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Markovian inhomogeneous closures for Rossby waves and turbulence over topography

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Abstract

Manifestly Markovian closures for the interaction of two-dimensional inhomogeneous turbulent flows with Rossby waves and topography are formulated and compared with large ensembles of direct numerical simulations (DNS) on a generalized β-plane. Three versions of the Markovian inhomogeneous closure (MIC) are established from the quasi-diagonal direct interaction approximation (QDIA) theory [6, 15, 8] by modifying the response function to a Markovian form and employing respectively the current-time (quasi-stationary) fluctuation dissipation theorem (FDT), the prior-time (non-stationary) FDT and the correlation FDT. Markov equations for the triad relaxation functions are derived that carry similar information to the time-history integrals of the non-Markovian QDIA closure but become relatively more efficient for long integrations. Far from equilibrium processes are studied, where the impact of a westerly mean flow on a conical mountain generates large amplitude Rossby waves in a turbulent environment, over a period of 10 days. Excellent agreement between the evolved mean streamfunction and mean and transient kinetic energy spectra are found for the three versions of the MIC and two variants of the non-Markovian QDIA compared with an ensemble of 1800 DNS. In all cases mean Rossby wavetrain pattern correlations between the closures and the DNS ensemble are greater than 0.9998.

Introduction

Modern statistical dynamical closure theory, initially applied to the iconic problem of homogeneous isotropic turbulence, has its origin in the pioneering works of Kraichnan [10] who derived the equations for his Eulerian direct interaction approximation (DIA) closure on the basis of formal renormalized perturbation theory. His approach had some elements in common with the renormalized perturbation theory and functional approaches to quantum electrodynamics (QED) developed in the mid-20th century by Tomonaga, Schwinger and Feynman ([7] reviews the literature). Unlike QED, where the fine structure constant, measuring interaction strength, is small (1/137), turbulence at high Reynolds number is a problem of strong interaction. The DIA is a two-point non-Markovian closure for the renormalized two-time covariances or cumulants and response functions but the interaction coefficients, expressed as vertices in diagrammatic form, are unrenornalized or bare. Subsequent independent approaches to the problem of homogeneous isotropic turbulence were developed [9, 13] however, these two-point non-Markovian closures were later shown to differ from the DIA only in how a fluctuation-dissipation theorem (FDT) [4] is invoked. The prior-time FDT relates the two-time spectral covariance $C_{\mathbf{k}}(t,t')$ at wavenumber $\mathbf{k}$ to the response function $R_{\mathbf{k}}(t,t')$ and the prior single-time covariance $C_{\mathbf{k}}(t',t')$ through

$$C_{\mathbf{k}}(t,t') \equiv R_{\mathbf{k}}(t,t')C_{\mathbf{k}}(t',t') \tag{1}$$

for $t \geq t'$.

Eulerian [10, 9, 13], and also quasi-Lagrangian homogeneous turbulence closures [11], describe the evolution of the renormalized propagators, the response functions $R_{\mathbf{k}}$ and two-point cumulants $C_{\mathbf{k}}$, but have deficiencies which are most obviously manifest in an underestimation of the inertial range kinetic energy spectrum. These inaccuracies arise from a failure to systematically correctly account for interactions due to strong coupling across scales or more technically, the modification of the strength and form of the interaction coefficients due to an additional vertex renormalization. In fact the whole problem of strong turbulence is contained in a proper treatment of vertex renormalization [12], an unsolved problem for strongly interacting fields in general. However, it has been shown that removing some of the convection effects of the large scales on the small scales modifies the Eulerian DIA such that it becomes consistent with the Kolmogorov inertial range spectra. Specifically this involves zeroing the interaction coefficients dependent on a cut-off ratio α such that the interactions between triads of wavenumbers are now localized in wavenumber space. We have found that there is an essentially universal value of α that gives close agreement at all scales of the energy spectra between ensembles of DNS [15]. This choice of α results in a 1-parameter empirical vertex renormalization applicable at all resolutions and independent of the strength of the interaction between waves and turbulence.

