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Linear and Nonlinear Lift Frequency Response of a Pitching Membrane Wing at Low Reynolds Numbers

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Abstract

A linear state-space reduced-order model for the lift generated by a two-dimensional pitching membrane at low Reynolds numbers is presented. The framework couples an unsteady aerodynamic model for the aerodynamic loads generated by rigid pitching and by the flexible degrees of freedom of the wing with a linear structural membrane equation. The lift frequency response of the pitching membrane with respect to the membrane's structural parameters is investigated, comparing the results with the response of a rigid wing. The analysis is extended by taking into account nonlinear structural effects using a harmonic balance method in the frequency domain. The nonlinear lift frequency response is obtained as a function of the amplitude of the pitching angular acceleration by iteratively solving a set of nonlinear equations for the membrane-fluid system and retaining only the first harmonic of the lift response. The effects of the structural nonlinearities on the lift response, together with the accuracy of the single-frequency harmonic balance approach, are discussed. Validations of the results are carried out using Direct Numerical Simulations.

Introduction

The fluid-structure interaction of membrane wings has gained a lot of attention in recent years, thanks to the growing interest in Micro Air Vehicles (MAVs). Many small-scale natural flyers with flapping wings take advantage of the aerodynamic benefits provided by compliant wings, such as delayed and softened stall and augmented lift at high angles of attack with respect to rigid wings [13], and they represent an inspiration for the design of MAVs. The mutual interaction between the membrane and the flow during flapping-flight is characterized by strong coupled nonlinear phenomena and it represents a major challenge for scientists and engineers. Experiments [1, 10, 13] and high-fidelity simulations [3, 4, 11] are considered reliable tools to investigate the problem, but they are expensive and often fail to provide general relationships between the structural parameters and the aerodynamic performance of the wing. Simplified theoretical models are present in the literature. Limiting the analysis to models based on the coupling between the equation of an extensible membrane and an aerodynamic model, it is worth to mention the works of Song et al. [13] and Waldman and Breuer [16] on nonlinear membranes subjected to static aerodynamic loads and the linear unsteady models of Tiomkin and Raveh [15], Sygulski [14] and Nardini et al. [9].

For natural flyers such as bats, the maximum camber of the wing can easily reach values higher than 10% of the chord during flight, resulting in strong nonlinear unsteady effects excited in the wing's response. Hence, there is a need for theoretical models that take into account the nonlinear unsteady behavior of the wing, while avoiding the direct numerical integration of the equations. Classical harmonic balance methods have been successfully used in many applications to model the nonlinear periodic response of dynamic systems to harmonic inputs in the presence of nonlinearities. Some examples related to fluid-structure interaction include aeroelastic limit-cycle oscillations in transonic flows [17], cylindrical vortex shedding and pitching airfoils [6] and unsteady nonlinear flows in cascades [5]. Harmonic balance methods are based on the idea of representing the input and the output of the system as a sum of harmonics using a truncated Fourier series and solving for the harmonic coefficients of the output through an iterative method.

In the present work, the harmonic balance method is used to generate a model for the fluid-structure interaction of a pitching membrane wing by coupling the quasi-linear equation of an extensible membrane with a linear aerodynamic model for low Reynolds numbers flows. The present approach can be considered as an extension of the linear model introduced by Nardini et al. [9] to the nonlinear regime. The present model will investigate the linear and nonlinear lift response of a pitching membrane, focusing on the effect of the structural nonlinearities and the structural parameters such as tension coefficient affect the response. An accurate representation of the frequency lift response is fundamental in the design of MAVs, which, due to their small inertia, present flight dynamic characteristics dominated by the aerodynamic forces, that allow them to perform rapid and precise maneuvers. The results from the model are validated against direct numerical integration of the governing equations and against Direct Numerical Simulations (DNS).

