21st Australasian Fluid Mechanics Conference
Adelaide, Australia
10-13 December 2018

Stability of the Non-Newtonian Asymptotic Suction Boundary Layer

S. O. Stephen[1] and P. T. Griffiths[2]

[1]School of Mathematics and Statistics
University of Sydney, Sydney NSW 2006, Australia

[2]School of Computing, Electronics and Mathematics
Coventry University, Coventry CV1 5FB, UK

Abstract

The stability of an asymptotic suction boundary layer is considered to investigate the effect of non-Newtonian viscosity on the transition process from a laminar flow to a turbulent flow at large Reynolds number. We present the derivation of the linear stability equations for a generalised Newtonian flow. The numerical solutions of a generalised Newtonian Orr-Sommerfeld equation are obtained for neutral two-dimensional and three-dimensional disturbances. The effect of non-Newtonian viscosity on the critical Reynolds number reveal the stabilising or destabilising effect for shear-thinning and shear-thickening fluids. The results are compared to our previous results in the absence of suction, presented in Griffiths *et al.* [4].

Introduction

We investigate the non-Newtonian flow past a flat plate, with uniform suction applied at the surface. This flow exhibits a constant boundary-layer thickness, making it attractive to numerical and theoretical analysis. For a Newtonian flow, the experimental and theoretical study of Fransson and Alfredsson [3] on the stability of the asymptotic suction boundary layer to Tollmien-Schlichting instabilities and free-stream turbulence, found that uniform suction delayed transition to turbulence. Recent interest for a Newtonian asymptotic suction boundary layer has been in identifying coherent structures by Deguchi and Hall [1] and a self-sustaining vortex/Tollmien-Schlichting wave interaction by Dempsey and Walton [2]. The focus here is the effect of a non-Newtonian fluid and is motivated by applications to use non-Newtonian flows to delay transition to turbulence.

As the first investigation for non-Newtonian flow in an asymptotic suction boundary layer this numerical and asymptotic study considers a Carreau model for the non-Newtonian viscosity. Our previous studies, [4] and [8], have shown that this is more appropriate for unbounded shear flows than the power-law model.

Formulation

Using Cartesian coordinates (x^*, y^*, z^*) we consider the flow of a viscous generalised Newtonian fluid with uniform velocity U_0^* and viscosity μ^* past a flat plate defined by $y^* = 0$. Far from the plate the x^* velocity component tends to the constant U_0^* and fluid is sucked through the plate with constant velocity V_0^*. The velocity field $\mathbf{u}^* = (u^*, v^*, w^*)$ and the pressure p^* satisfy the continuity and Navier-Stokes equations, subject to the boundary conditions

$$\mathbf{u}^* = (0, -V_0^*, 0) \quad \text{at} \quad y^* = 0, \tag{1}$$

and

$$\mathbf{u}^* \to (U_0^*, -V_0^*, 0) \quad \text{as} \quad y^* \to \infty. \tag{2}$$

Generalised Newtonian fluids are characterised by the viscosity being a function of the strain rate. They are appropriate to fluids exhibiting no yield stress (free from elastic effects). The Carreau model for a generalised Newtonian fluid states that

$$\mu^* = \mu_\infty^* + (\mu_0^* - \mu_\infty^*)[1 + (\lambda^* \dot{\gamma}^*)^2]^{\frac{n-1}{2}}. \tag{3}$$

Here μ_0^* is the viscosity at zero shear rate, μ_∞^* is the limiting constant viscosity at infinite shear rate, $\lambda^* > 0$ is a characteristic time constant and $\dot{\gamma}^*$ is the second invariant of the rate of strain tensor. When $\lambda^* = 0$ or $n = 1$ we have a Newtonian fluid. Values of $0 < n < 1$ correspond to shear-thinning fluids, while $n > 1$ is appropriate for shear-thickening fluids. Now write

$$\frac{\mu^*}{\mu_0^*} = \frac{\mu_\infty^*}{\mu_0^*} + \left(1 - \frac{\mu_\infty^*}{\mu_0^*}\right)[1 + (\lambda^* \dot{\gamma}^*)^2]^{\frac{n-1}{2}}, \tag{4}$$

where typically $\mu_\infty^*/\mu_0^* \ll 1$. Thus, in the analysis to follow, we consider the modified Carreau model where

$$\frac{\mu^*}{\mu_0^*} = [1 + (\lambda^* \dot{\gamma}^*)^2]^{\frac{n-1}{2}}. \tag{5}$$