Non-Markovian closures with potentially long time-history integrals present a significant computational challenge, particularly at high resolution, and even more so if the general inhomogeneous problem is attempted. Orszag [14] employed a heuristic approach to derive a simpler Markovian closure, denoted the eddy damped quasi-normal Markovian (EDQNM) closure, for the evolution of the single-time covariance associated with homogeneous isotropic turbulence. He recognized that in order to reproduce the Kolmogorov inertial range at small scales the molecular viscosity should be augmented by an eddy viscosity that has a functional form consistent with the Kolmogorov power law and a strength determined by an empirical constant. Interestingly, the EDQNM closure satisfies an H-theorem that guarantees the monotonic increase of entropy for the inviscid system and approach to a canonical equilibrium solution for two and three-dimensional systems [5]. The EDQNM closure can also be derived by modifying the Eulerian DIA; firstly the response function is replaced by a Markovian form with this eddy viscosity and secondly the two-time covariance is determined from the current-time FDT

$$C_{\mathbf{k}}(t,t') \equiv R_{\mathbf{k}}(t,t')C_{\mathbf{k}}(t,t) \tag{2}$$

for $t \geq t'$.

Bowman et al. [2, 3] showed that for anisotropic turbulence in the presence of linear wave phenomena EDQNM type closures may be potentially nonrealizable. They demonstrated that this was a possibility if the EDQNM was derived as a modification of the DIA closure with Markovian response functions and the two-time covariances determined by the current-time FDT (equation 2) or the prior-time FDT (equation 1). In order to eliminate the possibility of negative energies in the presence of large scale fluctuations, they established a realizable Markovian closure (RMC) by using a response function with positive damping and specifying the two-time covariance through the correlation FDT

$$C_{\mathbf{k}}(t,t') \equiv [C_{\mathbf{k}}(t,t)]^{1/2} R_{\mathbf{k}}(t,t')[C_{\mathbf{k}}(t',t')]^{1/2} \tag{3}$$

for $t \geq t'$. Here we report on application of these three versions of the FDT to formulate Markovian inhomogeneous closure (MIC) variants for the problem of general two-dimensional inhomogeneous turbulent flows interacting with Rossby waves and topography on a generalized β-plane but show only results for the correlation FDT variant in figure 1.

Two-dimensional flow over topography on a generalized β-plane

Following [8], we write the streamfunction in the form $\Psi = \psi - Uy$ where the small scales are determined by ψ and the large-scale westerly flow by U. The small scales evolve according to the barotropic vorticity equation in the presence of the large-scale flow and topography

$$\frac{\partial \zeta}{\partial t} = -J(\psi - Uy, \zeta + h + \beta y + k_0^2 U y) + \hat{\nu}\,\nabla^2 \zeta + f^0 \tag{4}$$

where ζ is the vorticity, h the scaled topography, $\hat{\nu}$ is the bare viscosity, f^0 a forcing function, β is the beta effect due to differential rotation, and k_0 is a wavenumber that, on a sphere, would determine the strength of the vorticity of the solid-body rotation. There is then a one-to-one correspondence between the spherical geometry and β-plane equations that extends to the statistical mechanics equilibrium solutions in the two geometries. The Jacobian is

$$J(\psi,\zeta) = \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x} \tag{5}$$

and the vorticity is related to the streamfunction through $\zeta = \nabla^2 \psi$. The large-scale flow U evolves according to the form-drag equation

$$\frac{\partial U}{\partial t} = \frac{1}{(2\pi)^2}\int_0^{2\pi} d^2\mathbf{x}\; h(\mathbf{x})\frac{\partial \psi(\mathbf{x})}{\partial x} + \alpha_U(\overline{U} - U). \tag{6}$$

The integrations of equations 4 and 6 are carried out for flows on the doubly periodic plane . Here $\mathbf{x} = (x,y)$ and the flow U is forced by relaxing it towards $\overline{U}$ with relaxation coefficient α_U. Spectral equations corresponding to this system were derived by expanding each of the small-scale functions in a Fourier series defined for a circular domain in wavenumber space $\mathbf{R}$ excluding the origin $\mathbf{0}$ i.e. $\mathbf{k} = (k_x, k_y)$, $k = (k_x^2 + k_y^2)^{1/2}$, with $\zeta_{\mathbf{k}} = \zeta^*_{-\mathbf{k}}$. The spectral equations for the small scales are then combined with the spectral representation of the form drag equation by extending the wavenumber space to include the origin via the relationship $\zeta_{-\mathbf{0}} = ik_0 U$, where $\zeta_{\mathbf{0}} = \zeta^*_{-\mathbf{0}}$ as the zero wavenumber spectral component. As well, generalised interaction coefficients A and K were derived allowing the spectral form of the vorticity equation to be written as in the same form as for flows on a non-rotating domain i.e.

