21st Australasian Fluid Mechanics Conference
Adelaide, Australia
10-13 December 2018

Realisability condition for the velocity structure function in the scaling range of turbulent flows

L. Djenidi[1], **R. A. Antonia**[1] **and S. Tang**[2]

[1]Discipline of Mechanical Engineering
The University of Newcastle, New South Wales 2308, Australia

[2]Institute for Turbulence-Noise-Vibration Interaction and Control,
Shenzhen Graduate School, Harbin Institute of Technology, Shenzhen, 518055, China

Abstract

Realisability conditions based on the Cauchy-Schwarz inequality are developed for the moments of the longitudinal velocity structure function, $\overline{(\delta u(r))^n}$ ($\delta u(r) = u(x,t) - u(x,+r,t)$), for homogeneous isotropic turbulence in the inertial range, when they are assumed to follow a power-law form ($\sim r^{\zeta_n}$). While these conditions cannot be used to assess the validity of any phenomenology behind the development of scaling laws in the form $\overline{(\delta u)^n} \sim r^{\zeta_n}$, they provide a simple and objective way to assess the realisability of the exponent ζ_n. In particular, they can also be used to assess the realisability of ζ_n obtained empirically from either experimental or numerical data. Application of these realisability conditions to existing data shows that the estimated exponents ζ_n as well as multifractal model expressions of ζ_n are not realisable. Interestingly, the β-model and the Kolmogorov prediction (*i.e.* $\zeta_n = n/3$) are both realisable, but only the latter is Reynolds number independent. This is expected since it is valid when the Reynolds number is infinitely large.

Introduction

Kolmogorov's second similarity hypothesis [1] leads to the following similarity law (hereafter denoted K41)

$$\overline{(\delta u)^n} = C_n(\bar{\varepsilon} r)^{n/3} \qquad (1)$$

where ($\delta u = u(x+r,t) - u(x,t)$) is the velocity increment, r the space increment, and C_n are universal constants independent of the Reynolds number and the turbulent flow configuration (the overbar denotes ensemble averaging). The range over which this law applies is called the inertial range and is defined as $\eta \ll r \ll L$, where η and L are the Kolmogorov length scale and integral length scale, and they are representative of the dissipative and energy containing large scales, respectively). However, almost 20 years later [2], acting on a remark made by Landau, he proposed the so-called refinement similarity hypotheses (hereafter denoted K62), which predicts that, in the inertial range, $\overline{(\delta u)^n}$ behaves as follows:

$$\overline{(\delta u)^n} \sim r^{\zeta_n}, \qquad (2)$$

where $\zeta_n \neq n/3$, except for $n = 3$. It should be pointed out that, independently of [2], Oboukhov [3] proposed a correction to $\overline{(\delta u)^2} \sim r^{2/3}$; in fact, K62 uses some elements of [3]. The lack of agreement between the experimental data and the prediction of K41, thought to vindicate Landau's remark and K62, and commonly attributed to the phenomenon of internal intermittency [4, 5], motivated the search for correcting or altering (1). Interestingly, while K41 predicts $\overline{(\delta u)^3} \sim r$, K62 simply uses it. This is because Kolmogorov assumed that $\overline{(\delta u)^3} \sim (4/5)\bar{\varepsilon} r$ is valid for K62. This points to a critical requirement for both theories, namely that the Reynolds number must be infinitely large (as seen later). This immediately raises a concern regarding the comparison between the experimental and numerical simulation data invariably obtained at finite Reynolds numbers and the predictions of both K41 and K62 for an infinitely large Reynolds number.

