20th Australasian Fluid Mechanics Conference
Perth, Australia
5-8 December 2016

Scattering Matrices, Water Wave Scattering, and Wave Energy Converters

M. H. Meylan[1], H. A. Wolgamot[2]

[1]School of Mathematical and Physical Science
The University of Newcastle
Callaghan, NSW, 2308, Australia
[2]Faculty of Engineering, Computing and Mathematics
University of Western Australia
35 Stirling Highway, Crawley, WA, 6009, Australia

Abstract

The scattering matrix or S-matrix is a widely used tool in many branches of physics and engineering and it gives a simple way to encapsulate the complex behaviour of a system which is characterized by the relationship between the incident and the scattered waves. It has proved to have many useful properties, for example the poles of the S-matrix in the complex-energy plane can be identified with bound states, virtual states or resonances. The authors have recently developed numerical methods which allow the analytic extension of wave scattering calculations for cylindrical wave energy converters (WECs) into the complex plane. We use this computational tool to investigate the poles of the S-matrix and show that these are identified with the near-trapped or resonant states for WECs. We give visualizations of the S-matrix. These results, while highly theoretical, have application in determining the wave response of WECs and could assist in the design of optimal arrangements.

Introduction

The linear wave-structure interaction problem is one of the best studied problems in hydrodynamics and it is the basis for numerous practical applications. Generally the solution is found by assuming the response is at a single frequency which is considered real. However, the solution can be extended to an analytic function of complex frequencies. Such an approach has found recent application to wave structure interaction problem through application of the singularity expansion method, [1, 4, 2, 6, 7, 5], although the theory can be traced back to [10, 3].

The scattering matrix or S-matrix has wide application in physics. It first appeared in [12] and it was developed by Werner Heisenberg independently in the 1940s [8]. In its simplest form the S-matrix is a unitary matrix which connects the incoming and outgoing waves. As we will see shortly it is close to the standard objects which are routinely calculated by engineers analyzing the motion of WECs (wave energy converters).

The S-matrix is unitary which imposes the condition that the zeros and poles must occur in complex conjugate pairs. This in turn can lead to insights about the structure of the solution. Most importantly the S-matrix encodes almost all the information about the solution. We focus here on the solution for a single cylinder. The solution method could be easily generalized to more complicated geometries and even to multiple bodies. This works builds on the results presented in [13].

The S-matrix is a complex function of a complex variable and it is only recently that methods have been developed to visualize what is essentially a four dimensional object - the study of such visualizations is a subject in its own right, e.g. [11] . This visualization offers an ability to appreciate the significance of the S-matrix which was not possible previously.

We consider here the simplest model for a WEC, a truncated cylinder which is free to move in heave only. From the symmetry of the body we can decompose the incident wave into *channels* which correspond to separation of variables in the angular coordinate. This has the effect of making the problem particularly simple since the incoming and outgoing channels have dimension one. We focus on the axisymmetric channel in particular because this is the one which excites the body motion in heave.

Theory

We non-dimensionalize so that the gravity and fluid density are both unity, and consider a single truncated cylinder, of radius a and draft d, free to move in heave only. The method of solution follows from that given by [14] and the code follows from that developed for [9].

Throughout this paper we use linear potential flow theory. Thus, the fluid motion is represented by a velocity potential ϕ, with the sinusoidal time dependence $e^{i\omega t}$ removed, which satisfies

$$\nabla^2\phi = 0 \tag{1a}$$

throughout the fluid (of depth h) with boundary conditions

$$\left[\frac{\partial\phi}{\partial z}\right]_{z=-h} = 0 \tag{1b}$$

on the seabed and

$$g\frac{\partial\phi}{\partial z} - \omega^2\phi = 0 \tag{1c}$$

on the free surface.

Considering a rigid body oscillating in the absence of incident waves, the total velocity potential may be decomposed into a radiation potential ϕ_3 proportional to the complex amplitudes of body displacement in heave x_3 and the incident and diffracted potential, i.e.

$$\phi = \phi_I + \phi_D + x_3\phi_3. \tag{2}$$

The radiation potentials may be determined using a further boundary condition on the surface of a moving rigid body; for translational modes

$$\frac{\partial\phi_3}{\partial n} = i\omega n_3 \tag{3}$$

where normal n points out of the fluid and into the body and n_3 is the normal associated with the heave motion.