$$\begin{aligned}(\frac{\partial}{\partial t} + \nu_{\mathbf{0}}(\mathbf{k})k^2)\zeta_{\mathbf{k}}(t) = \sum_{\mathbf{p}\in\mathbf{T}}\sum_{\mathbf{q}\in\mathbf{T}} \delta(\mathbf{k},\mathbf{p},\mathbf{q})[K(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}\zeta_{-\mathbf{q}} \\ + A(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}h_{-\mathbf{q}}] + f^0_{\mathbf{k}}\end{aligned} \tag{7}$$

where $\mathbf{T} = \mathbf{R}\cup\mathbf{0}$, $\delta(\mathbf{k},\mathbf{p},\mathbf{q}) = 1$ if $\mathbf{k}+\mathbf{p}+\mathbf{q} = 0$ and is otherwise 0. The Rossby wave frequency is given by $\omega_k = -\frac{\beta k_x}{k^2}$ and the scale dependent viscosity $\nu_0(\mathbf{k})k^2 = \hat{\nu}k^2 + i\omega_k$. The $\mathbf{k} = \mathbf{0}$ components of the forcing and viscosity are defined by $f^0_{\mathbf{k}} = \alpha_U \overline{\zeta}_{\mathbf{0}}$ and $\nu_0(\mathbf{0})k_0^2 = \alpha_U$.

QDIA closure equations

We consider next an ensemble of flows satisfying equation 7 where we express the vorticity component for a given realization by $\zeta_{\mathbf{k}} = \langle\zeta_{\mathbf{k}}\rangle + \hat{\zeta}_{\mathbf{k}}$ where the ensemble mean is denoted by $\langle\zeta_{\mathbf{k}}\rangle$ and angle brackets denote expectation value. The spectral equations can then be written in terms of mean and transients and we can define the two-point two-time cumulant as $C_{-\mathbf{p},-\mathbf{q}}(t,t') = \langle\tilde{\zeta}_{-\mathbf{p}}(t), \tilde{\zeta}_{-\mathbf{q}}(t')\rangle$.

The inhomogeneous non-Markovian QDIA closure equations comprise a self consistent field theory derived via renormalised perturbation methods where the second order terms can all be expressed in terms of single loop diagrammatic expressions [1]. Here we simply state the closed set of equations and refer the interested reader to the appropriate references [6, 7, 8, 15]. The tendency equations are: for the mean-field

$$\begin{aligned}(\frac{\partial}{\partial t} + \nu_{\mathbf{0}}(\mathbf{k})k^2)\langle\zeta_{\mathbf{k}}(t)\rangle = \sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q}) \\ \times\left[A(\mathbf{k},\mathbf{p},\mathbf{q})\langle\zeta_{-\mathbf{p}}(t)\rangle h_{-\mathbf{q}}\right. \\ \left.+K(\mathbf{k},\mathbf{p},\mathbf{q})[\langle\zeta_{-\mathbf{p}}(t)\rangle\langle\zeta_{-\mathbf{q}}(t)\rangle + C_{-\mathbf{p},-\mathbf{q}}(t,t)]\right] \\ +\overline{f}^0_{\mathbf{k}}(t),\end{aligned} \tag{8}$$

and for the two-time cumulant

$$\begin{aligned}(\frac{\partial}{\partial t} + \nu_{\mathbf{0}}(\mathbf{k})k^2)C_{\mathbf{k}}(t,t') \\ = \sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})A(\mathbf{k},\mathbf{p},\mathbf{q})C_{-\mathbf{p},-\mathbf{k}}(t,t')h_{-\mathbf{q}} \\ +\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})K(\mathbf{k},\mathbf{p},\mathbf{q})[\langle\zeta_{-\mathbf{p}}(t)\rangle C_{-\mathbf{q},-\mathbf{k}}(t,t') \\ +C_{-\mathbf{p},-\mathbf{k}}(t,t')\langle\zeta_{-\mathbf{q}}(t)\rangle + \langle\zeta_{-\mathbf{p}}(t)\zeta_{-\mathbf{q}}(t)\zeta_{-\mathbf{k}}(t')\rangle] \\ +\int_{t_0}^{t'} ds\; F^0_{\mathbf{k}}(t,s)R_{-\mathbf{k}}(t',s).\end{aligned} \tag{9}$$

