Prandtl number dependence of the onset of instability for differentially heated cavity flow

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Abstract
The differentially heated square cavity, consisting of a fluid filled square enclosure with a temperature difference imposed between the two side walls, is a classical problem in natural convection that is widely used to study fundamental features of buoyancy driven flow. For low Rayleigh numbers the flow is laminar and steady. However, as the Rayleigh number is increased above a critical value, the flow is unstable, and at still higher Rayleigh numbers it is fully turbulent.

The value of the critical Rayleigh number is strongly dependent upon the Prandtl number of the fluid, as is the frequency of the initial instability. In this study we use a combination of direct stability analysis and the linearised stability equations to find the critical Rayleigh number for Prandtl numbers in the range $0.011 < Pr < 1.4$. Over this range we find four distinct modes of oscillation at the onset of instability.

Introduction
The differentially heated cavity is a classical problem of heat transfer, where the two vertical walls of a rectangular enclosure are held at different temperatures, with buoyancy forces driving natural convection in the fluid within the enclosure. It is a simplified representation of a number of problems of interest to engineers, such as the insulating properties of cavity walls and double glazing, heat flow in fuel tanks, and the cooling of electrical equipment.

The simplicity of the geometry and boundary conditions belies the richness of the flow phenomena that can occur, and this has led it to being a well studied problem. Restricting ourselves to a square cavity, where the height equals the width, the first numerical computation of the flow was a spectral hand calculation of Poots [1]. This was followed by finite difference models run on digital computers by Wilkes and Churchill [2] and de Vahl Davis [3]. The problem’s simplicity has led to it being used as a test problem for Computational Fluid Dynamics (CFD) codes, and there are published benchmark solutions for laminar flow calculated with finite difference [4] and spectral [5] codes for air with a Prandtl number of $Pr = 0.71$.

For Rayleigh numbers below a critical number the flow is laminar and steady. Above this critical number it becomes unsteady. For two-dimensional flow in a square cavity filled with air ($Pr = 0.71$) Le Quéré and Behnia[6] found the critical Rayleigh number to be $Ra_c = 1.82 \times 10^8$ using a spectral code. Janssen and Henkes[7] studied the onset of instability for fluids in the range $0.25 < Pr < 7$, using a finite volume code. They determined the critical Rayleigh number and the frequency of the instability of the flow for 10 values of Prandtl number, finding that the behaviour of the supercritical flow was dependant on the Prandtl number. For $Pr \leq 2$ the velocity in the cavity experienced a regular sinusoidal fluctuation, the power spectra for which exhibited a clear single frequency. However, for $Pr > 2$ the supercritical flow was chaotic, with the velocity exhibiting a broad spectrum of oscillations. They found two unstable modes; one in the fluid exiting the side wall boundary layers, and the second in the horizontal boundary layer flowing along the upper and lower walls.

In this paper we extend the range of Prandtl numbers studied by Janssen and Henkes[7], and illustrate the different instabilities that occur for different Prandtl number fluids.

Numerical Methodology
We consider a square two-dimensional cavity of width and height $L$. The left wall is heated to temperature $T_H$ and right wall cooled to $T_C$, while the upper and lower walls are adiabatic, with $\partial T/\partial y = 0$. At the left the fluid is heated and it rises up the left wall and is discharged into the centre of the cavity. At the right the fluid is cooled and it descends the right wall and exits at the bottom. A schematic of the problem is shown in figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{The geometry for the differentially heated cavity.}
\end{figure}

The flow is characterised by two dimensionless groups, the Rayleigh and the Prandtl numbers,

$$Ra = \frac{g \beta \Delta T L^3}{\nu \alpha}, \quad (1)$$

$$Pr = \frac{\nu}{\alpha}, \quad (2)$$

where $g$ is the acceleration due to gravity, $\beta$ the thermal expansion coefficient, $\Delta T = T_H - T_C$ is the temperature difference between the hot and cold walls, $L$ the length scale (the width or height of the cavity), and $\nu$ and $\alpha$ the kinematic viscosity and the thermal diffusivity respectively.

The flow was modelled by a non-dimensional form of the two-dimensional incompressible Navier–Stokes equations, using the Boussinesq approximation to model buoyancy,

\begin{align}
\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x_j} &= \frac{Pr}{\sqrt{Ra}} \frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial p}{\partial x_i} + \delta_{i2} Pr \theta, \quad (3) \\
\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_j} &= \frac{1}{\sqrt{Ra}} \frac{\partial^2 \theta}{\partial x_j^2}, \quad (4) \\
\frac{\partial u_i}{\partial x_i} &= 0 \quad (5)
\end{align}
where distance is non-dimensionalised by the cavity width $L$, velocity by the buoyancy velocity $V_B = \sqrt{g\Delta \theta / \rho}$, pressure by $\rho V_B^2$, time by $L^2/\alpha \sqrt{Ra}$ and the temperature is non-dimensionalised as $\theta = (T - T_c)/\Delta T$.