Fluid-structure Interaction Model

Membrane's Nonlinear Dynamics

The structural dynamic behavior of the membrane wing is modeled as a nonlinear extensible string subjected to a normal force. This model has already been successfully used in previous works [3, 11, 12] and it is presented here in its nondimensional form. The following equation will be used as a starting point to develop a quasi-linear and a linear structural model. Indicating with T_S the tension of the membrane, with h and c the wing's thickness and chord respectively and with ρ_s the density of the material, the nonlinear membrane equation can be expressed as

$$2\mu\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} - c_T\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\left[1+\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2\right]^{-3/2} = \Delta C_p + C_f. \qquad (1)$$

$\mu = \rho_s h/\rho c$ is the density ratio and represents the ratio between the density of the membrane and the density of the fluid ρ, while $c_T = 2T_S/\rho U^2 c$ is the tension coefficient. U indicates the free-stream velocity and $\Delta C_p = 2\left(p^{low} - p^{up}\right)/\rho U^2$ represents the pressure difference between lower and upper surface of the wing, while $C_f = 2f/\rho U^2$ is the force coefficient, with f being an external force. The membrane's vertical displacement w has been normalized with the wing's chord c, obtaining $\bar{w} = w/c$, while $\bar{x} = x/c$ and $\bar{t} = tU/c$ represent dimensionless space and time, respectively. From now on, the bar will be omitted and t, x and $w(x,t)$ are assumed to be nondimensional. The tension of the membrane is represented as follows

$$T_s = E_s h\left(\delta_0 + \delta\right), \qquad \delta = \int_0^1 \sqrt{1+\left(\frac{\partial w}{\partial x}\right)^2}\,dx - 1, \qquad (2)$$

where E_s indicates the modulus of elasticity of the material, δ_0 the membrane pre-strain and δ the length increase of the membrane due to the deflection. The membrane is pinned at the two ends, resulting in $w(x=0,t)=0$ and $w(x=1,t)=0$ at the boundaries. Equations (1-2) will be referred to as the nonlinear membrane equations.

Linear Fluid-structure Interaction Model

A linear structural equation is obtained from equations (1-2) by performing a Taylor expansion of the nonlinear terms up to the first order, resulting in

$$2\mu\frac{\partial^2 w}{\partial t^2} - c_{T_0}\frac{\partial^2 w}{\partial x^2} = \Delta C_p + C_f, \tag{3}$$

$$c_{T_0} = \frac{2T_0}{\rho U^2 c}, \qquad T_0 = E_s h \delta_0, \qquad \delta \approx 0. \tag{4}$$

As stated in equation (4), the tension coefficient is constant and it only depends on the initial pre-strain. By decomposing the deflection w using a truncated Fourier series, equation (3) can be rewritten in terms of deflection coefficients $\mathcal{W}_k$ as

$$2\mu\ddot{\mathcal{W}}_k + k^2\pi^2 c_{T_0}\mathcal{W}_k = \Delta\mathcal{C}_{p_k} + \mathcal{C}_{f_k}, \tag{5}$$

where the deflection coefficients are defined as follows

$$w(x,t) = \sum_{k=1}^{N} \mathcal{W}_k(t)\cdot\sin(k\pi x), \text{ for } x\in[0,1]. \tag{6}$$

$\ddot{\mathcal{W}}_k$ represents the second derivative of the k-th deflection coefficient and N indicates the total number of modes considered. For the present work we use $N=10$. A Fourier decomposition is also performed on the pressure ΔC_p and on the forcing C_f, with the coefficients $\Delta\mathcal{C}_p$ and $\mathcal{C}_f$ defined as in equation (6).

The structural model represented by equation (3) is coupled to a linear aerodynamic model obtained from DNS at a chord-based Reynolds number $Re = 100$. The input to the aerodynamic model is represented by the motion of the wing, decomposed into rigid pitching about the leading edge and deflection. In order to capture the unsteady acceleration-related aerodynamic effects (e.g. added mass), the input to the system involves the wing's angular acceleration and the second derivative of the deflection coefficients [8]. The output is represented by the lift coefficient $C_L = 2L/\rho U^2 c$, where L is the lift of the wing, and by the Fourier modes of pressure. A schematic representation of the model is shown in figure 1, while details of the modeling procedure and of the coupling with the structure can be found in [8] and [9], respectively. When considering a maneu-

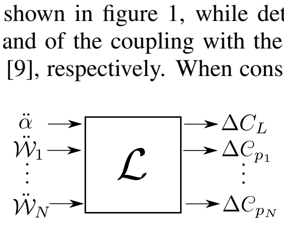

Figure 1: Schematic representation of the linear aerodynamic model. $\mathcal{L}$ indicates the transfer function of the system.

ver, the inertial force due to a rigid acceleration of the membrane (e.g. pitching or heaving) is taken into account as an external force. For the specific case of a membrane pitching about its leading edge, $C_f = 2\mu\ddot{\alpha}x$. Hence, the linear fluid-structure interaction resulting from the coupling of the structural and the aerodynamic model is a single-input single-output system with the angular acceleration as the only input and the lift coefficient as the output; the deflection and the pressure term are absorbed into the internal dynamics. The advantage of having a linear model lies in the amount of information that can be extracted from the system without requiring an expensive campaign of simulations, such as the frequency behavior or the stability as a function of the structural parameters [9].