The closure requires substitution for the two- and three- point expressions here given by

$$\begin{aligned}C_{\mathbf{k},-\mathbf{l}}(t,t') = \int_{t_0}^{t} ds\; R_{\mathbf{k}}(t,s)C_{\mathbf{l}}(s,t') \\ \times[A(\mathbf{k},-\mathbf{l},\mathbf{l}-\mathbf{k})h_{\mathbf{k}-\mathbf{l}} + 2K(\mathbf{k},-\mathbf{l},\mathbf{l}-\mathbf{k})\langle\zeta_{\mathbf{k}-\mathbf{l}}(s)\rangle] \\ +\int_{t_0}^{t'} ds\; R_{-\mathbf{l}}(t',s)C_{\mathbf{k}}(t,s) \\ \times[A(-\mathbf{l},\mathbf{k},\mathbf{l}-\mathbf{k})h_{\mathbf{k}-\mathbf{l}} + 2K(-\mathbf{l},\mathbf{k},\mathbf{l}-\mathbf{k})\langle\zeta_{\mathbf{k}-\mathbf{l}}(s)\rangle]\end{aligned} \tag{10}$$

and

$$\begin{aligned}\langle\hat{\zeta}_{-\mathbf{p}}(t)\hat{\zeta}_{-\mathbf{q}}(t)\hat{\zeta}_{-\mathbf{k}}(t')\rangle \\ = 2\int_{t_0}^{t'} ds\; K(-\mathbf{k},-\mathbf{p},-\mathbf{q})C_{-\mathbf{p}}(t,s)C_{-\mathbf{q}}(t,s)R_{-\mathbf{k}}(t',s) \\ +2\int_{t_0}^{t} ds\; K(-\mathbf{p},-\mathbf{q},-\mathbf{k})R_{-\mathbf{p}}(t,s)C_{-\mathbf{q}}(t,s)R_{-\mathbf{k}}(t',s) \\ +2\int_{t_0}^{t} ds\; K(-\mathbf{q},-\mathbf{p},-\mathbf{k})R_{-\mathbf{q}}(t,s)C_{-\mathbf{p}}(t,s)C_{-\mathbf{k}}(t',s).\end{aligned} \tag{11}$$

Here $f_{\mathbf{k}} = \overline{f}^0_{\mathbf{k}} + \tilde{f}^0_{\mathbf{k}}$, $\overline{f}^0_{\mathbf{k}} = \langle f^0_{\mathbf{k}}\rangle$ and $F^0_{\mathbf{k}}(t,s) = \langle\tilde{f}^0_{\mathbf{k}}(t), \tilde{f}^0_{-\mathbf{k}}(s)\rangle$. This treatment of the third moment agrees with the DIA, where the first two symmetric terms on the RHS of equation 11 contribute a nonlinear damping whereas the third term constitutes a nonlinear noise. At canonical equilibrium these terms cancel each other exactly.

Finally, we need an equation for the response of the system to infinitesimal fluctuations i.e. the off diagonal terms of the response function $R_{\mathbf{k},\mathbf{l}}(t,t') = \langle \frac{\delta \hat{\zeta}_{\mathbf{k}}(t)}{\delta \hat{f}^0_{\mathbf{l}}(t')} \rangle$ expressed in terms of diagonal cumulant $C_{\mathbf{k}}(t,t') = C_{\mathbf{k},-\mathbf{k}}(t,t')$ and response $R_{\mathbf{k}}(t,t') = R_{\mathbf{k},\mathbf{k}}(t,t')$ functions i.e.

