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Soliton Spectra of Random Water Waves in Shallow Basins

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Abstract

The data processing of random wave field on a shallow water is developed on the basis of the inverse scattering transform. The soliton component obscured in a random wave field is determined and the corresponding distribution function of number of solitons on their amplitudes is constructed. The approach developed is illustrated by means of a model quasi-random wave field and applied to the real data interpretation of wind waves generated in the laboratory wind tank.

Introduction

The traditional approach in the problem of wind wave study is based on the analysis of Fourier spectra and determination of their peculiarities. There is a vast number of both theoretical and experimental papers where this problem has been considered; it is impossible to list all of them here. Therefore, we refer only to the old review chapter "Wind waves" by Zaslavsky and Monin in the book [8] and recent Proceedings of IUTAM Symposium on Wind Waves [7] where a reader can find key references in this field. Many interesting and useful informations have been obtained about wind waves in terms of Fourier spectra. However one of the serious obstacles making the Fourier analysis ineffective in application to surface oceanic waves is the nonlinear character of such waves, whereas the Fourier analysis is the linear operation applicable to the systems obeying the superposition principle.

Osborne with co-authors (see, e. g., [11, 12, 13, 14, 15] and references therein) has developed the method of nonlinear spectral analysis of shallow water waves described by the Korteweg–de Vries (KdV) equation. Osborne's approach is based on the application of the inverse scattering method (ISM) to the analysis of random field data in one-dimensional space domain with the periodic boundary conditions. The main idea of his approach was in the presentation of complex initial disturbance in terms of a set of elliptic functions (cnoidal waves). These functions can be considered as the nonlinear eigenmodes which are preserved in the process of wave field evolution in contrast to the linear sinusoidal eigenmodes. This means that a nonlinear wave spectrum calculated on the basis of these modes is invariant in time while a usual Fourier spectrum is variable due to the nonlinear interactions between the different sinusoidal harmonics. The important feature of nonlinear eigenmodes is that the nonlinear spectrum naturally reduces to the Fourier spectrum, if the analyzed wave field is quasi-linear. However, the mathematical and numerical machinery used for the calculation of nonlinear eigenmodes is not so simple in contrast to the linear case.

Here we propose a very similar to Osborne's, but a bit different approach to the analysis of random water waves that is also based on the application of ISM. The essential feature of our approach is the interpretation of a random initial wave field in terms of the ensemble of solitons and quasi-linear ripples rather than the set of elliptic eigenmodes (a similar approach was also realised in [2]). The idea is illustrated by an example of shallow water waves described by the classical KdV equation with the random initial data. If one takes some portion of random field data (which should be long enough), the number of solitons, their amplitudes, speeds, characteristic durations, etc., can be calculated then by means of ISM or by the direct numerical simulation of the corresponding KdV equation. In the meantime, the knowledge of number of solitons obscured in the random wave field, their parameters and statistics is a matter of independent interest per se. We describe our approach below and give some examples.

As well-known, the soliton turbulence of rarefied soliton ensembles in strongly integrable systems is trivial to certain extent – the distribution function of solitons is unchanged in time [17] (the definition of strongly and weakly integrable systems is given in [17]). This is a consequence of a trivial character of soliton interactions in such systems. The solitons do not change their parameters after collisions, and only the paired collisions occur between them. However, the dynamics of a dense ensemble of solitons is more complicated and soliton turbulence is nontrivial even in the integrable nonlinear wave equations [4, 5]. The quantitative criterion of "dense soliton gas" was introduced in these papers and was shown that the density of soliton gas is bounded from above. The critical gas density, apparently, depends on soliton distribution function, which makes important the determination of such function in a particular physical problem such as a water wave turbulence. The numerical experiments supporting the developed theory [4] for the particular model distribution functions were reported in [1].

The KdV model considered here is the typical example of strongly integrable system applicable to the real physical field. This makes topical the development of handy methods of extraction of soliton distribution function from the natural complex wave fields. One of the experimental approaches to the solution of this practical problem has been considered for internal waves in Okhotsk Sea [10] and another example of processing of surface wave observational data has been reported in [2]. We hope, this publication will stimulate further interest to this important problem.

