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Supersonic jets with inverse square gravity and volcanoes

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Abstract

A number of celestial bodies are known to emit supersonic jets that can be so fast that they escape gravity. The complex features of such jets, that involve particulates, phase changes and rarefaction effects are modelled here by assuming a steady axisymmetric jet of a perfect gas. It is shown that the escape velocity of a gas is much smaller than that of a solid object. A closed form solution is obtained to the penetration height of such a jet with an isentropic expansion. Unsteady, inviscid computations confirm this result with the difference that the entropy increase across the normal shock in the jet causes the penetration height to be somewhat smaller than that given by the theory. The results are applied to a photograph of the volcano Tvashtar on the moon Io of Jupiter, in order to estimate the volcano's jet speed at the surface and its reservoir temperature.

Introduction

It was shown in [1] that, for steady isentropic axisymmetric flow of a perfect gas in an inverse square gravitational field, that the energy equation can be integrated, the only agency changing the total enthalpy of the gas being the work done against gravity. This derivation and its consequences for the escape velocity of a gas and for the penetration radius of a supersonic jet will be briefly reviewed. One of the important consequences is that the escape velocity of a gas is much smaller than that of a solid body. Examples of data for some moons in the solar system are shown in Table 1, showing solid body, u_{es}, and gas, u_{eg}, escape velocities. The results will then be applied to a particular volcano on the moon Io of Jupiter, in order to estimate characteristics of the jet.

Body	Gas	u_{es} km/s	u_{eg} km/s
Titan	Methane	2.65	0.80
Enceladus	Water	0.239	0.090
Io	SO_2	2.36	0.78
Ganymede	Oxygen	2.74	1.12
Callisto	CO_2	2.44	0.92
Europa	Oxygen	2.03	0.83

Table 1: Examples of parameters of moons in our solar system

Theoretical and numerical analysis

Consider a steady axisymmetric supersonic jet of a perfect gas issuing from the surface of a celestial body. Let the x-axis be the coordinate along the axis of the jet, y normal to this axis. Let u and v be the corresponding velocity components. Let h be the total enthalpy of the gas and p and ρ be its pressure and density. Thus

$$h = \frac{\gamma}{\gamma-1}\,\frac{p}{\rho} + \frac{u^2+v^2}{2}, \tag{1}$$

where γ is the ratio of specific heats. Then with t being the time, the unsteady energy equation is

$$\frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x} + v\,\frac{\partial h}{\partial y} = \frac{1}{\rho}\,\frac{\partial p}{\partial t} + u\,f_x + v\,f_y r. \tag{2}$$

With an inverse square gravity field, and acceleration of gravity g at the moon's surface ($r = r_0$),

$$f_x = -g\,\frac{r_0^2}{r^2}\,\frac{x}{r}, \quad f_y = -g\,\frac{r_0^2}{r^2}\,\frac{y}{r}, \quad r^2 = x^2+y^2. \tag{3}$$

Setting the time derivatives to zero for steady flow, observe that, along the jet axis, $v = 0$, $y = 0$ and $x = r$, so that, along the jet axis, which by assumption is also a streamline, the steady–flow energy equation becomes ($u \neq 0$):

$$\frac{dh}{dr} = -g\,\frac{r_0^2}{r^2}, \tag{4}$$

stating that the only agency that reduces h along $x = r$ is the work done against gravity. This may be integrated to give

$$h(r) = h(r_0) - g\,r_0\left(1 - \frac{r_0}{r}\right). \tag{5}$$

Assume that there is a constriction at the surface of the moon, so that the flow from a reservoir inside the moon to the outside passes through a minimum cross sectional flow area. Assume also that the pressure ratio between the reservoir and the moon's surface (p_r/p_0) is sufficiently large to make the constriction a sonic throat, so that (a = speed of sound)

