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Why tornado-like vortices are persistent?

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Abstract

The tornado-like vortices and vortical motions in turbulent flows have notable similarities. Both types of vortical flows have outer scales, with are subject to external influence; inner scales, which are affected by viscosity; and a range of inertial scales that display a degree of universality. Both types of the flows transfer kinetic energy from large to small scales, where the action of viscosity becomes significant. However, unlike turbulent flows, tornado-like flows are quite regular and, it seems, can remain laminar under some conditions. At the large scales, these flows have a special mechanism (the compensating mechanism) that prevents unrestricted generation of tangential vorticity and is, it seems, primarily responsible for preserving regularity of the flow. Unsteady small-scale axisymmetric disturbances, which may escape control of the *compensating mechanism*, are analysed in the present work. These disturbances are shown to represent inertial waves, which are generalised for arbitrary axisymmetric vortical flows and found to be stable provided the axial vorticity does not change its sign.

Motivation

A number of vortical flows — tornadoes, hurricanes, firewhirls and bathtub vortices — are characterised by centripetal motion that dramatically amplifies rotation in the flow. These flows are referred to here as *tornado-like flows*. The tornado-like flows are generally unsteady due to continuing accumulation of axial vorticity at the center but are quite stable and persistent: vorticity, which is a common source of instability and turbulence in fluid flows, acts as a factor that stimulates a regular rotation. Tornadoes and firewhirls can persist for hours, while the life span of a hurricane can be measured in days or even weeks.

The tornado-like flows are mostly turbulent, although this turbulence is not necessarily generated by the tornadic vorticity and usually is simply present in the atmosphere. In the late 1990s – early 2000s, a series of bathtub flow experiments were conducted in the CMM group at The University of Queensland [2, 3]. Figure 1 illustrates one of these experiments, where the flow in a rotating tank was seeded by small particles. The core of the flow seems laminar despite a large Reynolds number (>10000) and strong vorticity present in the flow. The existence of a 3-dimensional laminar vortical flow with very large Reynolds numbers indicates that vorticity can play a *stabilising role* in tornado-like flows. This issue is examined further in this work.

Note that the Ekman boundary layer at the bottom of the tank may be turbulent and this layer tends to swell near the drain – see [10, 1]. The experiments conducted at UQ involved a long settlement period to achieve a uniform (i.e. solid-body) rotation before draining the tank. Cristofano et. al. [4] recycled the water from the drain to establish a steady flow and detected turbulent fluctuations in their bathtub vortex experiment.

Stability of vortical flows

Tornado-like flows are complex flows, whose velocity fields are not known exactly. While comprehensive analytical treatment of stability of these flows thus may be difficult, stability of model flows that to some extent resemble realistic tornado-like flows has been repeatedly analised in publications [11]. The stability of a potential vortical sink or source was studied by Shusser and Weihs [12]. They concluded that the vortical source is always unstable, while the vortical sink is stable unless the Reynolds number is too low. The analysis was conducted using vorticity equations and was essentially based on the absence of axial vorticity in the bulk of the flow. The stability of the Burgers vortex was investigated by Gallay and Maekawa [5], who concluded that this vortex is stable.

Figure 1: Photo taken in experiments with bathtub vortex on a rotating table [2].

While the results mentioned above are undoubtedly relevant, the tornado-like flow has one feature that distinguishes it from the vortices mentioned above — the presence of axial vorticity at the periphery of the flow. This can bring effects (including unstable modes) that are not detected in the theories mentioned above.

Compensating mechanism

The evolution of vorticity in a tornado-like flow is controlled by the compensating mechanism [8]. On the one hand, this mechanism allows the flow to reach high levels of axial vorticity, that becomes no longer passive and affects the velocity field. A stable tornado-like vortex cannot form if vorticity in the flow is too weak and is not sufficiently amplified by the flow. On the other hand, the compensating regime prevents excessive increase of vorticity to the levels that would destabilise the flow. While the flow and vorticity are three-dimensional, the compensating mechanism limits the magnitude of tangential vorticity to values that modulate but do not alter the regular character of the flow.

