Flutter of a nonlinear-spring-mounted flexible plate for applications in energy harvesting

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Abstract

Previous work has investigated the linear fluid-structure interaction (FSI) of a spring-mounted cantilevered plate in an axial flow. In this paper we introduce a non-linear mount system that permits finite-amplitude oscillations of the support. However, the dynamics of the fluid-loaded flexible plate remain linear. This hybrid permits us to determine whether small (linear) flexible plate motions are able to drive non-linear motion of the spring mount and therefore energy production while also serving as an intermediate step before studying a fully non-linear system. We use numerical simulation for the non-linear system while our state-space solution of the corresponding linear system is used to guide the choice of parameters in the investigation. We show that above the flow speed of flutter-onset for small disturbances, amplitude growth leads to non-linear saturation so that the system settles into finite-amplitude oscillations. The frequencies of these oscillations evidence the dual-frequency characteristics of mount oscillation observed in physical experiments. When the natural frequency of the mount is low, hysteresis can occur and thereby the system supports sub-critical instability. When damping at the mount is introduced we show how energy is generated by the FSI system and that its efficiency as an energy-harvesting device is dependent upon the natural frequency of the mount.

Introduction

Recent practical motivation for the renewed study of cantilevered flexible plates in axial flow - a problem first studied in the modern era by [4] - is the potential to use flow-induced oscillations, or flutter, of the flexible plate to capture kinetic energy from the mean flow above a critical flow speed, examples of these recent studies being [11; 10] wherein the latter studies being [11] wherein the flow-induced oscillations of the flexible plate drive vertical oscillatory motion of a mass-spring support system that combined numerical simulation with eigen-analysis of the system equations; however, when the non-linear spring is implemented and the amplitude of motions is sufficiently large, the numerical simulation can be utilised. Thus, ideal two-dimensional flow is assumed wherein the rotationality of the boundary-layers is modelled by vortex elements on the solid-fluid interface and the imposition of the Kutta condition at the plate’s trailing edge. The Euler-Bernoulli beam model is used for the structural dynamics. The latter is appropriate because our overall objective is to design and optimise an energy-harvesting system that operates for low-amplitude deformations - to reduce material fatigue effects - of the flexible plate by tuning the support system such that the available flow speed coincides with the modified critical speed of flutter onset of the flexible plate.

Theoretical & Computational Modelling

The fundamentals of the current method that mixes numerical simulation with eigenvalue analysis are fully detailed in [2] for a fixed cantilever. The adaptation of that method to incorporate a linear spring-mount is detailed in [3].

The one-dimensional Euler-Bernoulli thin-beam model equation couched in finite-difference form is used for the structural dynamics

\[ \rho h \frac{d^2 \{ \eta \}}{dt^2} + \{ d \} \{ I \} \{ \eta \} + B \{ D \} \{ \eta \} + \{ K \} \{ I \} \{ \eta \} = \delta p. \]  

The flexible plate of length \( L \) is discretised into \( N \) mass points that are uniformly spaced so that \( \delta x = L/N \). \( B, \rho \) and \( h \) are...
The shear condition is used to solve for the second boundary value of the linear spring-stiffness dimensionally the spring-mount system, spring-mount properly a vertically oscillating flat plate with a vertically oscillating flexion of the plate about its leading edge is permitted. This means that the support mechanism can provide, without deformation, any level of moment reaction to the flexible plate at its upstream end. Therefore, the shear condition joins two separate systems: a vertically oscillating flat plate with a vertically oscillating flexible plate.

