation becomes nonhomogeneous. Combining these two cases, we obtain the following Cauchy problem with the velocity $c^2(x) = \mathbf{g}D(x)$ (where $\mathbf{g}$ is the gravitational acceleration):

$$\eta_{tt} - \langle\nabla, c^2(x)\nabla\rangle\eta = \mathcal{F}(x,t), \quad \eta|_{t=0} = \eta^0(x), \quad \eta_t|_{t=0} = 0, \quad (1)$$

We assume that $c^2(x)$ is a smooth function in Ω such that

$$c^2(x) > 0,\ x \in \Omega, \quad \textit{and} \quad c^2(x) = 0,\ \nabla c^2(x) \neq 0,\ x \in \partial\Omega. \quad (2)$$

For mathematical rigour, we assume that Ω is bounded; this assumption does not play any role for the final asymptotic formulas.

First, consider the case in which $\mathcal{F} = 0$. We fix a point $x^0 \in \Omega$ and assume that

$$\eta^0 = \Big(\mathsf{e}V\Big)\left(\frac{x - x_0}{l}\right), \quad (3)$$

where l is the characteristic size of the source, $V(y)$ is a smooth function decaying as $|y| \to \infty$, and $\mathsf{e}(z)$ is a smooth function such that $\mathsf{e}(\zeta) = 1$ for $\zeta \leq \zeta_0$ and $\mathsf{e}(\gamma) = 0$ for $|\zeta| \geq \zeta_1 > \zeta_0 > 0$. In fact, the choice of e is not essential for the asymptotic formulas, and we introduce it for mathematical rigor. Denote by L the characteristic size of the basin (or of the area in which we study wave processes). We introduce the dimensionless parameter $\mu = l/L$ and assume that μ is small (taking $l = 50$ km and $L \geq 1000$ km, we obtain $\mu \leq 1/20$).

In the second case, we set $\eta^0 = 0$ and take

$$\mathcal{F}(x,t) = \frac{1}{\tau^2} g'(\frac{t}{\tau})\Big(\mathsf{e}V\Big)\left(\frac{x - x_0}{l}\right). \quad (4)$$

Here τ is the characteristic time of the source action and $g(\zeta)$ is a smooth function decaying as $\zeta \to \infty$. We also assume that $\int_0^\infty g(\zeta)d\zeta = 1$; this gives $\tau^{-2}g'(\frac{t}{\tau}) \to \delta'(t - (+0))$ as $\tau \to +0$. Then it is easy to show that in the limit as $\tau \to +0$ the solution of the nonhomogeneous problem turns into the solution of the Cauchy problem for the homogeneous equation in the first case. We introduce the second dimensionless parameter $\lambda = \frac{L}{\tau c(x^0)}$ and assume that $\lambda\mu = \frac{l}{\tau c(x_0)} > \text{const} > 0$.

Note hat according to [6] to fix the unique solution to these problems it is not necessary (and even is wrong) to set any standard boundary conditions on $\partial\Omega$;, one needs to assume that the corresponding operator is self-adjoint. (This fact causes the main difficulties for the numerical analysis of these problems.) In the water wave theory this assumption means that the energy integral

$$J^2(t) = \frac{1}{2}\Big((\nabla\eta, c^2(x)\nabla\eta)_{L^2(\Omega)} + \|\eta_t\|^2_{L^2(\Omega))}\Big)$$

is finite. Our aim is to construct asymptotic solutions to both problems. In our construction, we use a serious modification of the Maslov canonical operator [7] to the case of localized asymptotic solutions [8, 9] and the Fock quantization of canonical transformations [10]. In the last section, we discuss the relationship of the constructed solutions with the solutions in nonlinear case based on the Carrier-Greenspan [11] transform and some ideas due to Mazova and Pelinovskii [12] (based on this transform) about the calculation of the run-up parameters (e.g., the uprush magnitude) in the 1D situation.