$$u_0^2 = a_*^2 = \frac{\gamma p_*}{\rho_*}. \tag{6}$$

Here the asterisk denotes the state at the sonic throat. Equation (6) identifies the jet speed at the surface (u_0) with the sonic speed within the jet at the throat (a_*). Note that the state at the throat (p_*, ρ_*) is determined by the isentropic expansion from the reservoir state (p_r, ρ_r) and is independent of the surface state (p_0, ρ_0). Substituting in equation (5) for $h(r_0) = h_*$ from equation(1), and setting $h(r) = 0$ at $r = \infty$ then gives

$$0 = \frac{\gamma}{\gamma-1}\,\frac{p_*}{\rho_*} + \frac{u_0^2}{2} - g\,r_0 = \frac{a_*^2}{\gamma-1} + \frac{u_0^2}{2} - g\,r_0 = \frac{\gamma+1}{\gamma-1}\,\frac{u_0^2}{2} - g\,r_0. \tag{7}$$

This represents the escape condition at which the jet speed at the surface is just large enough to overcome gravity. Solving for this special speed,

$$u_0^2 = u_{eg}^2 = 2\,g\,r_0\,\frac{\gamma-1}{\gamma+1}. \tag{8}$$

The "g" in the subscript "eg" distinguishes u_{eg} as the escape velocity of a gas, because it is significantly smaller than ($\gamma > 1$) the escape velocity $u_{es} = \sqrt{2gr_0}$ of a solid body. The reason is that the thermal energy of the gas at $r = r_0$ is converted to ordered kinetic energy in the isentropic expansion of the gas as it flows to increasing r; isentropic, because dissipative processes have been excluded by assumption.

When the jet velocity at the surface is smaller than the escape velocity, $u_0 < u_{eg}$, $h \to 0$ at a finite value of r, r_p, say. Substituting $h(r) = 0$ and $r = r_p$ in equation (5) and using equations (6) and (8), the maximum steady–state penetration radius is obtained:

$$\frac{r_p}{r_0} = \left(1 - \frac{u_0^2}{u_{eg}^2}\right)^{-1}. \tag{9}$$

This may be simplified in the limit when the penetration height above the surface H is small so that the surface may be considered to be plane, to

$$H = \frac{\gamma+1}{\gamma-1}\frac{u_0^2}{2g}, \tag{10}$$

which gives the penetration height of volcanoes when $H << r_0$.

The parameter space of the problem may be described by the formal dependence of any dimensionless quantity Q on dimensionless parameters

$$Q = Q\left(\frac{p_r}{p_0}, \frac{\rho_r}{\rho_0}, \frac{p_0}{\rho_0 g r_0}, \gamma, \frac{r_*}{r_0}\right), \tag{11}$$

where p_r is the reservoir pressure within the moon, ρ_r is the corresponding density and r_* is the radius of the opening at the surface. This parameter space was systematically explored by computation using the Euler equation. Figure 1 shows the computational domain at the initial condition. A 600×600 coarse grid was used with adaptive mesh refinement by a factor of 3 in both directions, so that the effective grid was 1800×1800. A threshold of the fractional density gradient was used as the refinement criterion. Solid boundaries are specified by a level set which is the smallest distance of a field point from the solid boundary.

An example of the result of a computation is shown in figure 2, illustrating the shock wave structure and other features of the flow in the meridional plane.

Figure 3 shows the dependence of the penetration radius on the velocity ratio for a particular set of parameters. Note that the computations give a lower value for this than that of the isentropic theory. This is thought to be caused by the entropy increase across the normal shock.

The volcano Tvashtar on Io

The New Horizons mission of NASA obtained a beautiful image of the moon Io of Jupiter in which an eruption of the volcano Tvashtar is clearly visible, see figure 4. This volcano reaches a height above the surface of Io of 300 km. With the radius of Io, this gives $r_p/r_0 = 1.16$.

Knowing that the atmosphere of Io is predominantly sulphur dioxide, we can approximate the behaviour of the gas by putting $\gamma = 1.25$. This would already give us a value of $u_o/u_{eg} = 0.37$ from equation (9), see figure 5, but it is necessary to account for the deficit due to non-isentropic effects by performing the appropriate computation.