We non-dimensionalise lengths, time, velocity and pressure with respect to ν^*/V_0^*, $\nu^*/(U_0^* V_0^*)$, U_0^* and $\rho^* {U_0^*}^2$, respectively, where ρ^* is the fluid density and $\nu^* = \mu_0^*/\rho^*$.

For a modified Carreau fluid then, following the above non-dimensionalisation,

$$\mu = \mu^*/\mu_0^* = \left(1 + \lambda^2 \dot{\gamma}^2\right)^{\frac{n-1}{2}}. \tag{6}$$

For this non-Newtonian flow, the non-dimensional basic flow quantities satisfy

$$\frac{\mathrm{d}v}{\mathrm{d}y} = 0 \quad \Rightarrow v = -1/R, \tag{7}$$

$$v\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{R}\left[1 + n\left(\lambda \frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\right]\left[1 + \left(\lambda \frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\right]^{\frac{n-3}{2}} \frac{\mathrm{d}^2 u}{\mathrm{d}y^2}, \tag{8}$$

where the Reynolds number $R = U_0^*/V_0^*$. The boundary conditons are

$$u = v + 1/R = 0 \quad \text{at} \quad y = 0,$$

and

$$u \to 1, \quad v \to -1/R \quad \text{as} \quad y \to \infty.$$

Here the viscosity

$$\mu = \left[1 + \left(\lambda \frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\right]^{\frac{n-1}{2}}. \tag{9}$$

For a Newtonian flow these equations have the simple solution $u = 1 - e^{-y}$ and $v = -1/R$. For a non-Newtonian flow these equations are solved numerically. The solutions of equation (8) for u for a range of shear-thinning flows with $\lambda = 1$ is shown in

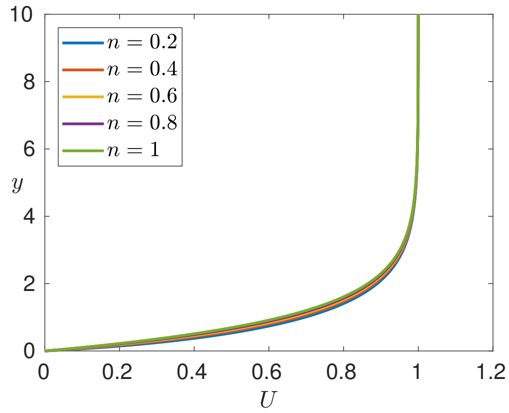

Figure 1: The basic flow solutions for u for a range of values of n with $\lambda = 1$.

figure 1. We see that the velocity tends to the free-stream velocity exponentially as y increases. Furthermore, as n decreases the boundary layer becomes thinner.

Linear stability analysis

We perturb the basic flow and write

$$\begin{aligned} u &= U_b + \tilde{u}(x,y,z), \\ v &= -1/R + \tilde{v}(x,y,z), \\ w &= 0 + \tilde{w}(x,y,z) \\ p &= \text{constant} + \tilde{p}(x,y,z). \end{aligned}$$

The viscosity is given by

$$\begin{aligned} \mu = \Bigg\{ & 1 + 2\lambda^2 \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \right] \\ & + \lambda^2 \left[\left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x}\right) \right]^2 + \lambda^2 \left[\left(\frac{\partial u}{\partial z}\right) + \left(\frac{\partial w}{\partial x}\right) \right]^2 \\ & + \lambda^2 \left[\left(\frac{\partial v}{\partial z}\right) + \left(\frac{\partial w}{\partial y}\right) \right]^2 \Bigg\}^{\frac{n-1}{2}}. \end{aligned}$$