$$
\begin{aligned}
&(\frac{\partial}{\partial t} + \nu_{\mathbf{0}}(\mathbf{k})k^2) R_{\mathbf{k}}(t,t') \\
&= \int_{t'}^{t} ds \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) R_{\mathbf{k}}(s,t') R_{-\mathbf{p}}(t,s) \\
&\times \big(4K(\mathbf{k},\mathbf{p},\mathbf{q})K(-\mathbf{p},-\mathbf{k},-\mathbf{q}) C_{-\mathbf{q}}(t,s) \\
&+ \big[2K(\mathbf{k},\mathbf{p},\mathbf{q})\langle \zeta_{-\mathbf{q}}(t)\rangle + A(\mathbf{k},\mathbf{p},\mathbf{q})h_{-\mathbf{q}}\big] \\
&\times \big[2K(-\mathbf{p},-\mathbf{k},-\mathbf{q})\langle \zeta_{\mathbf{q}}(s)\rangle + A(-\mathbf{p},-\mathbf{k},-\mathbf{q})h_{\mathbf{q}}\big]\big) \qquad (12)
\end{aligned}
$$

with $R_{\mathbf{k}}(t,t) = 1$ and $R_{\mathbf{k}}(t,t') = 0$ for $t < t'$.

Markovian inhomogeneous closure equations

The three versions of the fluctuation dissipation theorem (FDT) can be combined as follows

$$
C_{\mathbf{k}}(t,t') \equiv [C_{\mathbf{k}}(t,t)]^{1-X} R_{\mathbf{k}}(t,t') [C_{\mathbf{k}}(t',t')]^{X} \qquad (13)
$$

for $t \geq t'$ and $C_{\mathbf{k}}(t,t') = C_{\mathbf{k}}(t',t)$ or $t' \geq t$. Here $X = 0$ corresponds to the current-time FDT usually used in the EDQNM, $X = 1/2$ is the correlation FDT (equation (61) of [2]) and $X = 1$ is the prior-time FDT (Equation (3.5) of [4]). As mentioned previously, it has been shown [2] that in the presence of wave phenomena only the form with $X = 1/2$ (together with Markovian response functions with positive damping) will always be realizable. That is $C_{\mathbf{k}}(t,t)$ is real and non-negative when $X = 1/2$, in equation (65) of [4]. This applies equally in the inhomogeneous QDIA formalism which is also realizable. The important result of this study is the finding that it is possible to express the nonlinear noises and dampings of the inhomogeneous closure in forms that involve unique triad relaxation functions, Θ^X, Φ^X and Ψ^X i.e.

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Theta^X(\mathbf{k},\mathbf{p},\mathbf{q})(t) + \sum_{j=0}^{5}[D^j_{\mathbf{k}}(t) + D^j_{\mathbf{p}}(t) + D^j_{\mathbf{q}}(t)] \\
&\times \Theta^X(\mathbf{k},\mathbf{p},\mathbf{q})(t) = C^X_{\mathbf{p}}(t,t) C^X_{\mathbf{q}}(t,t) \qquad (14a)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Phi^X(\mathbf{k},\mathbf{p},\mathbf{q})(t) + \sum_{j=0}^{5}[D^j_{\mathbf{k}}(t) + D^j_{\mathbf{p}}(t)] \\
&\times \Phi^X(\mathbf{k},\mathbf{p},\mathbf{q})(t) = C^X_{\mathbf{p}}(t,t) \langle \zeta_{-\mathbf{q}}(t)\rangle \qquad (14b)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Psi^X(\mathbf{k},\mathbf{p},\mathbf{q})(t) + \sum_{j=0}^{5}[D^j_{\mathbf{k}}(t) + D^j_{\mathbf{p}}(t)] \\
&\times \Psi^X(\mathbf{k},\mathbf{p},\mathbf{q})(t) = C^X_{\mathbf{p}}(t,t) \qquad (14c)
\end{aligned}
$$

where $\Theta^X(\mathbf{k},\mathbf{p},\mathbf{q})(0) = \Phi^X(\mathbf{k},\mathbf{p},\mathbf{q})(0) = \Psi^X(\mathbf{k},\mathbf{p},\mathbf{q})(0) = 0$. Using these expressions manifestly Markovian closure terms may be formulated for the cumulant and response function equations i.e.