Quasi-linear Model - Harmonic Balance Method

Although the linear model offers an efficient and accurate representation of the membrane's dynamics in cases in which the change in the elongation is negligible, it neglects the tension increase due to the membrane's elongation, which is of fundamental importance in applications involving moderate and large deformations. To overcome this intrinsic limitation of the linear model, a quasi-linear structural equation can be derived from equation (1) by performing a Taylor expansion and retaining terms up to the third order, as shown in [7]. The result is a quasi-linear model that maintains the Fourier decomposition introduced by the linear model, while retaining the nonlinear increase of the tension given by the membrane's elongation. Following [7], the quasi-linear model can be expressed as

$$2\mu\ddot{\mathcal{W}}_k + k^2\pi^2 c_T\mathcal{W}_k = \Delta\mathcal{C}_{p_k} + \mathcal{C}_{f_k}, \tag{7}$$

$$c_T = \frac{2E_s h(\delta_0+\delta)}{\rho U^2 c}, \qquad \delta = \frac{\pi^2}{4}\sum_{m=1}^{N} m^2\mathcal{W}_m^2. \tag{8}$$

The harmonic balance method is based on the assumption that the variables present in equations (7-8), coupled with the linear aerodynamic model from Fig. 1, can be approximated by a truncated Fourier series involving multiples of the fundamental frequency ω, which is the fundamental frequency of the forcing term α:

$$\begin{aligned}
\alpha(t) &\approx \alpha_0 + \sum_{h=1}^{H}\left(\alpha_{2h-1}\cos(h\omega t) + \alpha_{2h}\sin(h\omega t)\right)\\
\mathcal{W}_k(t) &\approx \mathcal{W}_{k,0} + \sum_{h=1}^{H}\left(\mathcal{W}_{k,2h-1}\cos(h\omega t) + \mathcal{W}_{k,2h}\sin(h\omega t)\right)\\
\Delta\mathcal{C}_{p_k}(t) &\approx \Delta\mathcal{C}_{p_{k,0}} + \sum_{h=1}^{H}\left(\Delta\mathcal{C}_{p_{k,2h-1}}\cos(h\omega t) + \Delta\mathcal{C}_{p_{k,2h}}\sin(h\omega t)\right)\\
C_L(t) &\approx C_{L_0} + \sum_{h=1}^{H}\left(C_{L_{2h-1}}\cos(h\omega t) + C_{L_{2h}}\sin(h\omega t)\right).
\end{aligned}$$

α does not appear explicitly in equation (7), but it is contained in the term $\Delta\mathcal{C}_{p_k}$, which is obtained from the linear aerodynamic model summarized in figure 1. Given the harmonic coefficients of the input α, equation (7) coupled with the aerodynamic model can be solved for the $(2H+1)\times N$ coefficients $\mathcal{W}_{k,2h-1}, \mathcal{W}_{k,2h}$ (with N representing the total number of structural modes and H the total number of harmonics considered) using a Newton-Raphson iterative procedure. The harmonic coefficients of C_L are finally obtained from $\ddot{\alpha}$ and $\ddot{\mathcal{W}}_k$ through the aerodynamic model.

The number of harmonics retained in the solution is decided a priori and represents one of the parameters of the method. When large nonlinear effects are excited in the system, the number of harmonics that give a non negligible contribution to the response increases. When a single harmonic is used, the method is sometimes referred to as a *describing function* and it is helpful to compare the frequency response of linear and nonlinear systems through magnitude plots.