Fronts, the Hamiltonian system, and trajectories

For each fixed t, the solution of the homogeneous problem is localized in a neighborhood of the front, which is a closed, generally speaking, piecewise smooth curve $\gamma_t = x = X(t,\psi)$. To construct the front, one should find the trajectories $\Gamma_t = p = P(t;\psi), x = X(t;\psi)$ of the Hamiltonan system (in an appropriate 4D phase space [13])

$$\dot{x} = H_p, \qquad \dot{p} = -H_x, \quad H(x,p) = |p|c(x), \tag{5}$$

issuing at $t = 0$ from the point x_0 with the unit momenta $p = (\cos\psi, \sin\psi)$, $\psi \in [0, 2\pi]$, and then project them to Ω. The front consists of the endpoints of rays $x = X(t;\psi)$.

Regular points on the fronts are those inside Ω such that $X_\psi(t,\psi) \neq 0$. The points where $X_\psi(t,\psi) = 0$ are focal ones (they form "standard" space-time caustics), and the solution in their vicinity has a greater amplitude than in a neighborhood of regular points. On the coastline $\partial\Omega$, the momentum $p(t,\psi)$ takes infinite values (because $H \equiv |p|c(x) = c(x_0)$ on the trajectories and $c|_{\partial\Omega} = 0$), and (5) must be rewritten in other canonical variables. Locally we can always assume that Ω is given by the inequality $x_1 > f(x_2)$. Then the new coordinates (q,y) and the associated momenta (θ,ξ) are defined as follows:

$$x_1 = f(y) + q^2\theta, \quad x_2 = y, \quad p_1 = q^{-1}, \quad p_2 = \xi q^{-1} f'(x_2). \tag{6}$$

The values of $q < 0$ correspond to the part of the characteristic before its exit to the boundary $\partial\Omega$, $q > 0$, after the boundary, and $q = 0$, to the moment of the boundary. In the new coordinates, the Hamiltonian has the form

$$H = \sqrt{\mathbf{g}}\sqrt{\theta}\,\kappa(f(y) + \theta q^2, y)\sqrt{1 + \Big(q\xi - f'(y)\Big)^2}, \tag{7}$$

where $\kappa(x)$ is defined by $D(x) = (x_1 - f(x_2))\kappa^2(x)$. We denote by $q = Q(t;\psi)$, $y = Y(t;\psi)$, $\theta = \Theta(t;\psi)$, $\xi = \Xi(t;\psi)$ the trajectories in the coordinates $q;y;\theta;\xi$. Note that (5),(7) defines the same Hamiltonian system but written in different coordinates. The transition from (5) to (7) allows one to continue "correctly" the trajectories after their reflection from the shoreline, which can also be interpreted as a special type of caustic (a "nonstandard" one).

Aasymptotic solutions to the homogeneous problem outside of the shoreline $\partial\Omega$.

Denote by $y = y(x,t)$ the signed distance from the observation point x lying in a neighborhood of the front γ_t to γ_t, considering y positive if x lies outside the area bounded by the front and negative if x lies inside this area. Then the point x is characterized by two coordinates, the distance $y(x,t)$ and parameter (angle) $\psi(x,t)$ determined from the condition that the vector $x - X(\psi,t)$ is orthogonal to the front at the point $X(\psi,t)$, i.e. $\langle x - x(\psi,t), X_\psi(\psi,t)\rangle = 0$. Define a "phase"

$$S(x,t) = \langle P(\psi(x;t),t)x - x(\psi(x;t),t)\rangle. \tag{8}$$

In the vicinity of regular points of the front, the asymptotic function η is determined by the formula [8, 14]

$$\eta(x,t) \approx \sqrt{\frac{l}{|X_\psi(\psi,t)|}}\sqrt[4]{\frac{D(x^0)}{D(x)}}\mathrm{Re}\left[e^{-i\frac{\pi m(\psi,t)}{2}} F\left(\frac{S(x,t)}{l},\psi\right)\right]. \tag{9}$$

