Nonlinear Exact Coherent Structures in pipe flow and their instabilities

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Abstract

In this paper we present results from stability calculation of travelling waves of different kinds. All but one, of these states appear through saddle-node bifurcations. All these solutions are linearly unstable and have a low-dimensional unstable manifold. We also present numerical evidence that suggests solutions first identified by [21] exhibit Vortex-Wave Interaction discovered by [8] for large $R$.

Introduction

Research on flows in pipes, channels and boundary-layer flows have attracted many researchers. There has been particular interest in travelling wave (TW) states at large Reynolds number $R$ in channel ([11], [18], [19], [20], [6], [1]) and pipe geometries ([5], [9], [21], [10], [12], [16]) with different degrees of symmetry. These states do not arise from finite dimensional axis aligned along $z$ solutions first identified by [21] exhibit Vortex-Wave Interaction VWI (Vortex-Wave Interaction) states as it is descriptive of the initial conditions in phase space between those that return to laminar flow from those that don’t. Furthermore, when the unstable manifold of these states are low-dimensional, as suggested by earlier numerical calculations ([17]), of one of these states in pipe geometry, they correspond to exact coherent structures that are experimentally observable ([9]) in intermediate $R$ turbulence with the flow moving slowly from one TW state to another. In the same work, a new class of TW solutions, identified as NVC states, were found similar to the one observed earlier in boundary-layer flows by Deguchi & Hall [2]. The existence and stability of such travelling wave states, and their connection in phase space play important roles in understanding both transition and large $R$ behavior of pipe flows. These nonlinear TW states are also of potential technological importance if suitable controls can be inserted to stabilize a coherent state with a significantly smaller drag than an uncontrolled turbulent flow.

The aim of this short paper is to briefly describe already found ([14]) three different class of travelling wave solutions and present stability properties of them.

Computational Method

The TW solutions we are looking for satisfy the Navier-Stokes

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{R} \Delta \mathbf{u} , \nabla \cdot \mathbf{u} = 0$$

in the form (1).

Our computational method is based on Galerkin truncation in Fourier-modes in $\theta$ and $z - ct$ and a Chebyshev representation in $r$ similar to [21] which automatically accounts for the boundary condition. Details of numerical methods used in our calculations can be found in [14].

A similar representation is used for linear stability of equilibrium $\mathbf{u}(r, \theta, z - ct)$ states. We consider perturbations of the form $e^{i\theta} \tilde{u}(r, \theta, z - ct)$ with growth rate $\lambda$ moving at a rate $c$, equilibrium wavespeed. Stability analysis is performed by solving linearized Navier-Stokes equation (2) for infinitesimal small perturbation $\epsilon e^{i\theta} \tilde{u}(r, \theta, z - ct)$ added to the equilibrium solution. The number of unstable modes a TW has depends on the space in which the stability analysis was performed. In theory, one should allow all axial, azimuthal modes on an infinitely long pipe; however this is not possible numerically. We restrict our computations into the space in which solutions reside; i.e. $\tilde{u}(r, \theta, z - ct)$ possesses same symmetries as equilibrium solution unless we analyze stability around a bifurcation.
point where two equilibrium solutions meet. As in full non-linear Navier-Stokes problem, we use Galerkin truncation in Fourier-modes in $\theta$ and $z - ct$ and a Chebyshev representation in $r$. Linear stability computations are done using direct or Newton solver. Also, higher resolution calculations are performed by solving linear eigenvalue problem using Newton method with initial guess chosen from coarser resolution computations.

**Numerical Results**

Calculations described here are limited to $k_0 = 2$; i.e. two-fold azimuthally symmetric TW states. We assume TWs have $S$-symmetry (shift-and-reflect). In addition, a class of NVC states, identified as C2 possesses $\Omega_2$-symmetry (shift-and-rotate) as defined in [13]. For $k_0 = 2$, this results in rolls and streaks having four-fold azimuthal symmetry.

The first solution we found is called $WK$ state. Note that, in the previous paper ([14]) we were unable to get numerical convergence for $WK$ solutions for values of $R$ larger than about 11,000 and so agreement with asymptotic scaling results for VWI-states was only qualitative. Other solutions, C1 and C2, collapse towards the center of the pipe as $R \to \infty$. These were identified as NVC states. Despite localization of rolls and waves over a shrinking core at the center of the pipe, the streaks do not decay and remain the same size outside as inside the core, until wall effects become important.

Figure 1 shows the phase speed $c$ as a function of $R$ for $WK$, C1 and C2 solutions at axial-wave speed $\alpha = 1.55$. In [14], $WK$ solutions were found up to $R = 11,000$ and C1, C2 solutions were only displayed for the lower-branch since lower branch asymptotics were investigated. Here, we extend our computations to $WK$ solutions for larger $R$ and the upper-branch C1-C2 solutions.