Points from such computations are also shown in figure 5. Using these, the value of u_0/u_{eg} is seen to be closer to 0.41. Using $u_{eg} = 780$ m/s for Io, this result permits us to estimate the surface speed of the jet in Tvashtar to be $u_0 = 324$ m/s with an estimated error bar of 10%. It also permits the determination of the temperature of the reservoir from which the jet arises as $T_r = 730$ K with an error bar of 20%. Note that, using the escape

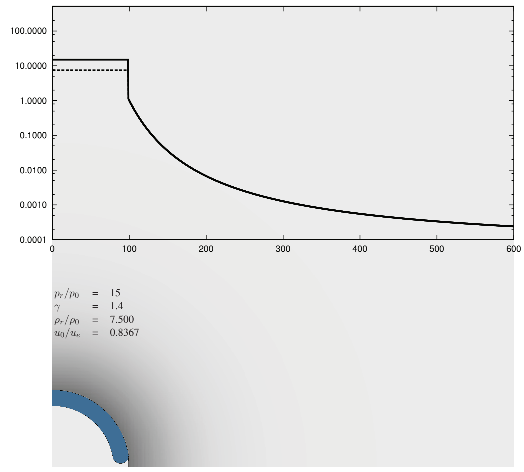

Figure 1: Meridional section through one quadrant of the flow field around a hollow spherical moon with an opening at a pole, showing the initial conditions for the computation. The greyshading is proportional to the magnitude of the fractional density gradient and illustrates the variation through the (here isothermal) atmosphere. The plot at the top shows pressure (heavy line) and density profiles.

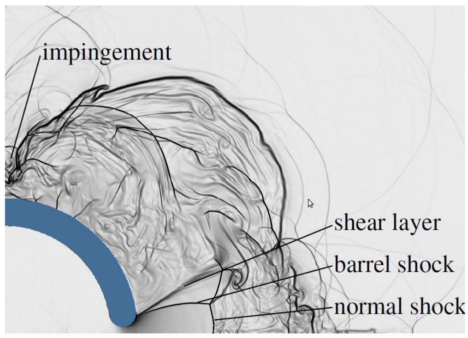

Figure 2: Example of results of computation of the flow. $p_r/p_0 = 60$, $u_0/u_{eg} = 0.837$, $\gamma = 1.4$, $r_*/r_0 = 0.1$.

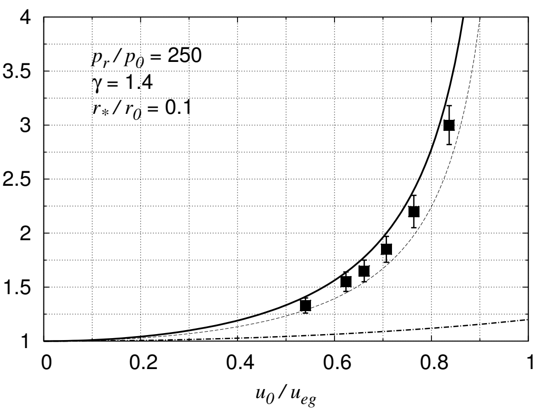

Figure 3: Plot of dimensionless steady–state jet penetration radius r_p/r_0 *vs.* u_0/u_{eg} for $\gamma = 1.4$, $r_*/r_0 = 0.1$. $p_r/p_0 = 250$. The full line shows the value given by equation (9). The dotted line indicates results with $p_r/p_0 = 15$. The chain-dotted line indicates the penetration radius of a solid body with the same surface speed.

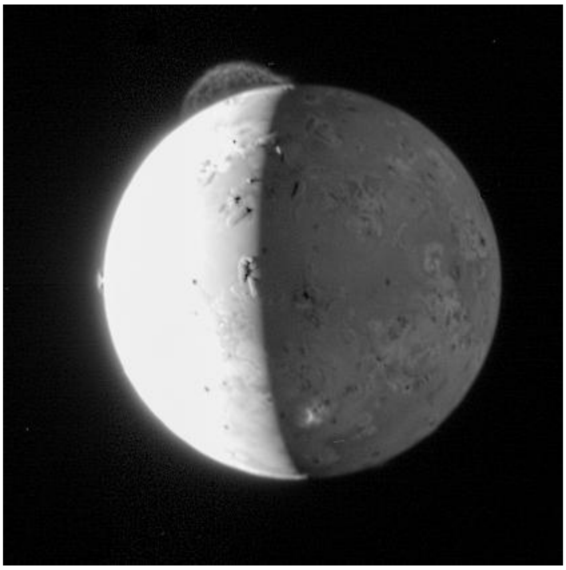

Figure 4: Image of the moon Io of Jupiter showing the volcano Tvashtar near the top. The volcano is probably visible because of the presence of particulates or of condensation.