Integrating fluid pressures over the body surface in the standard manner, the added mass and damping may be expressed in terms of the radiation potentials

$$-\iint_{S_B} \frac{\partial\phi_3}{\partial n}\phi_3 \mathrm{d}S = \omega^2 A - i\omega B \tag{4}$$

where A and B are the added mass and damping respectively, and S_B is the submerged body surface.

For the diffraction problem we need the boundary condition

$$\frac{\partial \phi_D}{\partial n} = -\frac{\partial \phi_I}{\partial n} \tag{5}$$

on the surface of the cylinder and obtain

$$F_e = -\mathrm{i}\omega \iint_{S_B} n_3(\phi_I + \phi_D)\mathrm{dS} \tag{6}$$

for the exciting force.

We cannot solve the problem without a further equation derived from the motion of the body. This is given by

$$(-\omega^2 M + C - \omega^2 A + \mathrm{i}\omega B)x_3 = F_e \tag{7}$$

where M is the body mass and C the hydrostatic restoring force. They are given by $M = \pi d a^2$ and $C = \pi a^2$.

Solution method for a Truncated Cylinder

We decompose the potential as

$$\phi(r,\theta,z) = \sum_{n=-N}^{N} \mathrm{e}^{\mathrm{i}n\theta}\chi_n(r,z) \tag{8}$$

which satisfies equations (1) We need to solve by imposing the appropriate condition on the surface of the cylinder. The problem is divided into an inner region, directly beneath the cylinder, and an outer region, as shown in Figure 1.

Outer region

For $r > a$ we can write

$$\chi_n^O(r,z) = \sum_{m=0}^{M} Z_m(z)\left(A_{mn}P_{mn}(r) + a_{mn}Q_{mn}(r)\right) \tag{9}$$

where

$$P_{mn} = \begin{cases} H_n^{(2)}(k_m r), m = 0 \\ K_n(k_m r), m \geq 1, \end{cases} \tag{10}$$

$$Q_{mn} = \begin{cases} J_n(k_m r), m = 0 \\ I_n(k_m r), m \geq 1, \end{cases} \tag{11}$$

where J_n etc. are Bessel functions and k_m are the solutions of

$$\begin{cases} k_m \tanh(k_m h) = \omega^2, \;\; n = 0 \\ -k_m \tan(k_m h) = \omega^2, n \geq 1. \end{cases} \tag{12}$$

with k_n ordered with increasing size. Depth variation is given by

$$Z_m(z) = N_m^{-1/2}\cosh(k_m(z+h)) \tag{13}$$

where the normalising function is

$$N_m^{-1/2} = \frac{1}{2}\left(1 + \frac{\sinh(2k_m h)}{2k_m h}\right) \tag{14}$$

Inner region

The inner region potential potential is written as

$$\chi_n^I(r,z) = \left(\alpha_{0n}\left(\frac{r}{a}\right)^{|n|} + \sum_{q=1}^{Q}\alpha_{qn}\frac{I_n(\lambda_q r)}{I_n(\lambda_q a)}\cos(\lambda_q(z+h))\right) \tag{15}$$

where

$$\lambda_q = \frac{q\pi}{h-d} \tag{16}$$

The nonhomogeneous part is given by

$$\chi = \begin{cases} u_3 \times \frac{1}{2(h-d)}\left((z+h)^2 - \frac{r^2}{2}\right), \;\; n = 0 \\ 0, \;\; |n| \geq 1 \end{cases} \tag{17}$$

where u_3 is the heave velocity $\mathrm{i}\omega x_3$.

Incident Potential

The incident velocity potential for an axisymmetric incident wave is given by

$$\phi_I = J_0(k_0 r)Z_m(z) \tag{18}$$

which is also the first term in the expansion of a plane incident wave in polar coordinates. This means that the heave response for this incident wave is identical to the heave response for a plane incident wave.