$$
\frac{\partial}{\partial t} C_{\mathbf{k}}(t,t) = 2\Re[\sum_{j=0}^{5}(\tilde{F}^j_{\mathbf{k}}(t) - D^j_{\mathbf{k}}(t) C_{\mathbf{k}}(t,t))] \qquad (15)
$$

$$
\frac{\partial}{\partial t} R_{\mathbf{k}}(t,t') + \sum_{j=0}^{5} D^j_{\mathbf{k}}(t) R_{\mathbf{k}}(t,t') = \delta(t-t'). \qquad (16)
$$

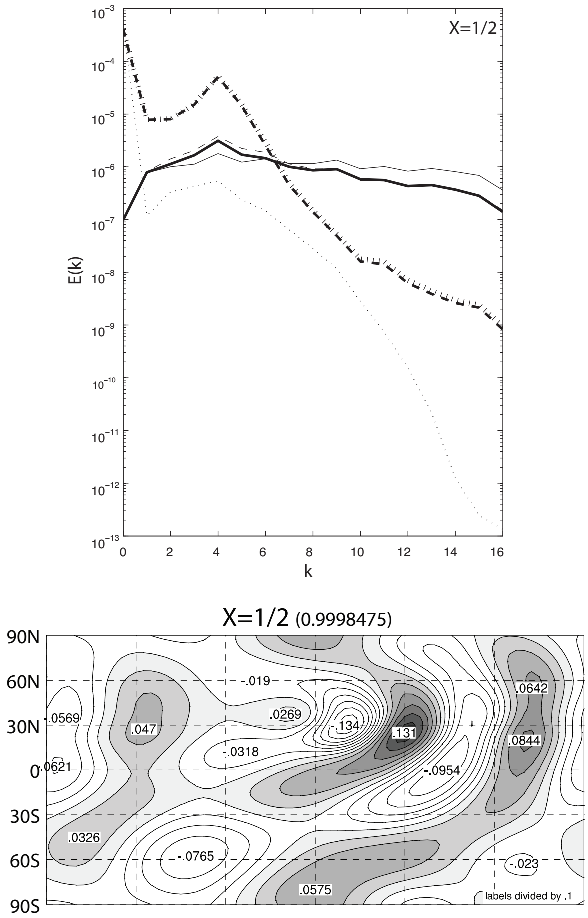

Figure 1: (a) The evolved day 10 spectra (top) and Rossby wave streamfunction (bottom) in non-dimensional units for MIC with $X = 1/2$ as compared to ensemble average of 1800 realisations of DNS (pattern correlation in brackets). Convert to physical values using (spectra) $1/4a_e^2\Omega^{-2} = 5.4 \times 10^4 m^2 s^{-2}$ and (streamfunction) $1/4a_e^2\Omega^{-1} = 740 km^2 s^{-1}$ respectively.

Here we do not write out the individual expressions in the summation but simply note that the integral terms in equations 15 and 16 may be replaced by explicitly Markovian expressions via use of the triad relaxation times equations 14.

Finally in order to close the Markovian equations for the single-time diagonal cumulant and response function, with auxiliary equations for the triad relaxation times, we need to formulate a Markov version of the mean field equation. This requires, in addition to Markovian expressions for the nonlinear damping, Markovian expressions for the eddy-topographic force i.e.

$$
\begin{aligned}
&\frac{\partial}{\partial t}\langle \zeta_{\mathbf{k}}\rangle = \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q})[K(\mathbf{k},\mathbf{p},\mathbf{q})\langle \zeta_{-\mathbf{p}}\rangle \langle \zeta_{-\mathbf{q}}\rangle\rangle \\
&+ A(\mathbf{k},\mathbf{p},\mathbf{q})\langle \zeta_{-\mathbf{p}}\rangle h_{-\mathbf{q}}] - [D^0_{\mathbf{k}} + D^M_{\mathbf{k}}(t)]\langle \zeta_{\mathbf{k}}\rangle + f^h_{\mathbf{k}}(t) + \overline{f}^0_{\mathbf{k}}, \qquad (17a)
\end{aligned}
$$