The compensating mechanism acts in the intensification region shown in Figure 2 and results in the 4/3 power law for the axial vorticity (more accurately, with a range of 4/3 to 3/2 for the vorticity exponent) [6, 8]. Hence, two major possibilities are present in three-dimensional flows with very intense vorticity and high Reynolds numbers: 1) development of primary and secondary instabilities followed by transition to turbulence (this does not mean that all flows with vorticity are linearly unstable but, practically, presence of sufficiently strong vorticity results in bifurcation and transition) or 2) preserving the regular character of the flow in a tornado-like vortex. It is interesting that both of these possibilities have conceptual similarities illustrated in Figure 2: the existence of outer and inner scales with an interval between them. The outer scales are affected by geometry and are not universal. Viscosity plays a role in the inner-scale effects. The intermediate intervals are inertial and transport 1) the kinetic energy down the scales in Kolmogorov turbulence or 2) the angular momentum to the centre of the flow in tornado-like vortices. When the Reynolds number is sufficiently high that the inertial interval of turbulence reaches two orders of magnitude, the 5/4 power law becomes clearly visible. The outer and inner scales in tornado-like flows are usually separated by a single order of magnitude. The tornadic flows are also affected by unsteady environment, so that the actual power exponent tends to fluctuate over its equilibrium value. For example, landfall of a hurricane may cause noticeable changes in exponents and it might take a few days before the exponents relax to their equilibrium values. The 4/3 to 3/2 range is clearly observed only when measurements are averaged over a large number of experiments [8].

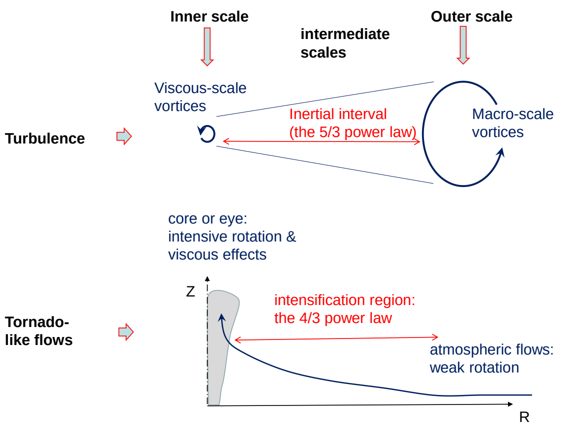

Figure 2: Conceptual similarities between two types of evolution of 3-D vortices in fluid flows: turbulence and tornado-like vortices: non-universal outer and viscous inner scales are separated by the inertial/intensification intervals.

Breaking the compensating mechanism

While being important for life-long evolution of the vortex, the compensating mechanism is rather slow and acts only at large scales. This mechanism can be broken if a small-scale disturbance is introduced into the flow [7]. Alternatively, the mechanism may fail to keep rotational components under control in the core of the flow, causing vortex breakdowns and flow reversals at the axis [8]. The simple action of the compensating mechanism through adjustments of vorticity vectors is illustrated in Figure 3. The vorticity vector AB is initially directed down but the flow over the bump turns it towards A′B′. The compensating mechanism acts to bring the vorticity vector towards its undisturbed direction A′B″ (see details in [7, 8]). If the scale of the disturbance Δr is sufficiently large (larger then a certain critical scale) , then the flow adjusts itself to preserve its structure. The critical scale is defined $\delta_{cr} \sim r_0/K$, where K represents the *vortical swirl ratio*, which is specified and used in the next section. The vortical swirl ratio K is large in flows with strong vorticity and the strong vortex approximation [10, 6, 8] is based on $K \gg 1$. If, however, the scale of the bump is small $\Delta r \ll \delta_{cr}$, the asymptotic matching carried out in ref. [7] indicates that compensating mechanism fails and a wave of disturbances propagates deeply into the flow. This indicates possible small-scale instabilities, but the asymptotic analysis of the ref. [7] was restricted only to time-independent perturbations. The evolution of unsteady short-scale disturbances is considered in the next section.