As mentioned above, the common feature of K41 and K62 is the four-fifths law (*i.e.* $\overline{(\delta u)^3} = (4/5)\bar{\varepsilon} r$), which is a rigorous result derived from the transport equation of $\overline{(\delta u)^2}$ when the Reynolds number is infinitely large [6]. Considering that this law is indisputable from a theoretical point of view, it is not surprising that it played a critical role in the turbulence theory, in particular in the assessment on both K41 and K62. Indeed, any theory of HIT at infinitely large Reynolds number must be consistent with this law. Unfortunately, the impossibility to reach an infinitely large Reynolds number makes it impossible to test through measurements or numerical simulations whether any proposed theory of HIT complies with this law. Nevertheless, despite this drawback, attempts were carried out, mostly from phenomenological arguments [4], to develop expressions for $\overline{(\delta u)^n}$. Without exception, all past and current expressions are of the form (2), which reflects the so-called anomalous behaviour of $\overline{(\delta u)^n}$ believed to be caused by the small-scale intermittency and which leads to ζ_n deviating from $n/3$ except for $n = 3$; $\zeta_3 = 1$ is imposed in order to recover the four-fifths law. Unfortunately, since the compliance with this law ($\overline{(\delta u)^3} \sim r$) by (2) cannot be tested at finite Reynolds numbers since the four-fifths law is not tenable at such Reynolds numbers, one cannot rule out the possibility that the proposed models (or more precisely the power-law exponents) may not be realisable in the sense that they may violate some kinematic and kinetic constraints.

In the present paper we develop realisability conditions for testing models of $\overline{(\delta u)^n}$, which yield $\overline{(\delta u)^n} \sim r^{\zeta_n}$ in the inertial range. We first consider an infinitely large Reynolds number which allows us to develop realisability conditions for the exponent ζ_n.

Realisability contitions

Infinite Reynolds number

If one assumes that $\overline{(\delta u)^n} \sim r^{\zeta_n}$ for any positive real n and the 4/5 law is verified, one can then easily derive the relationship between the exponents ζ_n. For example we can relate the exponents ζ_n to ζ_2 for $n \geq 3$. Indeed, writing $\overline{(\delta u)^3} = \overline{(\delta u)^2(\delta u)}$ and applying the Cauchy-Schwarz theorem, we have

$$|\overline{(\delta u)^3}| = |\overline{(\delta u)^2(\delta u)}| \leq \overline{(\delta u)^4}^{1/2}\,\overline{(\delta u)^2}^{1/2}. \qquad (3)$$

Thus, assuming $\overline{(\delta u)^n} \sim r^{\zeta_n}$ where ζ_n varies with n, one must have, by virtue of the 4/5 law (*i.e.* $\zeta_3 = 1$)

$$r \leq r^{\zeta_4/2} r^{\zeta_2/2} \qquad (4)$$

or, equivalently,

$$\zeta_4 \geq 2 - \zeta_2. \tag{5}$$

Also, for example, ζ_5, ζ_6, ζ_7 and ζ_8 can be obtained in a similar manner. Let us write $\overline{(\delta u)^4} = \overline{(\delta u)^3(\delta u)}$ and applying the Cauchy-Schwarz theorem, we find $\zeta_4 \leq (\zeta_6 + \zeta_2)/2$, and thus we have $\zeta_6 \geq 4 - 3\zeta_2$. writing $\overline{(\delta u)^5} = \overline{(\delta u)^3(\delta u)^2}$ we get $\zeta_5 \leq (\zeta_6 + \zeta_4)/2$. However, $(\zeta_6 + \zeta_4) \geq (3 - 2\zeta_2)$. Thus, one can have $\zeta_5 \leq 3 - 2\zeta_2$ or $\zeta_5 \geq 3 - 2\zeta_2$, while the constraint $\zeta_5 \leq (\zeta_6 + \zeta_4)/2$ must be satisfied. Further, writing $\overline{(\delta u)^5} = \overline{(\delta u)^4(\delta u)}$ and $\overline{(\delta u)^7} = \overline{(\delta u)^3(\delta u)^4}$, leads to $\zeta_7 \leq 5 - 4\zeta_2$ or $\zeta_7 \geq 5 - 4\zeta_2$ and $\zeta_8 \geq 6 - 5\zeta_2$. Finally, we obtain