Here $\psi = \psi(x;t)$, and the "profile" function F is related to the initial perturbation $\eta^0(\frac{x-x^0}{l})$ by

$$F(z,\psi) = \frac{e^{-i\pi/4}}{\sqrt{2\pi}}\int_0^\infty \widetilde{\eta}^0(\rho\mathbf{n}(\psi))e^{iz\rho}\sqrt{\rho}\,d\rho, \tag{10}$$

$$\widetilde{\eta}^0(k) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}^2}\eta^0(y)e^{-i\langle y,k\rangle}dy. \tag{11}$$

Formula (9) contains the following reasonable objects characterizing the solution. First, there are the fronts γ_t consisting of the points on the rays (characteristics); second, it is the multiplier $\sqrt[4]{\frac{D(x^0)}{D(x)}}$, which is a two-dimensional analogue of Green's law in channels linking the ratio of wave amplitudes and depths at different points of observation; third, the multiplier $|X_\psi(\psi,t)|^{-1/2}$, which is the ray "divergence" (or "convergence"): fronts propagate along the rays, and the the more they come in a neighborhood of a given point, the larger the amplitude of the wave in the vicinity of this point is; fourth, it is a function F linking the wave profile with the initial disturbance (source). Finally, there is a phase factorer containing the Maslov index m, which coincides with the Morse index defined as the number of irregular (focal) points on the trajectory coming to the point x. In the case when a few rays come to the neighborhood of the observation point, the solution is represented as a sum of such type of functions (see[8]).

The set of objects mentioned above is minimal from the point of view of the solution definition: if at least one of them is missing, the formula is either wrong or does not give a complete description of the solution. On the other hand, the involvement of other objects for a more detailed description of the solution can lead to a significant complication of the formulas.

It is also seen from (10) that the profile function F can be found as elementary or special functions in exceptional cases[1].

If the initial function η^0 is selected as (the asymmetric generalization of the function η^0 taken in [16, 17];, see also [18])

$$\eta^0 = \frac{A}{\left(1 + \left(\frac{\Delta x_1\cos\phi - \Delta x_2\sin\phi}{lb_1}\right)^2 + \left(\frac{\Delta x_1\sin\phi + \Delta x_2\cos\phi}{lb_2}\right)^2\right)^{3/2}}, \tag{12}$$

where $\Delta x = x - x^0$ and b_1, b_2, θ are real parameters, then $F(z,\psi)$ is the algebraic function

$$F(z,\psi) = \frac{Ae^{-\frac{i\pi}{4}}}{2\sqrt{2}\left(\beta(\psi,\phi) - iz\right)^{3/2}}; \tag{13}$$

here $\beta(\psi;\phi) = b_1^2\cos^2(\psi - \phi) + b_2^2\sin^2(\psi - \phi)$ and $\mathrm{Arg}(\beta(\psi;\phi) - iz) \in (-\pi,\pi)$ (which corresponds to the choice of argument in Wolfram Mathematica).

The asymptotic of the solution in a neighborhood of the focal points (from "standard" caustics) has different form; the corresponding formulas and their illustration can be found in [15].

The asymptotic solutions to homogeneous problem near $\partial\Omega$ (the shore)

Let us briefly describe the asymptotics in a neighborhood of the shoreline $\partial\Omega$, which obtained with the help of combination of the modified Maslov canonical operator [8] and the Fock quantization of canonical transformations (see [10, 19, 21, 22]). We restrict ourselves to the simplest case. Namely, let t^*, ψ^* be the

[1] This fact is related to the Heisenberg uncertainty principle well known in quantum mechanics. Indeed, since the solution is localized in the vicinity of the front, the "width" of the localization Δx turns out to be small, but then the corresponding momentum variable takes large values, and therefore, the definition of the solution must contain Fourier components of the solution with a large range of the dual momentum variable. This essentially means representing the solution in integral form; one can express the asymptotic solution in terms of elementary or special functions only in exceptional cases.