The Korteweg–de Vries model and data processing

Let us assume that there is a data of recorded surface waves at some fixed point x_0 of a shallow-water basin. So that the elevation η of the water level at this point is the known function of time: $\eta(x_0,t) = f(t)$ where $f(t)$ is a random function. For the description of further space evolution of data recorded at the point x_0, the timelike KdV (TKDV) equation [14] is used:

$$\frac{\partial \eta}{\partial x} + \frac{1}{c_0}\frac{\partial \eta}{\partial t} - \alpha\eta\frac{\partial \eta}{\partial t} - \beta\frac{\partial^3 \eta}{\partial t^3} = 0, \qquad (1)$$

where $c_0 = \sqrt{gh}$; $\alpha = 3/(2c_0h)$; $\beta = h^2/(6c_0^3)$ with g being the acceleration due to gravity and h being an unperturbed water depth.

In the process of evolution of an initial perturbation one can expect emergence of a number of solitons with different amplitudes and phases. Soliton solution to the TKdV equation (1) has the form:

$$\eta(x,t) = A\,\mathrm{sech}^2\,\frac{t-x/V}{T}, \qquad (2)$$

where the parameters V and T are related to the amplitude A:

$$V = \frac{c_0}{1-c_0\alpha A/3} \approx c_0\left(1+\frac{c_0\alpha A}{3}\right), \qquad T = \sqrt{\frac{12\beta}{\alpha A}}, \quad (3)$$

the approximate formula is valid for small amplitude solitons when $c_0\alpha A/3 \ll 1$.

Inasmuch as solitons are very stable with respect to interaction with others wave perturbations and influence of external effects (such as viscosity, inhomogeneity, etc), it is a matter of interest to extract them from the irregular components of a wave field and to describe their statistical properties and contribution to the total wave energy. This can be done by the following way. Let us consider a very long portion of recorded measurement data of surface perturbation at any given point x_0. The characteristic duration of this portion T_p is assumed to be much greater than the typical soliton time scale T. Let us represent a perturbation with the help of some dimensionless function $\varphi(t)$: $\eta(0,t) = U\varphi(t/T_p)$, where U is the characteristic wave "amplitude", e.g., the maximum value of perturbation $\eta(0,t)$ in the considered portion of data.

By means of the transformation $\tau = (t-x/c_0)/T_p$, $\xi = -\alpha Ux/T_p$, $u = \eta/U$, Eq. (1) and the corresponding initial condition $\eta(0,t)$ can be reduced to the standard form [9]:

$$u_\xi + uu_\tau + u_{\tau\tau\tau}/\sigma^2 = 0, \qquad u(0,\tau) = \varphi(\tau) \qquad (4)$$

with one dimensionless parameter σ^2 known in the oceanography as the Ursel parameter and defined as $\sigma^2 = \alpha UT_p^2/\beta$.

As it was mentioned above, we consider the case when the duration of the perturbation is long enough, so that $\sigma^2 \gg 1$. In this case the number of solitons obscured in the "initial" perturbation is also very big in general, and it is reasonable to describe them by the distribution function $f(A)$. This function determines the number of solitons dN within the interval $(A, A+dA)$ [9]: $dN = f(A)dA$. According to the theory developed in [9], the distribution function can be calculated at large values of σ by means of the formula:

$$f(A) = \frac{\sigma}{4\pi\sqrt{3U}}\int_L \frac{d\tau}{\sqrt{2U\varphi(\tau)-A}}, \qquad (5)$$

where the interval of integration L is determined by the condition $2U\varphi(\tau) > A$.

As follows from Eq. (5), soliton amplitudes are distributed in the interval [9] $0 < A < 2U\max[\varphi(\tau)]$, and their total number can be found from the formula

$$N = \int_0^\infty f(A)\,dA = \left(\frac{\sigma}{\pi\sqrt{6}}\right)\int_{\varphi(\tau)>0}\sqrt{\varphi(\tau)}\,d\tau. \qquad (6)$$

Therefore for large σ the total number of solitons is determined only by those intervals of τ-axis where function $\varphi(\tau)$ is nonnegative!