Results

Linear Lift Frequency Response

The frequency behavior of the linear model in terms of magnitude of the lift response for $\mu = 0.5$ and for different values of the tension coefficient c_T is represented in figure 2. The plot

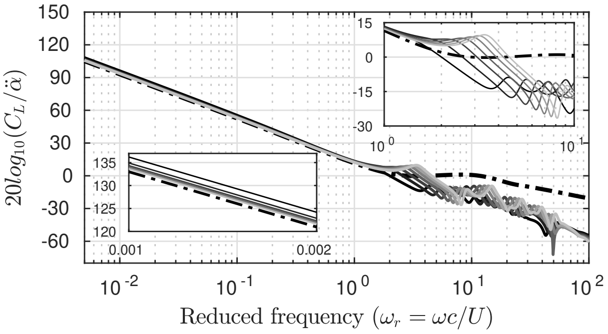

Figure 2: Magnitude of the linear lift frequency response for c_T from 0.5 to 4 with increments of 0.5 (lines from black to grey). The dash-dot line represents the response of the rigid wing.

also includes the lift response of a rigid wing, indicated with a dash-dot line. The flexibility of the membrane introduces a resonance in the response, as indicated by the peaks in the magnitude response. The resonance frequency depends on the tension coefficient: the lower c_T (hence, the higher the compliance of the membrane), the lower the resonance frequency. At low frequencies, a higher compliance results in higher lift coefficient generated, as often mentioned in the literature [3, 13]. As the tension coefficient increases, the low frequency C_L approaches the C_L of the rigid wing.

Quasi-linear Lift Frequency Response

When the quasi-linear model is considered, for a given set of structural parameters the output C_L is not only a function of the frequency, as in the linear case, but it also depends on the amplitude of the input $\ddot{\alpha}$. Using the harmonic balance method with a single harmonic ($H = 1$), the lift frequency magnitude plot is obtained for different values of the input $\ddot{\alpha} = -\alpha_2\omega^2 \sin(\omega t)$, where α_2 is the coefficient α_{2h} when $h = 1$. The lift plot is obtained with a Newton-Raphson method starting from an initial frequency ω_0 and marching in frequency with increments of $\Delta\omega$; the solution at a generic ω_i is used as the initial guess for the following iteration at $\omega_{i+1} = \omega_i + \Delta\omega$. Results for $\mu = 0.5$, $Eh = 250$ and $\delta_0 = 0.002$ (resulting in $c_T = 1$) and α_2 from 0.5 to 10 degrees are shown in figure 3. The top plot shows the solutions found by starting from $\omega = 0.01$ and marching forward with positive increments of $\Delta\omega$, while the bottom plot has been obtained by decreasing the frequency from $\omega = 100$ with a negative $\Delta\omega$.

For small values of α_2, the linear solution approaches the quasi-linear solution. As the amplitude of the oscillations increases, the response is dominated by the nonlinear membrane dynamics, which affects both the magnitude of the response and the resonance frequency. More importantly, the jumps present in the lift for some values of α_2 are an indicator of a hysteretic behavior [2]. The two branches of the solution for $\alpha_2 = 2.5$ degrees obtained by marching the frequency forward or backwards are represented in figure 4; the discontinuities in the lift response are highlighted with a dash-dot line. For this particular case, there are two frequency intervals that present hysteresis, corresponding to different structural modes excited by the input. By observing equations (7-8), it is possible to notice an analogy with a cubic Duffing oscillator, which hints to the presence of an additional unstable branch [2]. The unstable branch, represented with a black solid line in figure 4 for the low frequency hysteresis, can be found with the harmonic balance method by initializing the initial guess of the Newton-Raphson iteration with random numbers until a new solution, distinct from the two stable cases, is reached. That solution can

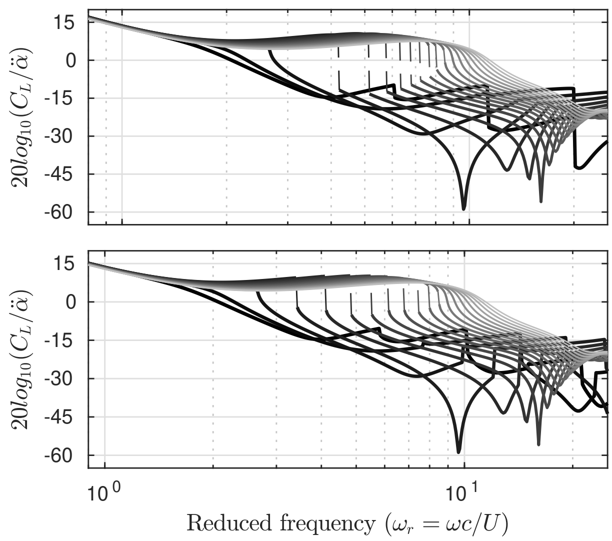

Figure 3: Lift frequency response of the quasi-linear model for α_2 from 0.5 to 10 degrees with increments of 0.5 degrees. The top plot represents the solution obtained marching forward in frequency, the bottom plot contains the solution obtained marching backwards.

then be marched by updating the frequency $\Delta\omega$ to find the unstable branch. Because of its instability, the unstable solution cannot be generally obtained by direct integration in time of the membrane equation.