In figures 2, 3 and 4 we display roll, streak and wave components of these TW states in a plane perpendicular to the pipe axis at $R = 50000$ when $\alpha = 1.55$. In each plot the rolls $(U, V, 0)$, radial and azimuthal waves $(u, v, 0)$ are depicted using arrows whilst the axial velocity intensity of streaks $W$ and axial waves $w$ are represented in colors where the lighter color corresponds to positive values of $W$ and $w$, while darker colors correspond to negative such values. Axial wave velocity $w(r, \theta, z_0)$ is shown at a fixed $z_0 = 2\pi/\alpha$. It is clear from figure 2b that, most action of waves are occurring around a small region which is consistent with VWI theory. On the other hand, roll, wave actions of C1 and C2 solutions are concentrated at the core while streaks are roughly the same size outside the core.

Figure 3: (a) Roll $(U, V, 0)$, streak $(0, 0, W)$ and (b) $(u, v, w)$ wave profiles at $R = 50000$ for $C1$ at $\alpha = 1.55$. 15 equispaced contour levels are plotted between minimum and maximum (a) $W(r, \theta)$ and (b) $w(r, \theta, z_0)$ where min/max taken over $(r, \theta)$.

Figure 4: (a) Roll $(U, V, 0)$, streak $(0, 0, W)$ and (b) $(u, v, w)$ wave profiles at $R = 50000$ for $C2$ at $\alpha = 1.55$. 15 equispaced contour levels are plotted between minimum and maximum (a) $W(r, \theta)$ and (b) $w(r, \theta, z_0)$ where min/max taken over $(r, \theta)$.

In figure 5 some results on Reynolds stresses $S_1$ and $S_2$ as defined in [14] are shown. The contours of $\frac{S_j}{\bar{S}_j}$ in figure 5a appear accumulated around the critical curve (where $1 - c - r^2 + W(r, \theta) = 0$) shown in black as expected from VWI states. Figure 5b shows $S_{j,m} := \max_{(r, \theta)} S_j(r, \theta)$ at $j = 1, 2$ against $R$ for $\alpha = 1.55$ for VWI-states. Linear fittings(dotted-line) involved the range $1.2 \times 10^5 < R < 5 \times 10^5$. $S_{1,m}$, $S_{2,m}$ scale as $R^{-1.58}$, not very close to asymptotic scaling $R^{-5/3}$ of VWI states. However, it is clear that, as $R$ increases both $S_{1,m}$ and $S_{2,m}$ curves (solid lines) become steeper suggesting that $R$ is still not large enough to reach the asymptotic scaling regime.

**Stability Analysis**

Figure 1 shows results of linear stability analysis around the bifurcation points. Note that, even though calculations were carried out up to $R = 50000$, we display solutions up to $R = 14000$ since there is no bifurcation detected beyond $R = 11000$. All solutions, except C1, appear through saddle-node bifurcations with ‘upper’ and ‘lower’ branches which are observed when an eigenvalue passes through the origin. In all cases we found that there is at least one positive eigenvalue; thus all states are unstable from their onset.

As seen in figure 1, $WK$, C2 solutions are born at about $R = 1658$ and $R = 2300$ respectively. When $R$ increases, lower-branch of $WK$ solution goes through 2 saddle node bifurcations at around $R = 11600$ where it makes an inverse S shape when an unstable real eigenvalue becomes stable, followed by a stability change of a stable eigenvalue. As a side note, this bifurcation is important on its own right because $WK$ solution start exhibiting VWI behaviour with a single dominating axial mode only after this S-shape bifurcations so we could find numerical evidence that supports VWI-theory.
higher wave speed branch has lower energy dissipation. Labels above/below turning points indicate properties of unstable eigenvalues: number of unstable, real- or complex- \( \lambda \)s.

On the other hand, \( C_1 \) is being created out of \( C_2 \) through another bifurcation which can be seen through linear stability analysis of \( C_2 \) in \( S \)-symmetric subspace without imposing additional shift-and-rotate symmetry. As \( R \) increases changes in stability of \( C_1 \) is observed as it makes an \( S \) shape as seen in figure 1.

Also, we should note that, as you increase \( R \) in upper branch some real eigenvalue pairs will join to form a complex pairs, though they are not included here since we only focus on stability around bifurcation points. As a final remark we have recently found new connections on the upper-branch \( WK \) solution around \( R = 2300 \) as seen in figure 1. However, stability calculations are not included here.

**Conclusions**

In this paper, we report new numerical computations of travelling wave solutions with shift and reflect symmetry through a numerical continuation process. We present some features of these solutions, including scaling of the lower-branch \( WK \) with Reynolds number in the range \( 1.2 \times 10^4 < R < 5 \times 10^4 \). Quantitative evidence roughly suggests numerically calculated lower-branch \( WK \) solution is a finite \( R \) realization of VWI states. However, we need larger \( R \) calculations in order to achieve scalings that are consistent with asymptotic stress scaling of \( R^{-5/3} \) in VWI solutions. It is also worth mentioning that other scalings of \( WK \) in the same range give good approximations to expected the \( R^{-1} \) scale for rolls, \( O(1) \) scale for streaks and a maximal wave amplitude of \( O(R^{-5/6}) \) of self sustaining process and VWI theory described in [8] & [7].

We also discuss linear stability properties of computes travelling wave solutions near bifurcation points and identify number of unstable directions. We confirmed that these travelling waves have slow and low dimensional unstable manifolds suggesting their relevance to transition in turbulence. It is clear that control of these state are important matters for further research with both theoretical and practical implications.

**References**