As well known, the KdV equation possesses an infinite number of conserved densities I_n (see, e.g., [9, 12]). One of them, $I_2 = \int \eta^2(x,t)\,dt$ with the integration performed from minus to plus infinity is proportional to the wave energy and therefore is of a special physical interest. The fraction of energy of a non-soliton component of a perturbation to the total energy of initial perturbation can be determined by means of the following formula [9]:

$$\frac{I_2^{ns}}{I_2^{tot}} = \frac{\int_{\eta(0,\tau)<0}\eta^2(0,\tau)\,d\tau}{\int_0^{T_p}\eta^2(0,\tau)\,d\tau}. \qquad (7)$$

The total energy of a soliton component in the wave field can be readily calculated, if soliton amplitudes are known:

$$I_2^{sol} = 4\sqrt{\frac{\beta}{3\alpha}}\sum_{k=1}^N A_k^{3/2} = \frac{4h}{3\sqrt{3g}}\sum_{k=1}^N A_k^{3/2}. \qquad (8)$$

In the last expression the values of coefficients α and β for surface water waves were used (see above after Eq. (1)).

Laboratory experiment with wind waves on shallow water

The theory developed above was applied to the data processing on laboratory experiments with wind wave generation. The experiments were conducted in the Luminy (Marseilles) small tank having the following sizes (length × width × height): 865 cm × 64 cm × 50 cm. The water depth in the tank in different experiments ranged from 1 cm to 8 cm: $h = 1$, 2, 3, 4, 6, and 8 cm. Surface waves were generated by a wind blowing up over the water surface with the different mean velocities: $V_w = 5.29$, 6.45, 8.62 and 13.24 m/s. At the opposite end of the tank it was placed a wave absorber to exclude reflected waves. Two sensitive electric probes recording water level were placed at the distances 100 cm and 300 cm from the ventilator. The detailed description of experimental set up can be found in [6].

Here we present the analysis of only one of the series of experiments with the water depth $h = 1$ cm and wind velocity $V_w = 5.29$ m/s. Other experimental series were analysed in a similar way. Wind waves generated by permanently blowing wind is an active system, i.e., the system with the permanent energy pumping at each point of water surface. Moreover, a distributed external force due to wind is not a constant, it varies from some maximum value near the ventilator to a smaller value at the opposite end of the tank. This leads to the different soliton distribution functions measured at two different distances from the ventilator. A small water viscosity also affects the soliton distribution function.

Another difficulty with wind generated surface waves is the effective excitation of high frequency Fourier components, so that the essential portion of wave energy is contained in that part of the Fourier spectrum which is beyond the range of validity of TKdV equation. Therefore we were forced to restrict our analysis by only low-frequency components of the wave spectrum for $\omega \ll \omega_{cr} = 76.7$ s^{-1}. Figure 1 presents the Fourier spectrum of wind waves recorded at two distances from the ventilator as indicated above. Therefore, to satisfy this condition, we cut the Fourier spectra of wind waves at $\omega_{lim} = 13.4$ s^{-1} (see dashed vertical line in Fig. 1) and ignored the high frequency portions of wave spectra above ω_{lim}.

The filtered fragments of 60-second duration records in two spatial points at the distances 100 cm and 300 cm from the ventilator are shown in Fig. 2. There is a visible difference between the wave fields shown in frames (a) and (b). This is a manifestation of the wave fetch effect on longer distances. In the result of this the statistics of obscured solitons in the record of frame b) is essentially richer than in the frame a).

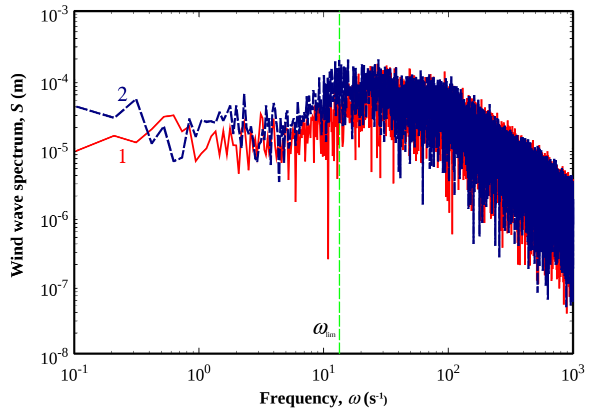

Figure 1: Fourier spectrum of wind waves generated in the laboratory tank at two different distances from the ventilator. Solid line 1 – pertains to the spectrum at the distance 100 cm from the ventilator, and dashed line 2 – to the distance 300 cm.