Substituting the disturbed flow into the governing continuity and Navier–Stokes equations, and linearising for small disturbances, yields the following perturbation equations

$$\begin{aligned} & \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} = 0, \\ & \left(\frac{\partial}{\partial t} + U_b \frac{\partial}{\partial x} - \frac{1}{R}\frac{\partial}{\partial y}\right)\tilde{u} + \frac{\partial U_b}{\partial y}\tilde{v} = -\frac{\partial \tilde{p}}{\partial x} + \frac{1}{R}\left\{ 2\frac{\partial}{\partial x}\left(M_b \frac{\partial \tilde{u}}{\partial x}\right) \right. \\ & \left. + \frac{\partial}{\partial y}\left[N_b\left(\frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[M_b\left(\frac{\partial \tilde{u}}{\partial z} + \frac{\partial \tilde{w}}{\partial x}\right)\right]\right\}, \\ & \left(\frac{\partial}{\partial t} + U_b \frac{\partial}{\partial x} - \frac{1}{R}\frac{\partial}{\partial y}\right)\tilde{v} = -\frac{\partial \tilde{p}}{\partial y} + \frac{1}{R}\left\{ 2\frac{\partial}{\partial y}\left(M_b \frac{\partial \tilde{v}}{\partial y}\right) \right. \\ & \left. + \frac{\partial}{\partial x}\left[N_b\left(\frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[M_b\left(\frac{\partial \tilde{v}}{\partial z} + \frac{\partial \tilde{w}}{\partial y}\right)\right]\right\}, \\ & \left(\frac{\partial}{\partial t} + U_b \frac{\partial}{\partial x} - \frac{1}{R}\frac{\partial}{\partial y}\right)\tilde{w} = -\frac{\partial \tilde{p}}{\partial z} + \frac{1}{R}\left\{ 2\frac{\partial}{\partial z}\left(M_b \frac{\partial \tilde{w}}{\partial z}\right) \right. \\ & \left. + \frac{\partial}{\partial x}\left[M_b\left(\frac{\partial \tilde{u}}{\partial z} + \frac{\partial \tilde{w}}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[M_b\left(\frac{\partial \tilde{v}}{\partial z} + \frac{\partial \tilde{w}}{\partial y}\right)\right]\right\}, \end{aligned}$$

where

$$\begin{aligned} M_b &= \left[1 + \lambda^2 \left(\frac{\partial U_b}{\partial y}\right)^2\right]^{\frac{n-1}{2}}, \\ N_b &= \left[1 + n\lambda^2 \left(\frac{\partial U_b}{\partial y}\right)^2\right]\left[1 + \lambda^2 \left(\frac{\partial U_b}{\partial y}\right)^2\right]^{\frac{n-3}{2}}. \end{aligned}$$

If the disturbances are assumed to have the normal-mode form, so being proportional to $\exp(\mathrm{i}(\alpha x - \omega t) + \mathrm{i}\beta z)$, the linear disturbance equations become

$$\mathrm{i}\alpha\hat{u} + \hat{v}' + \mathrm{i}\beta\hat{w} = 0, \tag{10}$$

$$\begin{aligned} & R[\mathrm{i}(\alpha U_b - \omega)\hat{u} + U_b'\hat{v} + \mathrm{i}\alpha\hat{p}] - \hat{u}' \\ & \quad = \bar{\mu}(\hat{u}'' - \gamma^2\hat{u}) + \bar{\mu}'(\hat{u}' + \mathrm{i}\alpha\hat{v}) + \bar{\nu}(\gamma^2\hat{u} + \mathrm{i}\alpha\hat{v}'), \end{aligned} \tag{11}$$

$$\begin{aligned} & R[\mathrm{i}(\alpha U_b - \omega)\hat{v} + \hat{p}'] - \hat{v}' \\ & \quad = \bar{\mu}(\hat{v}'' - \gamma^2\hat{v}) + 2[\bar{\mu}'\hat{v}' - (\bar{\nu}\hat{v}')'] - \bar{\nu}[\beta(\mathrm{i}\hat{w}' - \beta\hat{v})] \end{aligned} \tag{12}$$