where

$$
\begin{aligned}
D^M_{\mathbf{k}}(t) &= -4 \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) K(\mathbf{k},\mathbf{p},\mathbf{q}) K(-\mathbf{p},-\mathbf{q},-\mathbf{k}) \\
&\times C^{1-X}_{-\mathbf{q}}(t,t) \langle \zeta_{-\mathbf{k}}(t)\rangle^{-1} \Phi^X(-\mathbf{p},-\mathbf{q},-\mathbf{k})(t) \qquad (17b) \\
D^0_{\mathbf{k}}(t) &= \nu_0(\mathbf{k})k^2 \qquad (17c)
\end{aligned}
$$

where the eddy-topographic force satisfies

$$f_{\mathbf{k}}^h(t) = 2\sum_{\mathbf{p}}\sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q})K(\mathbf{k},\mathbf{p},\mathbf{q})A(-\mathbf{p},-\mathbf{q},-\mathbf{k}) \times C_{-\mathbf{q}}^{1-X}(t,t)h_{\mathbf{k}}\Psi^X(-\mathbf{p},-\mathbf{q},-\mathbf{k})(t) \quad (17d)$$

This then closes the equations for the three MIC's where, rather than performing the integrals over time, only the partial differential equations involving the unique triad relaxation times (equations 14) must be solved. The Markov approximation is equivalent to the Markovian response function equation 16 and replacement of the two-time covariance by the generalised FDT of equation 13.

Comparison of Markovian Inhomogeneous Closure models far from equilibrium

The performance of the three MIC's were compared to those of an 1800 member ensemble DNS and with the non-Markovian QDIA closure for the case of an initial eastward mean flow of $7.5ms^{-1}$ impinging on a $2500m$ conical mountain centred at $30°$N and with smaller amplitude kinetic energy in the mean and transient small scales. Parameters for the calculations are as for case 1 described in table 1 of [8]. The impact of the mean flow on the topography results in the rapid generation of large amplitude Rossby waves in a highly turbulent environment over 10 day integrations. The calculations are performed at circular truncation in wavenumber space at $k = 16$ resolution, sufficient to resolve these relatively large scale dynamics, and constitutes a far from equilibrium process and a severe test of the closures. The performance of each of the MIC's and the QDIA is excellent. Here we show only the $X = 1/2$ MIC and the DNS (figure 1) results for a) spectra: initial mean energy (dotted), initial transient energy (thin solid), evolved DNS transient energy (thick solid), evolved DNS mean energy (short wide dashed), evolved closure transient energy (thin dashed) and evolved closure mean energy (thick dashed), and b) streamfunction. In all cases the pattern correlations of the day 10 mean Rossby wave streamfunction for the closures with DNS are greater than 0.9998. Over the 10 days, there is significant evolution of the mean and transient energy spectra particularly between wavenumbers 2 to 6 where the Rossby waves amplify by orders of magnitude in the mean with only slight differences in the transient kinetic energy between the $X = 0$ MIC calculation (not shown) and the DNS largely due to suppressed energy transfer to these scales due to the presence of the Rossby waves.

Conclusions

The performance of the three versions of the MIC models, and of the non-Markovian QDIA closures, was found to be remarkably accurate in the parameter regime of large-scale Rossby wave dispersion over topography in a turbulent environment. For higher resolution and higher Reynolds numbers we expect that the Markovian inhomogeneous closures, like the non-Markovian DIA and QDIA, will need to incorporate a regularization, or empirical vertex renormalization, in order to yield the correct small scale spectra. The Markovian inhomogeneous closures differ from the non-Markovian QDIA closure in that the response function has been modified to a form that is Markovian and the time history integrals have also been modified by the FDTs in such a way that their information can be characterized by three triad relaxation functions (for each variant) that satisfy auxiliary Markovian tendency equations. Thus the MIC's contain much the same information as the non-Markovian QDIA but whose time history integrals are replaced by differential equations that become relatively more efficient for long integrations. These calculations point to the prospect of developing analytical forms of the triad relaxation functions, or underpinning response functions, as is the case for isotropic EDQNM closures, that would not only increase the computational efficiency enormously but provide a more complete understanding of the relative importance of time-history effects and the most efficient methods to approximate this information.

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