$$\zeta_n \leq or \geq (n-2) - (n-3)\zeta_2,\ when\ n = 2p+1 \tag{6}$$

$$\zeta_n \geq (n-2) - (n-3)\zeta_2,\ when\ n = 2p \tag{7}$$

where p is an integer. These expressions can be interpreted as the realisability conditions for the exponents ζ_n of the power-law form (2). Note that at this stage there is nothing to suggest or help us determine the value of ζ_2. If we follow the general practice and assume that ζ_2 departs slightly from the value 2/3, we can, to a first approximation, write $\zeta_2 = 2/3 + \xi$, with ξ small and real. Then (6) and (7) become

$$\zeta_n \leq or \geq \frac{n}{3} - (n-3)\xi,\ when\ n = 2p+1 \tag{8}$$

$$\zeta_n \geq \frac{n}{3} - (n-3)\xi,\ when\ n = 2p \tag{9}$$

Recall that ζ_n must increase monotonically with n. Thus, we must have, for example,

$$2 - \zeta_2 < 3 - 2\zeta_2 < 4 - 3\zeta_2. \tag{10}$$

since $\zeta_4 < \zeta_5 < \zeta_6$. It is important to remember that these relations are valid if and only if the 4/5 law is satisfied and $\overline{(\delta u)^n} \sim r^{\zeta_n}$. Using Hölder's inequality, [4] proposed a convexity inequality for the even exponents (see his expression (8.11)) to explain the trend exhibited by experimental data of [7]. Noteworthily, the Cauchy-Schwarz inequality is a special case of Hölder's inequality.

Interestingly, replacing the inequality sign by the equality sign in the above expressions (this is a valid possibility compliant with the Cauchy-Schwarz inequality) yields a more stringent realisability condition

$$\zeta_n = \frac{n}{3} - (n-3)\xi, \tag{11}$$

which can be interpreted as a limiting realisability condition for the exponent ζ_n. Notice a resemblance to several relations for ζ_n, derived from phenomenological arguments of small-scale intermittency, in particular with the β-model. This model yields

$$\zeta_n = \frac{n}{3} + (3-D)(1 - \frac{n}{3}), \tag{12}$$

where D is a (fractal) dimension. When $n = 2$, $\zeta_2 = \frac{2}{3} + (3-D)/3 = \frac{2}{3} + \xi$. Thus, (11) is identical to (12) if $\xi = 1 - D/3$. The β-model, like (11), predicts a linear variation of ζ_n with n and has been abandoned as it does not agree with experimental results for large values of n and replaced by more sophisticated models, such as multifractal models (*e.g.* [4, 5]), developed to achieve better agreement between model predictions and experimental data. Despite their differences, all small-scale intermittency models (non fractal, fractal and multifractal) must adhere to $\zeta_3 = 1$ since they must reproduce the 4/5 law. This indicates that these models also assume, at least implicitly, an infinitely large Reynolds number, which raises an obvious concern. This is discussed in the below.

Finite Reynolds number

The realisability conditions (8) and (9) hold, in principle, for an infinitely large Reynolds number, which is an essential requirement for the 4/5 law but also for both (1) and (2). When the Reynolds number is finite, the 4/5 law is not strictly valid. This is illustrated in Figure 1 which shows distributions of $-\overline{(\delta u)^3}/(\bar{\varepsilon} r)$ in terms of r/η in forced turbulence in a periodic box at $Re_\lambda = 732$ and 1131 (the data have been extracted from Figure 3b of [8]); also included in the figure are the experimental data of [7] which will be commented on later in this section. Although 0.8 is relatively well approached by the maximum of -$\overline{(\delta u)^3}/(\bar{\varepsilon} r)$ for the largest Re_λ, there is no convincing plateau, indicating that the 4/5 law is not yet satisfied and implying the absence of an inertial range. The consequence of the absence of a plateau is evident: ζ_3 cannot be equal to 1.