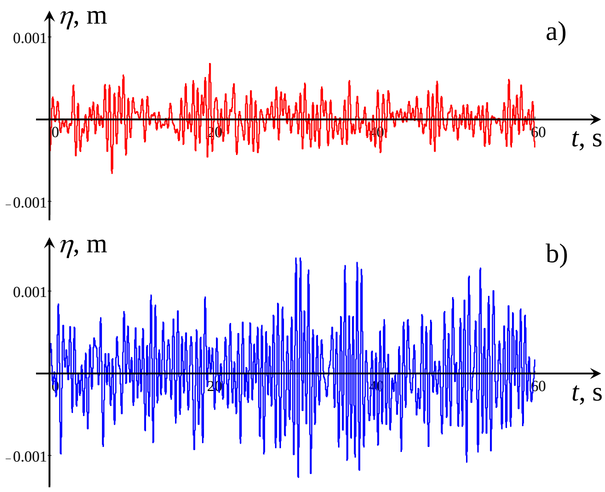

Figure 2: Fragments of low-frequency components of surface perturbations generated by wind in two spatial points of measurements in the tank. Frame a) – surface perturbation at the distance 100 cm; frame b) – at the distance 300 cm.

The recorded data shown in Fig. 2 were used as the input for the TKdV equation (4). In the statistically equilibrium state each 60-second portion of recorded data is equivalent to the same portion taken at a different time interval, therefore one can expect that the number of solitons obscured in each portion of data is the same in average and their distribution function is invariant with respect to the time shift. This was confirmed in the data processing.

The TKdV equation (4) was solved numerically using the recorded data of 60 sec duration from the total time interval of 208 sec. After a while solitons emerged from the quasi-random data and their amplitudes were easily determined with the help of a special subroutine. This allowed us to determine the histogram of soliton numbers in the each particular interval of amplitudes $A+\Delta A$ (the analogue of a differential distribution function). On the basis of this histogram we determined also the integral (cumulative) distribution function – the total number of solitons with the amplitudes less than A normalised by the total number of all solitons. Figure 3 demonstrates the histogram of soliton numbers versus amplitudes for the time series shown in Fig. 2. The experimental data can be approximated by the Poisson distribution function $P(n) = \lambda^n e^{-\lambda}/n!$, where the parameter $\lambda = 3.85$ for the histogram shown in frame (a) and $\lambda = 4.94$ for the histogram shown in frame (b).

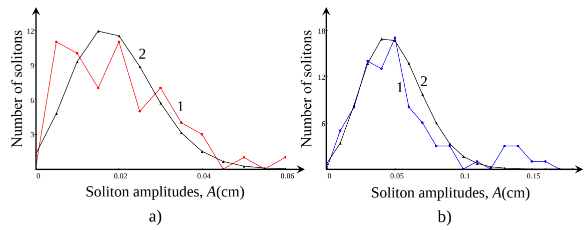

Figure 3: The histogram of soliton numbers versus soliton amplitudes for the time series shown in Fig. 2. Lines 1 reflects experimental data, lines 2 are the best fit of these data by the Poisson distribution functions.

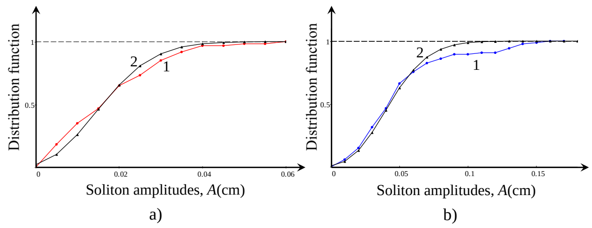

Figure 4: The integral (cumulative) distribution functions (lines) for the experimental data shown in Fig. 2. Lines 2 represent the best fit approximation with the Poisson cumulative distribution functions with the same parameters λ as in Fig. 3.