values such that $x^* = X(t^*, \psi^*) \in \partial\Omega$. We assume that the Jacobian $\mathbf{J} = \det \frac{\partial(Q,Y)}{\partial(t;\psi)}(t^*, \psi^*)$ is nonzero; this means that the point $X(t^*;\psi^*)$ is not a (strong) focal point on $\partial\Omega$. Then we can introduce the new coordinates $\Big(\sigma = 2Q(t;\psi)\sqrt{\Theta(t;\psi)}, y = Y(t;\psi)\Big)$ in a neighbourhood of the point x^*. These coordinates are related x_1, x_2 with the help of two first equations (6). Let us find $\tilde{t} = \tilde{t}(\sigma, y), \psi = \psi(\sigma, y)$ (in the neighborhood of the $t^*;\psi^*$) from the equations $\sigma = \Sigma(\tilde{t};\psi), y = Y(\tilde{t};\psi)$ and set

$$\tilde{t}_{od}(\sigma;y) = \frac{\tilde{t}(\sigma;y) - \tilde{t}(-\sigma;y)}{2}, \quad \tilde{t}_{ev}(\sigma;y) = \frac{\tilde{t}(\sigma;y) + \tilde{t}(-\sigma;y)}{2}.$$

Then we have

$$\eta(\sigma;y;t) \approx Ab_1b_2\left(\frac{\tilde{t}_{od}(\sigma,y)}{2\sigma|\mathbf{J}(\tilde{t}(\sigma;y), \psi(\sigma;y))|}\right)^{1/2} \times \tag{14}$$

$$\mathrm{Re}\left[\frac{e^{-\frac{i\pi m}{2}}\left(\beta(\psi(\sigma,y),\phi) + \frac{i}{\mu}\Big(-\tilde{t}(-\sigma,y) + c_0 t\Big)\right)}{\left(\left(-\beta(\psi(\sigma;y);\phi) + \frac{i}{\mu}\Big(\tilde{t}_{\mathrm{ev}}(\sigma;y) - c_0 t\Big)\right)^2 + \frac{(\tilde{t}_{\mathrm{odd}}(\sigma;y))^2}{\mu^2}\right)^{3/2}}\right].$$

Here $\beta(\psi;\theta)$ is the same as in (13), m is the corresponding Maslov index; see details and formulas in more general situations in [21, 22].

Asymptotic solutions in the nonhomogeneous case.

It was shown in [23]) that the solution of the nonhomogeneous problem is represented as the sum of two terms. One of them describes the transient part: it remains localized in a neighborhood of the point x^0 and decays in time. The second term describes a propagating wave (it is of the greatest interest to us) having behavior similar to the solution of the homogeneous problem. Namely, it looks like $\eta = \eta_1 + \frac{\partial\eta_2}{\partial t}$, where

$$\eta_j\Big|_{t=0} = W_j\left(\frac{x - x_0}{\mu}\right), \quad \frac{\partial\eta_j}{\partial t}\Big|_{t=0} = 0, \quad j = 1,2, \tag{15}$$

and the Fourier transform of the functions $W_j(\frac{x-x_0}{l}) =$ have the form:

$$\widetilde{W}_1(p) = \sqrt{2\pi}\,\mathrm{Re}\left[\tilde{g}\Big(\tau c(x^0)|p|\Big)\right]\tilde{V}(p), \tag{16}$$

$$\tilde{W}_2(p) = \frac{\sqrt{2\pi}}{\lambda|p|}\,\mathrm{Im}\left[\tilde{g}\Big(\tau c(x^0)|p|\Big)\right]\tilde{V}(p). \tag{17}$$

Here the bar stands for complex conjugation.

Thus, for the propagating part of the solution in the nonhomogeneous part one can use the formula for the homogeneous one, replacing the initial source with an "equivalent" one. For the case of (12), one can express the final formulas via the erfc function (see [23]). More pragmatic representation could be obtained in the following way [24, 22]. Let us approximate the Fourier transform of the function g: $\tilde{g}\Big(\tau c(x^0)\rho\Big)$ by some polynomial $G_N(\rho)$ of degree N so that $\overline{\rho\tilde{g}\Big(\tau c(x^0)\rho\Big)\tilde{V}(\rho\mathbf{n}(\psi)(\psi))}$ and $\rho\tilde{G}_N(\rho)\tilde{V}(\rho\mathbf{n}(\psi)(\psi)$ be very close to each other; here $\mathbf{n}(\psi)$ is the 2-D vector with components $\cos\psi, \sin\psi)$. Since these $\tilde{g}_n$ are polynomials, it is natural to take a linear combination of Laguerre polynomials. Let $\tilde{G}_N(\rho) = \sum_{m=0}^{N} s_m\rho^m$. Then, as shown in [24], the solution can be approximately written in the form