While, as stated above, a power-law form as expressed as (1) or (2) cannot be tenable at finite Reynolds numbers, it was nevertheless tested in finite Reynolds number turbulent flows (*e.g.*[7, 9, 10]). It is in this context then pertinent to assess its realisability when the Reynolds number is finite. Staring with $\overline{(\delta u)^3}$, if for a large but finite Reynolds number the departure from the 4/5 law is assumed to be small, one can then, under a linearisation approximation, write $\zeta_3 = \zeta_{3,exact} + \chi_3$, where $\zeta_{3,exact}$ should be 1 by virtue of the 4/5 law and χ_3 is a small real number. Let us further assume that $\overline{(\delta u)^n} \sim r^{\zeta_n}$ can also be approximately verified for all n (strictly not valid; at least, it remains to be proven). Then, applying the same procedure as above involving the Cauchy-Schwarz inequality, we obtain the new realisability conditions

$$\zeta_n \leq or \geq (n-2)(1-\chi_3) - (n-3)\zeta_2,\ when\ n = 2p+1 \tag{13}$$

$$\zeta_n \geq (n-2)(1-\chi_3) - (n-3)\zeta_2,\ when\ n = 2p, \tag{14}$$

or the more stringent one

$$\zeta_n = (n-2)(1-\chi_3) - (n-3)\zeta_2. \tag{15}$$

As $Re_\lambda \to \infty$, one must recover the 4/5 law, then $\zeta_3 \to 1$, and consequently $\chi_3 = f_3(Re_\lambda) \to 0$, which implies $\zeta_n = f_n(Re_\lambda)$. At this stage, ζ_2 remains an unknown; it may or may not depend on Re_λ. If it depends on Re_λ, it must approach a limiting value, say $\zeta_{2,exact}$, as $Re_\lambda \to \infty$. If $\zeta_{2,exact} = 2/3$ at infinitely large Re_λ, then (15) indicates that $\overline{(\delta u)^n} \sim r^{n/3}$, *i.e.* the K41 prediction, is realisable when $Re_\lambda \to \infty$. The Re_λ-dependence of ζ_n has been extensively discussed in [13] who show a clear Re_λ-dependence on ζ_n in various flows if one assumes that (2) holds at finite Reynolds numbers. We report in Figure 2 the values of ζ_n, based on (15), for finite Re_λ; for ζ_2, we used the experimental results of [7], also reported in the figure, where $\zeta_2 = 0.71$; although these data are relatively old, they played a pivotal role in the development and testing of the current intermittency models of $\overline{(\delta u)^n}$, which is why we reproduce them here; also since all studies carried out after this seminal paper reproduce the same trend as seen in the figure, we only reproduce a few results obtained via DNS of forced HIT [9, 10]. To plot (15), we have arbitrarily used the asymptotic limit $\zeta_3 = 1$ only for the purpose of illustration; changing ζ_3 changes only the "anchor point" or the point around which the family of curves calculated with (15) for different ζ_2 would rotate. Note that the experimental values of ζ_n for $n \geq 10$ may be questionable; we show them only because they were included in the data reported in [7]. According to the present analysis, ζ_n for $n = 2p$ should lie on or above