Referring to [8], the shear force in the flexible plate at the leading edge is calculated through the following equation of motion for \( \eta_0(t) \),

\[
B \frac{d^2 \eta}{dx^2} \bigg|_{x=0} = - \left[ M^*_s \eta_0 + d^*_s \eta_0 + K^*_s \eta_0 \right],
\]

where \( M^*_s \) is the mass at the mount system. To non-dimensionalise the spring-mount system, spring-mount properties must be applied in their 'per-width' form; for example, the per-width value of the linear spring-stiffness \( K_s \) is \( K^*_s = \int K_s dx \). The shear condition is used to solve for the second boundary condition mass point \( \eta_{-1}/2 \) to solve for the first boundary condition mass point \( \eta_1 \). We enforce that the clamp extension is flat by setting that the first mass point in the system is horizontally in line with the clamp and so \( \eta_{-1} = \eta_1 \). These boundary conditions are applied where necessary in the leading-edge values of \( [D] \).

The flow field is found using a linearised BEM with \( N \) first-order vortex panels on the flexible plate. At the centre of each panel is a control point; these coincide with the mass points in equation (2). Vortex singularities are used because of the discontinuity of tangential fluid velocity across the plate that makes it a lifting surface; the distributed lift drives the motion of the flexible plate. The singularity strengths are determined by enforcing the no-flux boundary condition at every panel control point and continuity of the distributed vorticity between adjacent panels in the discretisation. In addition, the boundary condition of zero vorticity at the plate’s trailing edge is applied, thus enforcing the standard Kutta condition of zero pressure difference at the trailing edge for linear displacements.

The unsteady Bernoulli equation is utilized to determine the pressure distribution across the flexible plate i.e. the pressure perturbation that drives the plate motion; it is here equated as

\[
\delta p = 2 \rho U^2 \left[ B \right] \left\{ \eta \right\} + \rho U \left[ B \right] \left\{ \dot{\eta} \right\} + \rho U \left[ B \right] \left\{ \ddot{\eta} \right\}.
\]

This transmural pressure is then used as the forcing term in equation (2). \( \rho \) is the fluid density and \( [B] \) are matrices of singularity influence coefficients: the \( [B] \) matrices marked with a + or - have been suitably rearranged to have the equation in terms of \( \eta \) instead of linearised panel slope \( \theta \) and averaged values of \( \eta \). The fluid pressure terms that depend on plate displacement, velocity and acceleration in equation (4) can be interpreted as the hydrodynamic stiffness, damping (two terms) and inertia respectively.

The motions of the plate and the fluid flow are fully coupled through deflection, vertical velocity and acceleration of the two media at their interface. This is achieved by equating equations (2) and (3) that allows the following single system (matrix) equation to be written

\[
\{ \ddot{\eta} \} = [E] \{ \dot{\eta} \} + [F] \{ \eta \}.
\]

We take two approaches to the solution of equation (5). In the first approach that is only applicable to the linear model, we reduce the second-order ordinary differential equation in \( \eta \) to first-order using the state-space variables \( w_1(t) = \eta(t) \), \( w_2(t) = \dot{\eta}(t) = \dot{w}_1(t) \) that therefore allows \( \eta(t) = w_2(t) \). Re-arranging in companion-matrix form, single-frequency time-dependent response is assumed at \( \omega \) which is a complex eigenvalue of the companion-form. Positive \( \omega_0 \) and \( \omega_\infty \) respectively represent the oscillatory and amplifying parts of the response. As the flexible plate is discretised into \( N \) mass points we therefore extract \( 2N \) system eigenmodes. This form of analysis is used to first identify the linear-stability characteristics of the system prior to studying the dynamics of finite-amplitude motions.

For the non-linear study, we perform a time-discretisation of the system and then numerically time-step the equation using a fully-implicit method to determine the system response to an applied form of initial perturbation in the form of an arbitrary deflection of the flexible plate with maximum displacement \( \eta_0 \); this method is fully detailed in [3]. We implement the non-linear mount model by using the mount displacement from the previous time step to calculate the contribution of the non-linear component of the mount spring in equation (2). In doing so we are able to study transient behaviour and reveal localised flow-structure dynamics that when summed contribute to the system response.