$$\begin{aligned} & R[\mathrm{i}(\alpha U_b - \omega)\hat{w} + \mathrm{i}\beta\hat{p}] - \hat{w}' \\ & \quad = (\bar{\mu} - \bar{\nu})(\hat{w}'' - \gamma^2\hat{w}) + (\bar{\mu} - \bar{\nu})'(\mathrm{i}\beta\hat{v} + \hat{w}'), \end{aligned} \tag{13}$$

where $\gamma^2 = \alpha^2 + \beta^2$, the primes denote differentiation with respect to y and the viscosity functions $\bar{\mu}$ and $\bar{\nu}$ are defined as

$$\bar{\mu} = [1 + n(\lambda U_b')^2][1 + (\lambda U_b')^2]^{(n-3)/2}, \tag{14}$$

$$\bar{\nu} = (n-1)(\lambda U_b')^2[1 + (\lambda U_b')^2]^{(n-3)/2}. \tag{15}$$

Numerical results

The system of equations (10–15) represents a coupled quadratic eigenvalue problem for α. This problem is solved subject to no-slip at the wall and decaying perturbations far from the wall.

The neutral temporal and spatial stability of the system was determined using a Chebyshev polynomial discretization method. The Gauss–Lobatto collocation points were transformed into the physical domain via an exponential map. The boundary conditions were then imposed at $y = 0$ and $y = y_{max}$, where the value of y_{max} is such that the steady mean flow results had converged to within some desired tolerance, typically, 10^{-10}. In each case considered it was found that the value of y_{max} that ensured mean flow convergence also ensured that both the real and imaginary parts of the solution for α had converged to at least four decimal places.

In all cases 100 collocation points were distributed, via the exponential map, between the upper and lower boundaries. Further increasing the number of collocation points revealed no discernible difference in the numerical results.

The stability equations were then solved in terms of primitive variables at each of the collocation points, excluding those on the boundary edges. This global eigenvalue solution method is favourable when compared to local (Orr–Sommerfeld) approaches as it is possible to simultaneously obtain all of the eigenvalues and eigenvectors.

We present the numerical results for shear-thinning fluids with $\lambda = 1$ for $0 < n < 1$ and the Newtonian case $n = 1$. The neutral results for wavenumber α and frequency $F = \omega/R$ are shown in figures 2 and 3 for the two-dimensional case $\beta = 0$. The corresponding results for the three-dimensional case with $\beta = 0.1$ are shown in figures 4 and 5. The numerical results show that in both cases the critical Reynolds number increases as n decreases. Thus, this shear-thinning non-Newtonian flow is more stable than the corresponding Newtonian one.

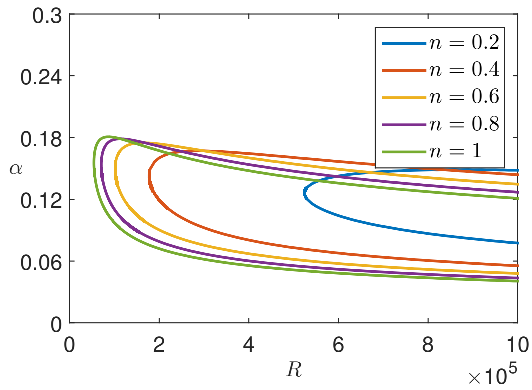

Figure 2: The neutral wavenumber α versus R for $\beta = 0$.

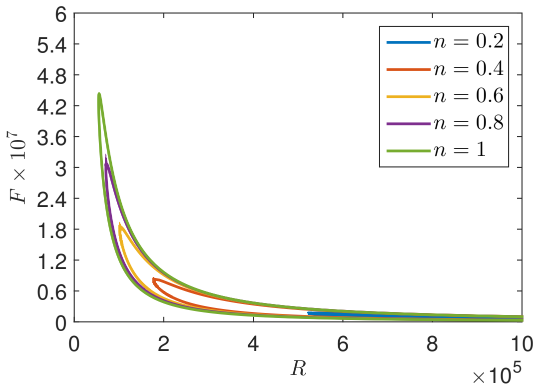

Figure 3: The neutral frequency F versus R for $\beta = 0$.