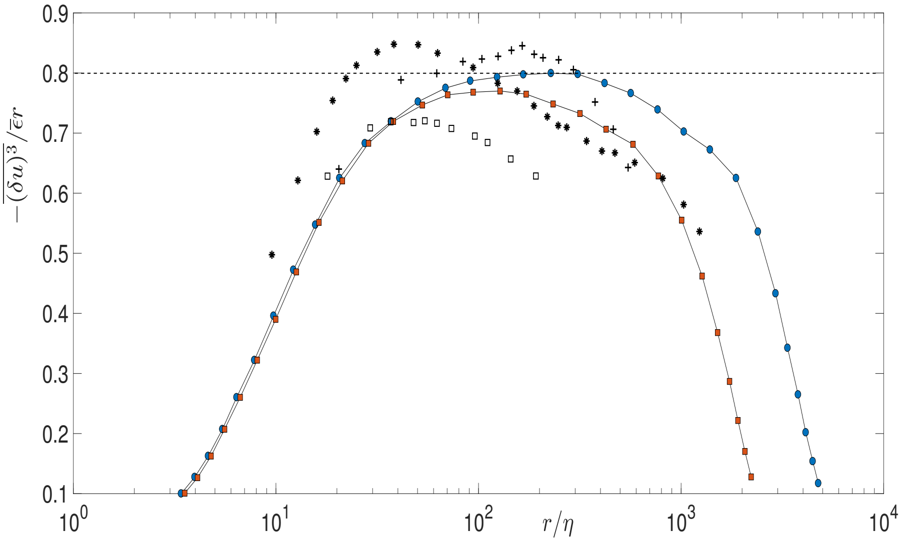

Figure 1: Distributions of $-\overline{(\delta u)^3}/(\bar{\varepsilon}r)$ versus r/η. Forced turbulence in a periodic box [8] at $Re_\lambda = 732$ (red squares) and 1131 (blue circles). Experimental data of [7]; open squares: $Re_\lambda = 515$; +: $Re_\lambda = 536$; *: $Re_\lambda = 852$.

the values calculated with (15) while ζ_n with $n = 2p+1$ could lie on, above or below the values calculated with (15), with the constraint that $\zeta_n < \zeta_{n+1}$; a lack of compliance with this represents a violation of the realisability conditions. Clearly, the data (for the even exponents with $n \geq 4$) reported by [7] violate this latter condition. A similar conclusion can be drawn for the DNS data of forced HIT.

The value of ζ_3 reported in [7] warrants a comment as it conflicts with a lack of a plateau in their data shown Figure 1. [7] states that the 4/5 law is free of any intermittency assumption. While this is a correct statement, which might have misled the authors in assuming that the deviation of a plateau was due to measurement uncertainties, it unfortunately ignores the finite Reynolds number effects, which, to be fair to the authors, was not known at that time. Figure 1 conclusively provides evidence that ζ_3 cannot be equal to 1 in the data reported in [7]; this should not be surprising since as, we have seen, $\zeta_3 = 1$ requires an infinitely large Reynolds number and is not strictly observed in the more reliable DNS data of [8] at larger Re_λ than those attained in [7]. Further, the data for $Re_\lambda = 536$ and 852 exhibit an unrealistic behaviour, *i.e.* $-\overline{(\delta u)^3}/(\bar{\varepsilon}r)$ exceeds the limiting value of 0.8 over a non negligible range of r; [11] attributed this unrealistic behaviour of $-\overline{(\delta u)^3}/(\bar{\varepsilon}r)$ to an underestimation of $\bar{\varepsilon}$. A possible alternative reason for this behaviour relates to the fact that [7] used turbulent shear flows (*e.g.* jet and pipe flows) which are not only subjected to the finite Reynolds number effects, but also to large scale anisotropy. Thus, caution should be exercised regarding these measurements and their use for testing K41 and K62 is rather dubious. This caution can in fact be generalised to all finite Reynolds number measurements and numerical simulations.

Figure 2 also shows the β-model, one of the earlier intermittency models. We have used $D = 2.8$, the value generally accepted [4]. If we use $D = 2.87$ ($= 3 - 3\xi$, so that (12) is identical to (11)) corresponding to $\zeta_2 = 0.71$, the β-model collapses perfectly well with (15), as expected. Not surprisingly, taking $D = 3$, which implies $\xi = 0$, one recovers K41. Note that any linear model (in n) satisfies the realisability condition (15).

Multifractal models were developed with the objective to improve agreement with the empirically estimated values of ζ_n of [7] or at least to reproduce the curvature exhibited in Figure 2 that the β-model fails to capture (e.g. [14]). However, notwithstanding that no power law of r for $\overline{(\delta u)^n}$ can be strictly observed for finite Reynolds numbers, since the exponents ζ_n for the multifractal models follow the concavity exhibited by the experimental data, one can only conclude that they violate the realisability conditions. Oddly enough, the realisability for the odd exponents may appear to be verified by the experimental data. However, this is only fortuitous. Indeed, consider for example ζ_9. Since ζ_n should conform with a nondecreasing trend with increasing n, ζ_9 must be larger than ζ_8, whose value must be larger than or equal to that calculated with (15). The experimental data present a concavity which cannot satisfy *simultaneously* the realisability conditions and the nondecreasing requirement.