These equations were solved using the in-house non-staggered Cartesian finite volume solver SnS [8, 9]. The momentum and temperature equations were discretised in space using a second order central difference scheme, and in time using a fractional step scheme using Adams-Bashforth differencing for advection and Crank-Nicolson differencing for the diffusion terms [10]. The time step was varied to ensure that the maximum Courant number at no time exceeded 0.25. Regular meshes of $160 \times 160$, $320 \times 320$ and $640 \times 640$ were used depending on the Rayleigh number.

To solve for the linear stability problem, the mean fields were perturbation fields following [7]. The lowest Prandtl numbers ($\text{Pr} \approx 0.5$) were applied to solve for fluids with the Prandtl numbers given in table 1. For each mesh two Rayleigh numbers are given which bracket the limit of stability; $Ra_{\bar{u}}$ is the highest Rayleigh number for which stable flow was found, and $Ra_{\bar{u}}$ is the lowest Rayleigh number for unstable flow. The column $f$ lists the frequency of the perturbation to the velocity.

The calculated stability limit was dependant on the grid resolution. For the $160 \times 160$ mesh the onset of instability was found to be at $Ra = 1.72 \times 10^8$. However, the direct stability analysis using the $320 \times 320$ and $640 \times 640$ meshes both found the onset of instability at $Ra = 1.82 \times 10^8$, although as with Le Quéré and Behnia long runtimes were needed to pinpoint the onset of instability, as shown in figure 2.

To perform a direct stability analysis, for a given Prandtl number the test case was run for successively higher Rayleigh numbers to establish the onset of instability. The state of the flow was evaluated by monitoring the RMS value of the fields $u_i$, $\theta$ and $p$, and their values at the coordinate $(0.05L, 0.9L)$, following [7]. However, as noted by Le Quéré [6] when close to the critical Rayleigh number the growth of the unstable modes in the flow is slow, and so a flow that appears stable might be unstable to small perturbations.

To determine the stability of a seemingly steady solution, the SnS code as modified to calculate the linear stability equations. Decomposing the velocity, pressure and temperature into their mean and fluctuating components, $u_i = \bar{u}_i + u'_i$, $p = \bar{p} + p'$, $\theta = \bar{\theta} + \theta'$, the linear stability equations are,

$$\frac{\partial u'_i}{\partial t} + u_i \frac{\partial u'_i}{\partial x_j} + \frac{\partial \bar{u}_i}{\partial x_j} = \frac{Pr}{\sqrt{Ra}} \frac{\partial^2 u'_i}{\partial x_j^2} + \delta_{ij} Pr \theta'$$  \hspace{1cm} (6)

$$\frac{\partial \theta'}{\partial t} + u_i \frac{\partial \theta'}{\partial x_j} + \frac{\partial \bar{\theta}}{\partial x_j} = \frac{1}{\sqrt{Ra}} \frac{\partial^2 \theta'}{\partial x_j^2}$$ \hspace{1cm} (7)

$$\frac{\partial u'_i}{\partial x_i} = 0.$$ \hspace{1cm} (8)

To solve for the linear stability problem, the mean fields were loaded from a stable solution to the direct stability analysis. The perturbation fields $u_i$, $\theta'$ and $p'$ were set to zero, and then a random perturbation was applied to the initial temperature field $\theta'$. The equations were then stepped forward in time using the same fractional step solver as was used in the direct stability analysis. The use of this solver more accurately determined if a seemingly stable flow was steady or not.

In order to determine the frequency of the instability of the flow, the velocity was monitored at $(0.05L, 0.9L)$. A Fourier transform of the time signal of the $u$ component of velocity was used to find the dominant frequency of the instability.

**Results and Discussion**

The solver was first applied to the stability for air at $Pr = 0.71$, to allow a ready comparison with the solution of [6]. It was then applied to solve for fluids with the Prandtl numbers given in table 2. The lowest Prandtl numbers ($Pr > 0.1$) are liquid metals, $Pr = 0.71$ air, $Pr = 1.40$ gaseous ammonia, and the range $0.1 < Pr < 0.3$ mixtures of noble gasses.

Stability of Air, $Pr = 0.71$

Le Quéré and Behnia[6] performed a detailed study of the stability of air using a spectral code, and determined the critical Rayleigh number for $Pr = 0.71$ in a square cavity to be $Ra_c = 1.82 \times 10^8$. To validate the ability of SnS to determine the onset of instability, it was used to calculate flow with Rayleigh numbers around this value, on a range of meshes varying from $160 \times 160$ to $640 \times 640$. The results for these meshes are summarised in table 1. For each mesh two Rayleigh numbers are given which bracket the limit of stability; $Ra_{\bar{u}}$ is the highest Rayleigh number for which stable flow was found, and $Ra_{\bar{u}}$ is the lowest Rayleigh number for unstable flow. The column $f$ lists the frequency of the perturbation to the velocity.

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Figure 2: Time history of $u_{RMS}$ for $Pr = 0.71$ on a $320 \times 320$ mesh.