**Results**

Our results are presented in non-dimensional form using the scheme detailed in [2] whereby reference time and length are

\[
\tau_t = (ph)^2 / (\bar{\rho}B^2) \quad \text{and} \quad L_t = ph / \bar{\rho} \tau_t.
\]

Therefore non-dimensional velocity, time and plate oscillation frequency are calculated as

\[
\bar{\theta} = \frac{\theta}{L_t} \quad \bar{\phi} = \frac{\phi}{L_t} \quad \bar{\tau} = \frac{\tau}{L_t}.
\]

The non-dimensional displacement, non-linear spring coefficient and length (or mass ratio) of the flexible plate are defined by

\[
\bar{\eta} = \frac{\eta}{L_t} \quad \bar{\theta} = \frac{\theta}{L_t} \quad \bar{L} = \frac{L}{L_t}.
\]

Therefore, the merit of the scheme in equation (9) is that for given plate and fluid properties, the non-dimensional length and flow speed \( \bar{U} \) and \( \bar{L} \) become independent control parameters.

To characterise the mounting system, we non-dimensionalise \( \omega_0 \) so that \( \bar{\omega}_0 = (K_s^* / M^*_s) \bar{\tau}_t \), where \( M_s^* = \rho \bar{\theta} L \) is the total plate mass. Damping is included at the mount in multiples of critical damping \( \bar{d} = 2 \sqrt{K_s^* M^*_s} \) so that \( d^* = d / \bar{d} \).

In summary, the critical velocity and frequency of the system \( \bar{U}_c \) and \( \bar{\omega}_0 \) take the functional dependence on the system’s control parameters \( f (\bar{L}, \bar{\omega}_0, d^*, \bar{\gamma}) \). All results presented herein are for ‘short’ plates with a mass ratio of \( \bar{L} = 1 \) and an illustrative value of \( \bar{\gamma} \) set equal to 10. Therefore \( \bar{U}_c \) and \( \bar{\omega}_0 \), in turn, depend only upon \( \bar{\omega}_0 \) and \( d^* \).

We discretise the flexible plate into \( N = 50 \) panels, following [2] wherein the precursor linear methods were validated. Finally, the start-up procedure for the non-linear system is from
a small, but finite, initial displacement that, we have checked, does not influence the final saturated state to which it amplifies.

System Dynamics

A frequency analysis of the mount-oscillation displacement in the absence of damping is presented in table 1(a) for different increments $\Delta$ above the linear flutter-onset flow speed $U_\ell$ for spring-mounting systems with $\bar{\omega}_s = 1$ and 10. These results show that there are two different frequencies present in the time series of the mount oscillation. The term $d_{U}$ is the intensity of the higher frequency divided by the intensity of the lower frequency. It can therefore be seen that the lower frequency, associated with single-mode flutter, dominates the oscillation. Its value increases with flow speed above $U_\ell$ (i.e. increasing $\Delta$) because this generates increased amplitudes of the mount oscillation that, through the strain-hardening of the spring, increases its stiffness. We remark that these frequencies increase towards the value for the corresponding fixed cantilever at 15.3, as shown in 2, because the fixed cantilever is analogous to a mounting with infinite spring stiffness. The higher frequency is the coalescence of the second and third eigenmodes of the flexible plate in vacuo which the spring mounted clamp promotes at lower flow speeds than for the fixed cantilever case.

The development of the non-linearly saturated states summarised in table 1(a) from a small-amplitude initial disturbance are illustrated in figure 2 for each of $\bar{\omega}_s = 1$ and 10 when $\Delta = 0.6$. In each figure both the mount- and flexible-plate tip-displacements are shown. Both figures feature exponential amplitude growth at early times due to the linear-instability mechanism after which the non-linear forces of the spring mount take effect to arrive at the finite-amplitude state of oscillatory equilibrium state (or limit-cycle oscillations). Contrasting the time series for $\bar{\omega}_s = 1$ and 10, it is noted that the former (less stiff) mount results in higher amplitudes, as would be expected, with a greater component of the higher-frequency content as seen in table 1(a). We also remark that the tip deflection relative to the mount displacement is lower for $\bar{\omega}_s = 1$ and this suggests that the configuration is a more viable energy-harvesting device when attempting to minimise plate flexure.