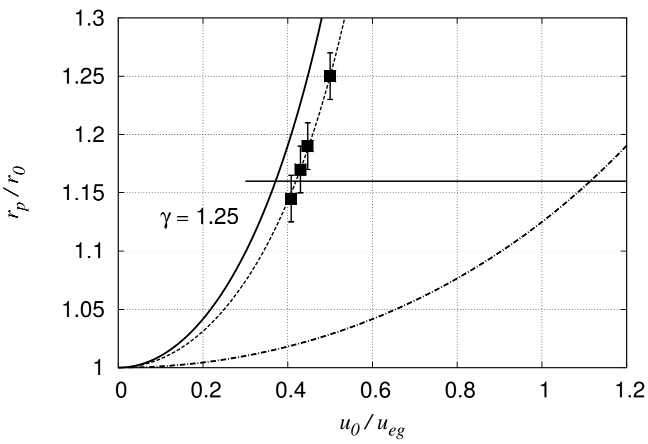

Figure 5: Computation of penetration radius on Io to determine the surface speed of the volcano Tvashtar. The observed penetration radius ratio is shown as a horizontal line. The full line is equation (9). The dotted line is a line of best fit through the computed results and the dash dotted line is the value for a solid object with the same surface speed.

velocity of a solid object instead of that of a gas would incorrectly put these two values at 870 m/s and more than 3000 K! An example image of such a computation is shown in figure 6. Clearly the smooth dome shape of the jet at this lower velocity ratio is very different from the violent features of the image in figure 2 and quite similar to the image of Tvashtar.

Conclusions

The results of a previous investigation, [1], on the features of supersonic jets from moons, in which the flow is modelled as a steady axisymmetric jet of a perfect gas, are reviewed. One of the conclusions of that work was that the escape velocity of a gas is much smaller than that of a solid body, because the thermal energy of the gas at the surface is converted to ordered kinetic energy in the isentropic expansion of the jet, just as in a rocket nozzle. A consequence is that, when the surface speed

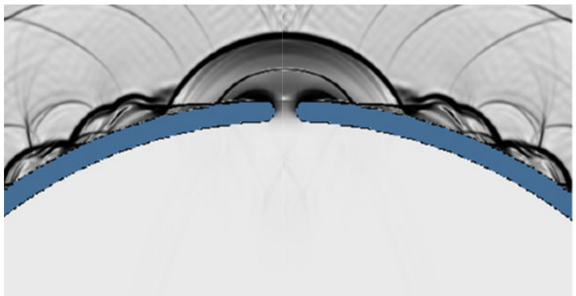

Figure 6: Image from one of the computations of the volcano Tvashtar. Note how the structure of this flow differs from cases with larger u_0/u_{eg} such as the one in figure 2. It is like a smooth dome, similar to that in the image of Tvashtar, figure 4. The shock waves visible above the dome appear stronger than they are, because the greyshading is proportional to the *fractional* density gradient.

is smaller than the escape velocity, the penetration radius of a jet is much larger than that of a solid object. The findings are applied to an image of the volcano Tvashtar on the moon Io of Jupiter, which rises to a height of 300 km, in order to estimate the surface speed of the jet of Tvashtar, as well as the temperature of the reservoir from which it issues. These two values are much smaller than what one would obtain by incorrectly using the escape velocity of a solid object.

Acknowledgments

Thanks are due to NASA for permission to use the image of Io.

References

[1] Hornung, H. G. *J. Fluid Mech.*, **795**, 2016, 950–971.