The corresponding integral distribution functions for the experimental data of Fig. 2 are shown in Fig. 4 (lines 1) together with the approximative Poisson integral functions with the same parameters as in Fig. 3. The total number of solitons emerged from the wave field shown in Fig. 2a) was 60, and emerged from the wave field shown in Fig. 2b) was 86. As expected, the time series recorded closer to the ventilator (Fig. 2a) contained less number of solitons than the time series recorded further from the ventilator (Fig. 2b). In the latter case the wave field was much better developed due to the influence of wave fetch.

If we assume that all 60 solitons in the time series shown if Fig. 3a) are uniformly distributed in the time interval of 60 s, then we obtain that the time interval per each soliton is $\Delta T_1 = 1$ s. As follows from the histogram shown in Fig. 3a) the maximal number of solitons have amplitudes $A_{m1} = 0.02$ cm and the duration $T_{s1} = 0.026$ s. Therefore $\Delta T_1/T_{s1} \approx 3.85$. The similar estimates for the time series shown if Fig. 3b) give the time interval per each soliton $\Delta T_2 \approx 0.7$ s. As follows from the histogram shown in Fig. 3b) the maximal number of solitons in this time series have amplitudes $A_{m2} = 0.05$ cm and the duration $T_{s2} = 0.165$ s. Therefore $\Delta T_2/T_{s2} \approx 4.2$. This shows that in both cases the "soliton gas" is very dense (cf. [4, 16]). The fragments of numerical calculations presented in Fig. 5 illustrate the soliton gas density in both time series.

Conclusions

To analyze long random time series of water waves in shallow basins we have proposed an approach which differs from the traditionally used Fourier analysis. Our approach is based on the extraction of obscured solitons from the complex wave fields and construction of histograms of solitons at different points of observation. According to the theoretical conception, a soliton component of a wave field in the well-developed nonlinear perturbations should dominate. The number and individual parameters of solitons are preserved in the conservative statistically

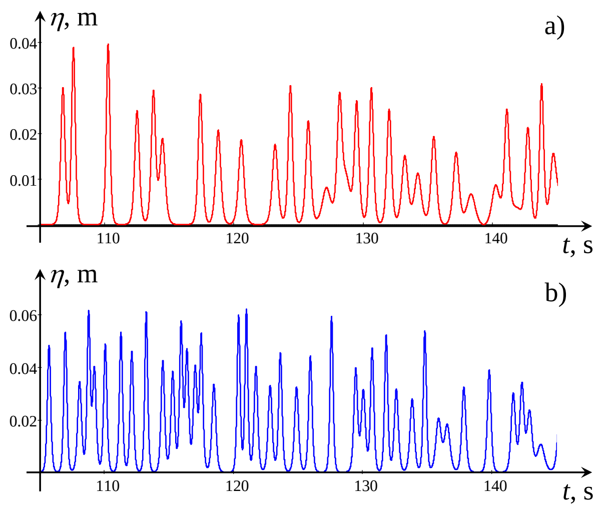

Figure 5: The fragments of numerical calculations with the input data taken from Fig. 2 illustrating the soliton gas density in both time series.

homogeneous systems [17], therefore the distribution function (or histograms of solitons) remains the same at different points of observation, if the dissipative factors (i.e., viscosity or external sources of energy) are negligible. In contrast to that the Fourier spectrum changes due to nonlinearity.

Our approach is in line with the contemporary development of the theory of strong turbulence in the integrable or near-integrable systems [1, 2, 3, 4, 5, 16, 17]. Experimentally constructed distribution function can be used for the determination of degree of density of a soliton gas and its closeness to the critical value defined in [4, 16]. The results obtained here are supplementary to the results of field experiments reported in [2].

A small dissipation can cause a gradual decay of soliton histograms and their distortion, in general. This effects can depend, apparently, on the specific type of dissipation. The approach briefly described in this paper (the details can be found in the recent publication [6]) can provide some additional information about the energy distribution in natural wave fields such as the relationship between the soliton and nonsoliton components obscured in the fields. This can help researchers to estimate the intensity of external sources or sinks of energy in the wave turbulence. The approach in its current form is applicable to the KdV-like systems, e.g., shallow-water waves (see, e.g., [3] where the turbulence of soliton gas was studied both within the integrable KdV and non-integrable KdV-BBM equations). Its generalization to the internal waves described by the Benjamin–Ono equation or surface deep-water waves described by the nonlinear Shrödinger equation is an interesting and challenging problem (see [13]).

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