Results such as those of figure 2 have been used to generate figure 3 wherein the amplitude of the saturated oscillatory state is plotted against non-dimensional flow speed using each of mount and plate-tip deflections and at each of $\bar{\omega}_s = 1$ and 10. For the case $\bar{\omega}_s = 1$, the expected bifurcation diagram is seen for the mount motion; this is similar to that in 4 for flow over one side of a flexible plate non-linearly deformed due to divergence instability. However, the result for the flexible-plate tip shows an increasing rate of amplitude growth for $\bar{U} > U_\ell$ and this is probably due to the nature of the instability mechanism changing (from pure single-mode flutter to include modal-coalescence flutter) with increases to flow speed. This effect is also seen for $\bar{\omega}_s = 10$ for both the mount and tip deflections. Included in all the panels of figure 3 are the instability-onset flow speeds for the corresponding linear analysis as dashed lines. Accordingly, for the case $\bar{\omega}_s = 1$ a hysteresis loop can be seen that indicates sub-critical instability; the existence of sub-critical flutter instability of fixed cantilevers has often been noted in experimental studies. Clearly the existence of sub-critical instability would be advantageous in energy harvesting because the device could operate at flow speeds lower than those of flutter onset predicted by linear-stability analysis.

Energy Harvesting

In order to model energy-harvesting of the present FSI system, damping is now introduced at the spring mount. To illustrate its effect we use a value of $d^+ = 0.1$. Its effect on oscillation frequency and amplitude, as a modification to the corresponding

<table>
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<th>$\Delta$</th>
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<td>0.8</td>
<td>$\bar{\omega}_1$</td>
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Table 1: Mount oscillation frequencies for increasing $\bar{U} = U_\ell(1 + \Delta)$ for varying $\bar{\omega}_s$ with $L = 1$ & $\bar{\sigma} = 1 \times 10^4$: (a) $d^+ = 0$, (b) $d^+ = 0.1$.

![Figure 2](image-url)

Figure 2: Displacement in time for $\Delta = 0.6$ with $d^+ = 0$, $L = 1$ & $\bar{\sigma} = 1 \times 10^4$: — Mount, - - Mid; Top - $\bar{\omega}_s = 1$, Bottom - $\bar{\omega}_s = 10$. Small windows show traces from $\bar{t} = 16$ to 17.
We have extended our model for predicting the two-dimensional linear-stability characteristics of spring-mounted cantilevered plates in uniform flow by incorporating a non-linear spring at the mount. This allows finite amplitudes of the mount displacement to be modelled. The dynamics of the system have been investigated for cases that, for a rigid mounting, would succumb to single-mode flutter at stability onset. It has been shown that the introduction of a non-linear spring at the mount permits dual frequency mount oscillation and saturation of this oscillation with increasing amplitude for increasing flow speed above the linear $U_c$. The value of the main, low frequency, component of the response lies between the natural frequency of the mount $\omega_0$ and the frequency at flutter onset of a fixed cantilever; the higher frequency component arises from a modal-coalescence coupling between the in vacuo second and third eigenmodes of the flexible plate. This higher frequency has less effect on the mount oscillation at higher values of $\omega_0$. For low $\omega_0$, there exists hysteresis that yields a region of flow speeds supporting sub-critical flutter instability. The inclusion of (linear) dashpot damping at the mount does not lead to a change from the fundamental dynamics of the corresponding elastic system. Its inclusion allows the energy-generating capability of the system to be quantified. Preliminary results indicate efficiencies that would be useful in engineering applications and that the performance of the system benefits from a relatively flexible spring mounting.

**References**