Matching conditions

We match the potential and its derivative on the boundary beneath the cylinder. That is,

$$\chi^I = \chi^O \tag{19}$$

and

$$\partial_r\chi^I = \partial_r\chi^O \tag{20}$$

on $r = a$ and $-d \geq z \geq -h$.

Analytic extension to complex frequencies

We want to extend the solution analytically (i.e. as convergent power series) for complex values of ω. The only place where the solution depends on the frequency is the solution of the dispersion equation (12). We have developed a method to achieve this so that in practice only a small modification of the method for real frequencies is required to achieve this extension.

Scattering Matrix

The scattering matrix is the solution calculated using an incident wave of the form $H_n^{(1)}(kr)$. We could develop the theory for this incident wave with little difficulty but we show here how to calculate it from the standard incident waves $J_n(kr)$. We assume that the total potential is of the form

$$J_0(kr) + aH_0^{(1)}(kr) \tag{21}$$

The scattering matrix $S(\omega)$ is given by

$$H_0^{(2)}(kr) + S(\omega)H_0^{(1)}(kr) \tag{22}$$

We know that (by definition)

$$H_0^{(1)}(kr) = J_0(kr) + \mathrm{i}Y_0(kr) \tag{23}$$

and

$$H_0^{(2)}(kr) = J_0(kr) - \mathrm{i}Y_0(kr) \tag{24}$$

We can solve for the scattering matrix by equating the two expressions

$$J_0(kr) + a_0H_0^{(1)}(kr) = C_1H_0^{(1)}(kr) + C_2H_0^{(2)}(kr) \tag{25}$$

which then means that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 + a_0 \\ a_0 \end{pmatrix} \tag{26}$$

This has solution

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+2a_0 \\ 1 \end{pmatrix} \tag{27}$$

The scattering matrix is therefore given by

$$S = 1+2a_0 \tag{28}$$

Results

Our code allows us to calculate all the necessary information, and hence the scattering matrix, exactly in principle. In practice the number of internal and evanescent modes, Q and M, must be chosen to achieve converged results. For these calculations we use M=Q=40.

We consider a cylinder of unit radius in water of depth $h = 4$. We consider two values for d, $d = 1$ and $d = 3$. Figure 2 is the case when $d = 1$. The top plot shows the absolute value of the heave velocity $|u_3|$ as a function of real frequency. The large plot underneath is a visualization of the S-matrix. This is created using the method of [11]. The colour represents the phase information and the hue is proportional to the logarithm of the absolute value. We get dramatic changes in the figure near poles and zeros. We can find the poles and zero of the S-matrix and we mark these on the figure as well. We also have a smaller inset plot which shows a close up of the S-matrix in the region around the poles and zeros. Figure 3 is the same plot but for $d = 3$. Comparison of the heave response and the scattering matrix shows how strongly connected the two objects are. The larger, deeper cylinder has a much stronger and sharper resonance, which is apparent in the velocity plot and manifested in the S-matrix plane as a smaller separation between the zero and pole. Resonances are critical for most wave energy devices, so the locations of the poles and zeros in the complex plane, capturing both radiation and scattering behaviour, is of great interest. However, this is by no means the only physical quantity which is encoded in the S-matrix, and it is possible to derive identities for the scattering problem using the understanding generated.

Conclusions

We have shown how for a non-trivial wave scattering problem in which there is a moving body we can define a scattering matrix and compute its values in the complex plane. We have shown that the behaviour in the complex plane determines the solution for real frequencies. We believe that this method has a range of applications and that this alternative method of looking at water wave scattering could lead to significant insights into the system behaviour.

Further results will be presented at the conference.

Acknowledgements

This material is based upon research supported by, or in part by, the U. S. Office of Naval Research under award number N00014-15-1-2611 (MHM).