The linear stability code was used to confirm stability limits, and for the $320 \times 320$ mesh the solution was found to be neutrally stable at $Ra = 1.82 \times 10^8$ (figure 3), agreeing with the values of Le Quéré.

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Figure 3: The linear stability solutions for $u_{RMS}$ for $Pr = 0.71$, $Ra = 1.81 \times 10^8$, $Ra = 1.82 \times 10^8$, and $Ra = 1.83 \times 10^8$ on a $320 \times 320$ mesh.

A spectral analysis of the $u$ velocity at the coordinate $(0.05L, 0.9L)$ was used to determine the frequency of the instability. Figure 4 shows the power spectra of this velocity for the linear stability and direct stability solutions, which were both calculated on a $320 \times 320$ mesh for $Ra = 1.83 \times 10^8$. As can be seen both solvers calculate the same frequency of the instability, 0.046, which compares well with the value of 0.044 found by Janssen and Henkes[7].
Stability Limits for $0.011 \leq Pr \leq 1.4$

The solver was then applied to fluids with the Prandtl numbers listed in table 2. As with the $Pr = 0.71$ case, the critical Rayleigh number was first found using the direct stability solver, and then the linear stability equations were solved to confirm the threshold of instability. As with table 1 two Rayleigh numbers are listed which bracket the stability limit, and the frequency of the instability of the $v$ velocity field is also given. For different Prandtl numbers different modes of instability were observed in the solutions, and these are classified in the “Mode” column of the table.

![Graph showing power spectra for the $v$ component of velocity at (0.05,0.9), calculated for $Ra = 1.83 \times 10^8$, $Pr = 0.71$ using the direct and linear stability solvers, on a 320 x 320 mesh.]

Table 1: Summary of results for $Pr = 0.71$.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>$Ra_{L1}$</th>
<th>$Ra_{L2}$</th>
<th>$f$</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>160 x 160</td>
<td>1.70 x 10^8</td>
<td>1.75 x 10^8</td>
<td>0.0428</td>
<td>A</td>
</tr>
<tr>
<td>240 x 240</td>
<td>1.78 x 10^8</td>
<td>1.80 x 10^8</td>
<td>0.0453</td>
<td>A</td>
</tr>
<tr>
<td>320 x 320</td>
<td>1.81 x 10^8</td>
<td>1.83 x 10^8</td>
<td>0.0460</td>
<td>B</td>
</tr>
<tr>
<td>640 x 640</td>
<td>1.80 x 10^8</td>
<td>1.84 x 10^8</td>
<td>0.0463</td>
<td>C</td>
</tr>
</tbody>
</table>

Table 2: Summary of results for $0.011 \leq Pr \leq 1.4$.

The Rayleigh numbers bracketing the onset of instability are shown in figure 5, and agree well with the values from [7, 6]. The trend line has been broken into separate segments for each mode of instability. The critical Rayleigh number increases with increasing Prandtl number, but the trend is not smooth, and this is due to the different modes of instability encountered at different Prandtl numbers.

The frequencies of instability are shown in figure 6, and compares well with the data of [7]. For all the cases shown here the initial unstable mode had a single dominant frequency, which agrees with [7] who found this was the case for $Pr < 2.5$.

Preliminary data for $Pr = 4$ showed no single dominant frequency in the unstable flow, again agreeing with [7] who found that for $Pr > 4$ the flow at the onset of instability was chaotic.

![Graph showing critical Rayleigh number for Prandtl numbers in the range $0.011 < Pr < 0.14$. Current data compared with [7, 6].]

Unstable modes

Four different modes of instability were identified for the range of Prandtl numbers studied. Contour plots of the mean velocity and its variance are shown in figure 7 for examples of each mode, with the mean temperature fields being given in figure 8.

For high Prandtl numbers $Pr \geq 0.7$ a well known instability occurs where the boundary layers on the side walls discharge into the cavity, named mode $D$. At lower Prandtl numbers $Pr < 0.3$, mode $C$ described by [7] occurs when the horizontal boundary layer detaches from the wall in mid-cavity and flaps vertically. Mode $B$ occurred for $Pr = 0.1$ when there was distinct boundary layers on all four walls of the cavity. The instability manifested itself as a vertical oscillation in the stagnation point where the horizontal boundary layers impinge on the side walls. Mode $A$ occurred for low Prandtl numbers ($Pr < 0.03$) when the flow consisted of a circular rotating body of fluid at the centre of the cavity. A perturbation rotated slowly within the core of the flow.

While modes $B$ to $D$ might be considered as the same mode, with the location of the oscillation migrating across the width of the cavity, the discontinuities in the critical Rayleigh number and frequency of instability suggests that they should be considered as distinct modes.
Conclusions
A stability analysis has been made for two-dimensional flow in a differentially heated cavity, using direct and linear stability methods, for fluids with Prandtl numbers in the range $0.011 < Pr < 1.4$. Four distinct modes of instability were found. The critical Rayleigh number was found to increase with increasing Prandtl number, but the relationship is not smooth due to the different modes of instability encountered for different Prandtl numbers. The frequency of instability varied with Prandtl number, and again this was not a smooth function.

References