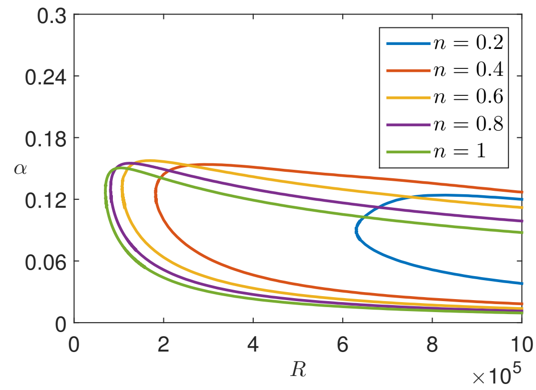

Figure 4: The neutral wavenumber α versus R for $\beta = 0.1$.

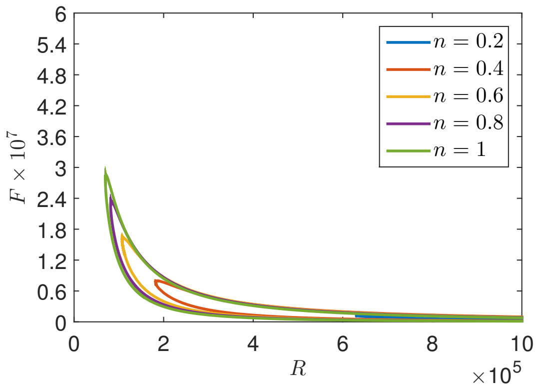

Figure 5: The neutral frequency F versus R for $\beta = 0.1$.

This result for the asymptotic suction boundary layer is opposite to the behaviour found for the Blasius boundary layer in [4]. There it was found that the effect of non-Newtonian viscosity was to destabilise the flow for shear-thinning fluids, i.e. the critical Reynolds number decreases as n decreases.

The fact that the effect of non-Newtonian viscosity differs between Blasius flow and the asymptotic suction boundary layer warrants further investigation. In order to make comparisons results have been obtained for the same values of viscosity at the wall, as was considered by Griffiths [8]. Specifically, solutions of equations (8) and (9) were obtained for u and μ for a range of values of n, where λ was determined for each n such that the viscosity at maximum shear rate (at the wall) was the same value. The solutions for μ where $\mu(0) = 0.5$ are shown in figure 6. With this value of viscosity at the wall the results for λ for each n considered are shown in table 1.

n	λ
0.8	6.5826
0.6	1.4219
0.4	0.9097
0.2	0.7157

Table 1: Values of λ for a range of values of n for $\mu(0) = 0.5$.

The corresponding neutral curves for fixed $\mu(0)$ for the two-dimensional case $\beta = 0$ for α and $F = \omega/R$ (not shown) reveal the same behaviour illustrated in figures (2–3), namely that the critical Reynolds number increases as n decreases. This is not unexpected since the values of λ are not too different.

Conclusions

Our numerical solutions show that for the asymptotic suction boundary layer the critical Reynolds number increases as n decreases for shear-thinning fluids. These preliminary results suggest that transition to turbulence in an asymptotic suction boundary layer may be delayed for a non-Newtonian fluid compared to a Newtonian one. Thus, this could provide a mechanism for controlling turbulent flow.

For the current problem, ongoing calculations of the growth rates are underway. It was found in [4] that the growth rates increase as n decreases for the Blasius boundary layer. We expect to report on these in the presentation. The effect of the non-Newtonian flow on the growth rates will provide a clearer picture of the overall effect on the stability of the asymptotic suction boundary layer.

Further results need to be obtained for the three-dimensional case when the viscosity at the wall is kept the same for different values of n. This exercise also needs to be repeated for the Blasius boundary layer to further investigate the seemingly opposite effect of non-Newtonian viscosity on the stability of the Blasius boundary layer and the asymptotic suction boundary layer. This is the focus of our current numerical investigations.