$$\eta_{\mathrm{inh},N} = \sum_{m=0}^{N} s_m\left(\frac{il}{c(x^0)}\right)^m \frac{\partial^m\eta_{\mathrm{hom}}}{\partial t^m}, \tag{18}$$

where η_{hom} is the solution of the homogeneous problem; outside of the "standard" focal points it is defined by (13) and (14).

The influence of non-linearity near the shore. The Carrier-Greenspan Transformation

The formulas from the previous sections have strict mathematical justification. The results of this section are obtained at the physical level of rigor, and its mathematical justification is currently being developed. It was shown in the famous paper [11] that in the one-dimensional case the system of shallow water equations with the bottom profile $D \equiv x$

$$u_t + uu_x + \eta_x = 0, \quad \eta_t + [(\eta + D)u]_x = 0. \tag{19}$$

can be linearized ino the characteristic variables. Later it was shown in the paper [12] that in fact this linearization is related to the formal discarding of nonlinear summands in system (19), which gives a system of equations or a one-dimensional wave equation with the velocity squared $c^2 = z$ $(z \geq 0)$

$$U_s + N_z = 0, \quad N_s + (zU)_z = 0, \qquad \Leftrightarrow \quad N_{ss} - (zN_z)_z = 0. \tag{20}$$

Here z is a "linear problem" coordinate and s is a "linear problem" time. The correspondence between the solutions of systems (19) and (20) is defined by the point change of variables (see [25])

$$x = z - N + \frac{1}{2}U^2, \quad t = s + U, \quad u = U, \quad \eta = N - \frac{1}{2}U^2,$$

directly linking systems (19) and (20).

Next, we use the following ideas. We simplify the formula for solving (14) in a sufficiently small neighborhood of the shore using the fact that the characteristics approach the shore at right angles and that the wave front when approaching the shore is locally rectilinear. We rewrite the formula for solving the linear system (20) in the vicinity of the shore in the curvilinear coordinates, taking x to ben the normal direction to the shore and y to be the coordinate along the shore. By dropping small summands and freezing the value of y on each characteristic, we approximate the solution of the two-dimensional linear problem by a family of solutions of one-dimensional problems (with respect to the variable x) parameterized by the variabl ye. Thus, we obtain an approximate formula for the solution of the linear system (20),

$$N(z,s) = P\left(\frac{\partial}{\partial s}\right)\mathrm{Re}\,\frac{A_0(z+ib)}{B_0(z,s)}, \quad U(z,s) = P\left(\frac{\partial}{\partial s}\right)\mathrm{Re}\,\frac{A_0}{B_0(z,s)},$$

where

$$B_0(z,s) = \left(z - \frac{(s+ib)^2}{4}\right)^{3/2}, \quad P\left(\frac{\partial}{\partial t}\right) = \sum_{m=0}^{N} s_m\left(\frac{i\mu}{c_0}\right)^m \frac{\partial^m}{\partial t^m}.$$

Here A_0 and B are constant along the characteristic; i.e. they depend only on y. Finally, the transformation $z \to x$, $s \to t$, $N \to \eta$, $U \to u$ by the formulas (21), applied to (21) gives an approximation to solve the nonlinear two-dimensional problem in the framework of the shallow water model.

The resulting formula is a working algorithm that can be implemented using programs such as Wolfram Mathematica or Matlab reducing the whole question to the solution of the Hamiltonian system (5).

Conclusions

We present effective asymptotic formulas for the solutions of the Cauchy problem for homogeneous and nonhomogeneous wave equations with spatially localized initial data and right-hand side, which describe the generation, propagation, and run-up of the long wave in the ocean. We also discuss the relationship of these asymptotic solution with nonlinear effects.

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