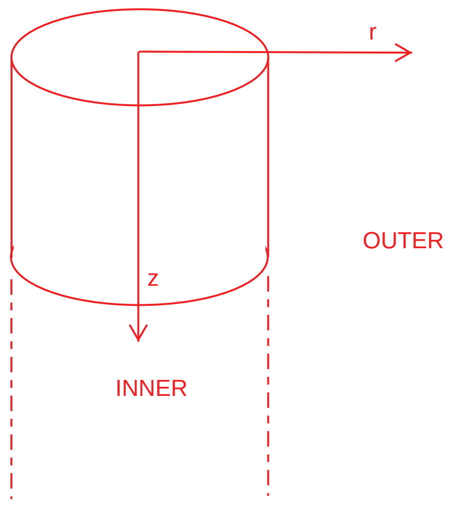

Figure 1: Layout of the inner and outer regions relative to the cylinder.

References

[1] Beale, J. T., Eigenfunction expansions for objects floating in an open sea, *Commun. Pure Appl.Maths*, **30**, 1977, 283–313.

[2] Hazard, C. and Loret, F., The singularity expansion method applied to the transient motions of a floating elastic plate, *Math. Modelling. Numer. Anal.*, **41**, 2007, 925–943.

[3] Maskell, S. and Ursell, F., The transient motion of a floating body, *J. Fluid Mech.*, **44**, 1970, 303–313.

[4] Meylan, M. H., The wave response of ice floes of arbitrary geometry, *J. Geophys. Res.*, **107**, art. No. 3005.

[5] Meylan, M. H., The time-dependent motion of a floating elastic or rigid body in two dimensions, *Journal of Applied Ocean Research*, **46**, 2014, 54–61.

[6] Meylan, M. H. and Eatock Taylor, R., Time-dependent water-wave scattering by arrays of cylinders and theapproximation of near trapping, *J. Fluid Mech.*, **631**, 2009, 103–125.

[7] Meylan, M. H. and Fitzgerald, C. J., The singularity expansion method and near-trapping of linear water waves, *Journal of Fluid Mechanics*, **755**, 2014, 230–250.

[8] Rechenberg, H., *The historical development of quantum theory*, volume 1, Springer Science & Business Media, 2001.

[9] Siddorn, P. and Eatock Taylor, R., Diffraction and independent radiation by an array of floating cylinders, *Ocean Engineering*, **35**, 2008, 1289–1303.

[10] Ursell, F., The decay of the free motion of a floating body, *J. Fluid Mech.*, **19**, 1964, 303–319.

[11] Wegert, E., *Visual complex functions: an introduction with phase portraits*, Springer Science & Business Media, 2012.

[12] Wheeler, J. A., On the mathematical description of light nuclei by the method of resonating group structure, *Physical Review*, **52**, 1937, 1107.

[13] Wolgamot, H. A., Meylan, M. H. and Reid, C. D., Linear water wave propagation through multiple floating elastic plates of variable properties, *J. Fluid. Struct.*, in press.

[14] Yeung, R. W., Added mass and damping of a vertical cylinder in finite-depth waters, *Applied Ocean Research*, **3**, 1981, 119–133.