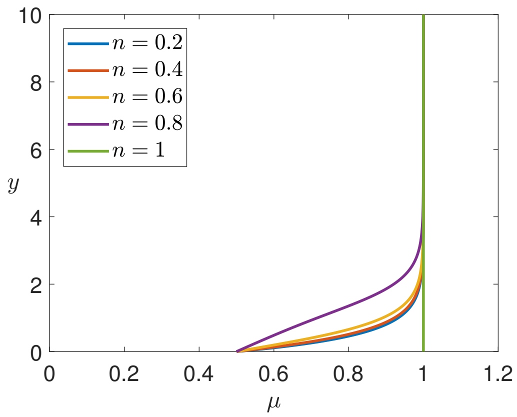

Figure 6: The basic flow solutions for μ for a range of values of n with $\mu(0) = 0.5$.

An asymptotic description of the linear stability modes for this problem will be useful. We are currently conducting such a study. For a Newtonian flow the linear stability of the asymptotic suction boundary layer has been investigated recently by Dempsey and Walton [2]. Following their analysis for neutral lower-branch modes we introduce $\varepsilon = R^{-1/4}$ and the new variables $(X, Z) = \varepsilon(x, z)$, $T = \varepsilon^2 t$. The small disturbances are taken to be proportional to $E = \exp(i\alpha(X - cT) + i\beta Z))$. The disturbed flow is governed by a triple-deck structure.

For the Newtonian case, Dempsey and Walton [2] derived the following linear dispersion relation

$$i^{5/3}\frac{\mathrm{Ai}'(\xi_0)}{\kappa_0} + \alpha^{1/3}\sqrt{\alpha^2+\beta^2} = 0, \quad \xi_0 = -i^{1/3}\alpha^{1/3}c,$$

where Ai is the Airy function and $\kappa_0 = \int_{\xi_0}^{\infty}\mathrm{Ai}(s)\,ds$. This was solved for fixed β, yielding solutions for α and c which agree well with the lower branch of the neutral stability curve.

The asymptotic analysis for a non-Newtonian flow is similar, but it is complicated by the nonlinear viscous terms in the lower deck. We hope to report on the results shortly, which will allow preditions of the effect of non-Newtonian flow to be readily obtained. Asymptotic results have previously been obtained for lower-branch modes in the Blasius boundary layer for a modified Carreau fluid in [4], in agreement with the numerical results of that study.

Acknowledgements

The authors acknowledge Professor P. Hall for suggesting consideration of the stability of the asymptotic suction boundary layer for a non-Newtonian flow.

References

[1] Deguchi, K. and Hall, P., Free-stream coherent structures in parallel boundary-layer flows. *J. Fluid Mech.*, **752**, 602–625, 2014.

[2] Dempsey, L.J. and Walton, A.G., Vortex/Tollmien-Schlichting wave interaction states in the asymptotic suction boundary layer. *Q. Jl Mech. Appl. Math*, **70**, 187–213, 2017.

[3] Fransson, J.H.M. and Alfredsson, P.H., On the disturbance growth in an asymptotic suction boundary layer. *J. Fluid Mech.*, **482**, 51–90, 2003.

[4] Griffiths, P.T., Gallagher, M. and Stephen, S.O., The effect of non-Newtonian viscosity on the stability of the Blasius boundary layer. *Phys. Fluids*, **29**, 1–18, 2016.

[5] Griffiths, P.T., Stephen, S.O., Bassom, A.P. and Garrett, S.J., Stability of the boundary layer on a rotating disc for power law fluids, *J. Non-Newton. Fluid*, **207**, 1–6, 2014.

[6] Griffiths, P.T., Garrett, S.J. and Stephen, S.O., The neutral curve for stationary disturbances in rotating disk flow for power-law fluids, *J. Non-Newton. Fluid*, **213**, 78–81, 2014.

[7] Griffiths, P.T., Flow of a generalised Newtonian fluid due to a rotating disk, *J. Non-Newton. Fluid*, **221**, 9–17, 2015.

[8] Griffiths, P.T., Stability of the shear-thinning boundary-layer flow over a flat inclined plate, *Proc. R. Soc. A*, **473**, 20170350, 2017.