Conclusions

Small-scale intermittency, which has been a subject of active research for about 60 years, is believed to be responsible for the anomalous scaling of the moments of the velocity increments. Models have been proposed, using phenomenological arguments, to account for this anomaly. It is therefore important to ensure that these models are realisable so that their predictions comply with the equations of motion and do not lead to incorrect/impossible results. The realisability conditions proposed here provide a simple, reliable and objective means for testing the viability of past and future models (based on a power-law form (2)) in predicting the behaviour of the structure functions of any order when the Reynolds number is infinitely large. In that respect, they play a similar role to the realizability conditions for the Reynolds-stress turbulence models [12]. Importantly, they can also be used to assess the realisability of the power-law exponents ζ_n obtained empirically from either experimental or numerical data. For example, they show that ζ_n estimated by [7] are not realisable. Consequently, all power-law models which reproduce the experimental results of [7] cannot be realisable. Note however that these realisability conditions cannot be used to assess the validity of any phenomenology behind the development of scaling laws in the form $\overline{(\delta u)^n} \sim r^{\zeta_n}$.

It is important to stress that the realisability conditions proposed

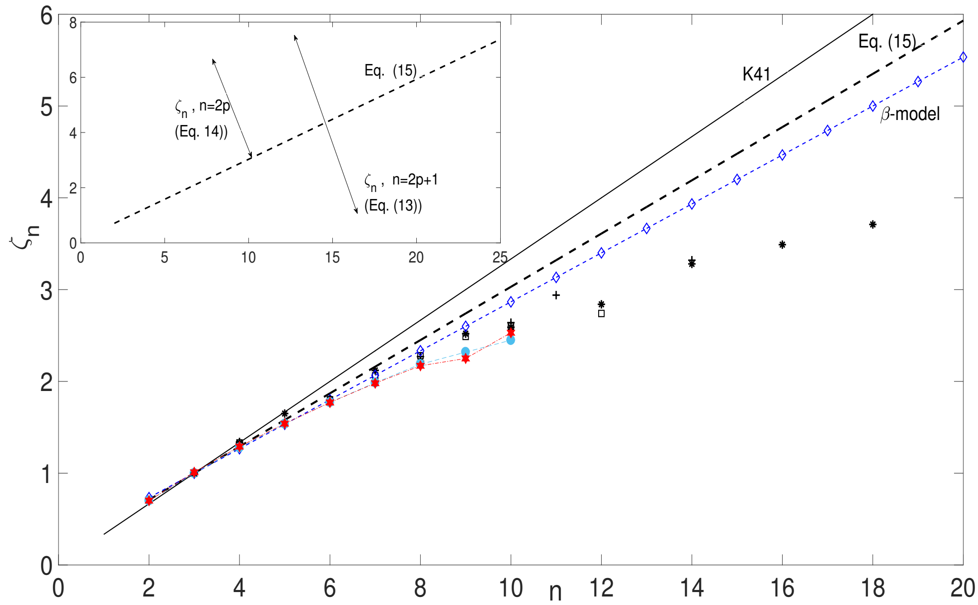

Figure 2: Variation of ζ_n as a function of order n. Solid line: $\zeta_n = n/3$, according to K41; thick dashed line: equation (15) with $\xi = 0$ (or equivalently $\zeta_3 = 1$) and $\zeta_2 = 0.71$; blue diamonds: β-model with $D = 2.8$. Experimental data of [7], square: $Re_\lambda = 515$; +: $Re_\lambda = 536$; *: $Re_\lambda = 852$. DNS data: [10], cyan bullets: $Re_\lambda = 250$ (this was a maximum value); [9], red stars: $Re_\lambda = 460$.