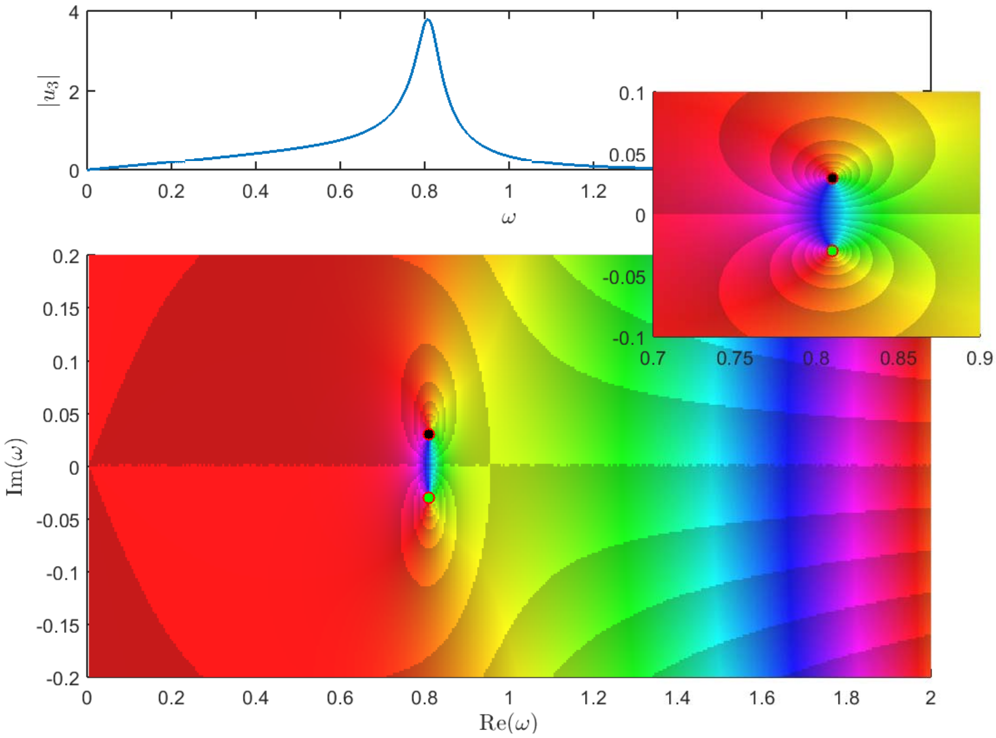

Figure 2: The top plot shows the heave velocity $|u_3|$ as a function of real ω. The bottom plot shows the S-matrix as function of ω. The values are calculated for a cylinder of radius one, with $h = 4$ and $d = 1$. Also shown are the zero of the scattering matrix (black dot) and the singularity of the scattering matrix (green dot). The smaller inset figure is a close up around the poles and zeros.

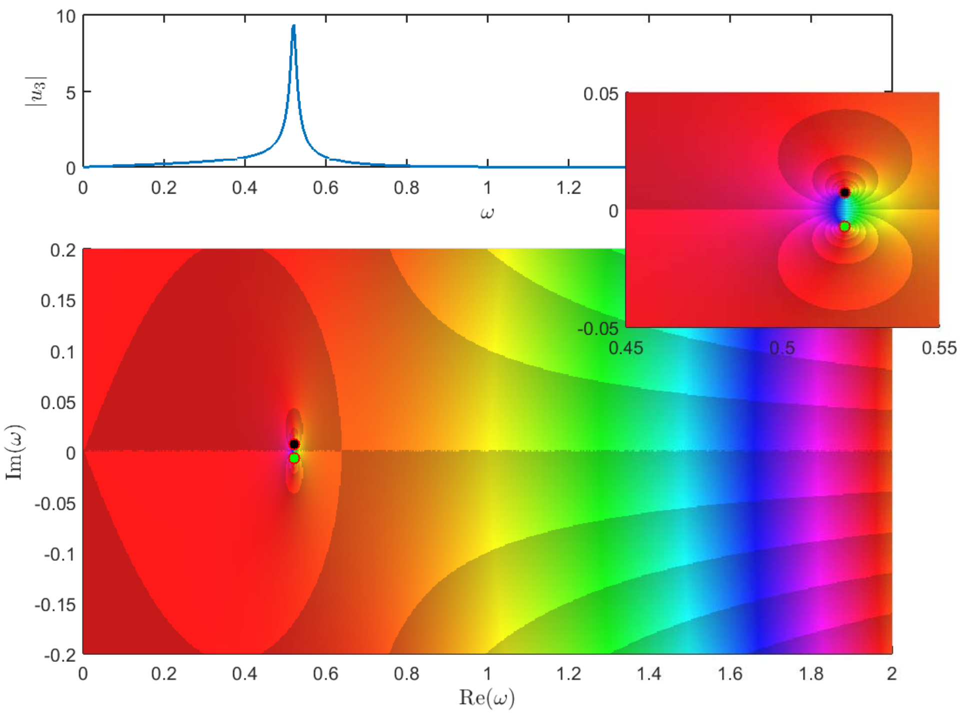

Figure 3: As in Figure 2 except $d = 3$.