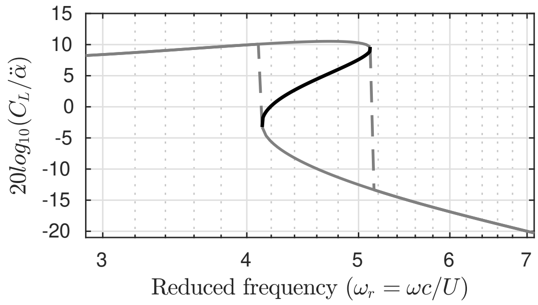

Figure 4: Hysteretic behavior of the nonlinear response for $\alpha_2 = 2.5$ degrees and representation of the two stable branches (grey lines) and the unstable branch (black line) of the response.

The periodic response of a nonlinear system usually contains more than one harmonic, with the higher harmonics as multiples of the fundamental frequency. The amplitude of the harmonics decreases with the harmonic index h, indicating that the lower harmonics give the largest contribution to the response, hence justifying the use of a truncated Fourier series to represent the system's response. Although a single harmonic approach often represents a good approximation of the full nonlinear response, for strong nonlinearities higher harmonics might be considered to improve the accuracy of the method. An example is shown in the next section.

Validation

A comparison between the lift response from the harmonic balance method and from the direct integration of the full nonlinear membrane equation for $\alpha_2 = 2.5$ degrees is represented in figure 5. The direct integration is performed by means of a 4^{th}-order Runge-Kutta method. Both linear and nonlinear aerodynamics are considered. The linear aerodynamics is represented by the linear reduced-order model summarized in figure 1, while for the nonlinear aerodynamics a well-validated DNS solver with an immersed boundary method to represent the wing's geometry is used. Details on the DNS solver and on the coupling with the nonlinear equation can be found in [11]. The membrane's structural parameters are the same as described in the previous section and the Reynolds number is 100.

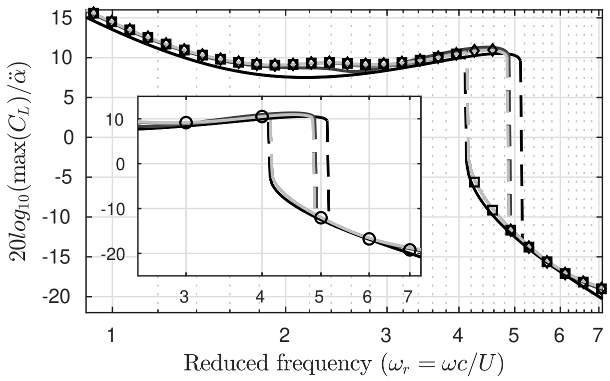

Figure 5: Comparison between harmonic balance solution and direct numerical integration for a membrane wing pitching at an angle of 2.5 degrees. Harmonic balance solutions are represented with lines from black to grey for an increasing number of harmonics (1-3-5-7 respectively); solid lines represent the solution obtained marching forward in frequency, dash-dot lines represent the solution obtained marching backwards. Results from direct numerical integration of the nonlinear equation coupled with the linear aerodynamic model are shown with diamonds (increasing frequency) and squares (decreasing frequency). DNS solutions are represented with circles in the box.

The model is in excellent agreement with both direct numerical integration and DNS. Such an agreement between linear and nonlinear aerodynamics indicates that, for the considered angle of attack, the strongest nonlinearity is represented by the structure. The quasi-linear solution converges with 5 harmonics.

Conclusions

Reduced-order models for the fluid-structure interaction of a pitching membrane at Reynolds number 100 have been introduced. The models are based on approximations of the nonlinear extensible string equation, coupled with a linear model for the fluid. The quasi-linear model is represented by using a harmonic balance method in the frequency domain. The performance of the two models are compared by analyzing the lift frequency response of the membrane. When strong nonlinear effects are excited in the response, the linear model fails, while the quasi-linear model is able to capture the behavior of the membrane, still maintaining a low computational cost. The presence of a hysteresis in the nonlinear lift response was found. Finally, using both a linear aerodynamic model and DNS, the quasi-linear model was shown to produce accurate results.

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