here are only valid when $\overline{(\delta u)^n}$ is expressed in a power-law form (2). At finite Reynolds numbers, one cannot have $\overline{(\delta u)^3} \sim r$, which automatically rules out any model that predicts this relation when the Reynolds number is finite; this includes not only intermittency models, but also non-intermittency ones such as power-law models based on K41, *e.g.*, relation (1). Accordingly, one cannot use (2) to assess the effects of finite Reynolds number on $\overline{(\delta u)^n}$. One must be rather careful when attempting to fit power-law models for $\overline{(\delta u)^n}$ to experimental and numerical simulation data. Caution should be exercised: if a power-law model is used, even as an empirically approximate model, to express $\overline{(\delta u)^n}$, then not only should the (empirical) power-law exponents comply with the realisability conditions, but, as shown in the present analysis and [13], they must vary with the Reynolds number; importantly too, the exponent ζ_3 cannot be assumed to be equal to 1. Interestingly, only K41, with $\zeta_3 = 1$, can satisfy realisability as well as independence on Re_λ.
Finally, the impossibility to carry out experiments and simulations at an infinite Reynolds number raises a major concern with regard to the so-called the anomalous behaviour (*i.e.* $\zeta_n \neq n/3$ when n is not 3 in (2)), which has been apparently confirmed by experimental and numerical results. However, as seen above, $\overline{(\delta u)^n} \sim r^{\zeta_n}$ is not tenable when the Reynolds number is finite. What is perhaps anomalous is to find $\zeta_3 = 1$ when the Reynolds number is finite.

References

[1] Kolmogorov, A. N., The locally structure of turbulence in incompressible viscous fluid for very large Reynolds numbers, Dokl. Akad. Nauk SSSR 30 (see also Proc. R. Soc. Lond. A (1991), 434, 9-13) (1941).

[2] Kolmogorov, A. N. 1962 A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. J . Fluid Mech. 13, 8285.

[3] Oboukhov, A.M. 1962 Some specific features of atmospheric turbulence. Journal of Fluid Mechanics 13 (1), 7781.

[4] Frisch, U. 1995 Turbulence: the legacy of AN Kolmogorov. Cambridge university press.

[5] Boffetta, G., Mazzino, A. and Vulpiani, A. 2008 Twenty-five years of multifractals in fully developed turbulence: a tribute to giovanni paladin. Journal of Physics A: Mathematical and Theoretical 41 (36), 363001

[6] Kolmogorov, A. N. 1941b Dissipation of energy in the locally isotropic turbulence. Dokl. Akad. Nauk SSSR 32 (see also Proc. R. Soc. Lond. A (1991), 434, 15-17).

[7] Anselmet, F., Gagne, Y. , E. J. Hopfinger, E.J. and Antonia, R.A. High-order velocity structure functions in turbulent shear flows, J. Fluid Mech 1984, 140,63–89.

[8] Ishihara, T., Gotoh, T. and Kaneda, Y. 2009 Study of High-Reynolds Number IsotropicTurbulence by Direct Numerical Simulation. Annu. Rev. Fluid Mech. 41, 16580.

[9] Gotoh, T., Fukayama, D. and Nakano, T. 2002 Velocity field statistics in homogeneous steady turbulence obtained using a high-resolution direct numerical simulation. Phys. Fluids 14 (3), 10651081.

[10] Chen, S.Y., Dhruva, B., Kurien, S., Sreenivasan, K.R. and Taylor, M.A. 2005 Anomalous scaling of low-order structure functions of turbulent velocity. J. Fluid Mech. 533, 183192.

[11] Lindborg, E. 1999 Correction to the four-fifths law due to variations of the dissipation. Physics of Fluids 11 (3), 510512.

[12] Schumann, U. 1977 Realizability of reynolds-stress turbulence models. Phys. Fluids 20 (5), 721725.

[13] Tang, S.L., Antonia, R.A., Djenidi,L., Danaila, L. and Zhou, Y. 2017 Finite Reynolds number effect on the scaling range behaviour of turbulent longitudinal velocity structure functions, J. Fluid Mech. 820, 341-369.

[14] Parisi, G. and Frish, U. 1985 On the singularity structure of fully developed turbulence. Turbulence and predictability in geophysical fluid dynamics and climate dynamics.