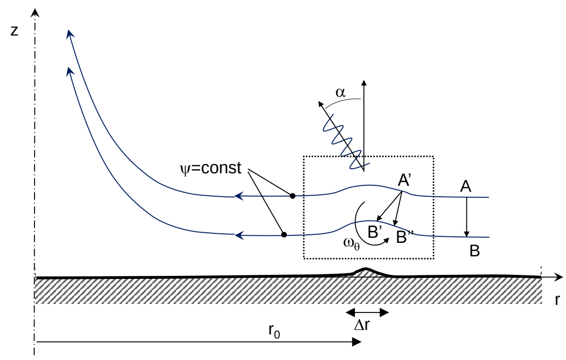

Figure 3: A small disturbance in a vortical flow with axial vorticity [7]

Unsteady waves

Dimensionless form of the governing equations

The dimensionless form of the equations governing vorticity evolution in axisymmetric unsteady flow is given by [8]

$$\frac{\partial^2 \Psi}{\partial Z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \Psi}{\partial R}\right) = -R\Omega_\theta, \qquad (1)$$

$$\mathrm{St}\frac{\partial \Omega_\theta/R}{\partial T} + V_z\frac{\partial \Omega_\theta/R}{\partial Z} + V_r\frac{\partial \Omega_\theta/R}{\partial R} = -2K^2\frac{\Gamma\Omega_r}{R^3} + \frac{(\ldots)}{\mathrm{Re}}, \qquad (2)$$

$$\mathrm{St}\frac{\partial \Gamma}{\partial T} + V_z\frac{\partial \Gamma}{\partial Z} + V_r\frac{\partial \Gamma}{\partial R} = \frac{(\ldots)}{\mathrm{Re}}, \qquad (3)$$

where

$$V_z = \frac{1}{R}\frac{\partial \Psi}{\partial R},\quad V_r = -\frac{1}{R}\frac{\partial \Psi}{\partial Z},\quad \Omega_\theta = \frac{\partial V_r}{\partial Z} - \frac{\partial V_z}{\partial R} \qquad (4)$$

$$\Omega_r = -\frac{1}{\mathrm{St}}\frac{1}{R}\frac{\partial \Gamma}{\partial Z},\qquad \Omega_z = \frac{1}{\mathrm{St}}\frac{1}{R}\frac{\partial \Gamma}{\partial R} \qquad (5)$$

The variables are normalised according to

$$R = \frac{r}{r_*},\ Z = \frac{z}{r_*},\ \Psi = \frac{\psi}{\psi_*},\ V_r = \frac{v_r}{v_*},\ V_z = \frac{v_z}{v_*},\ T = \frac{t}{t_*}$$

$$\Gamma = \frac{\gamma}{\gamma_*},\ \Omega_z = \frac{\omega_z}{\omega_*},\ \Omega_r = \frac{\omega_r}{\omega_*},\ \Omega_\theta = \omega_\theta\frac{r_*}{v_*}, \qquad (6)$$

where $\psi_* = v_* r_*^2$, $t_* = \gamma_*/(\omega_* r_* v_*)$ and all characteristic values are denoted by the subscript "$*$".The main parameters determining the character of the flow are the vortical swirl ratio

$$K \equiv \frac{1}{\mathrm{Rs}} \equiv \frac{(\gamma_*\omega_*)^{1/2}}{v_*} \qquad (7)$$

which is reciprocal of the *swirling (modified) Rossby number* Rs, and the Reynolds and Strouhal numbers

$$\mathrm{Re} \equiv \frac{v_* r_* \rho}{\mu}, \ \ \mathrm{St} \equiv \frac{r_*^2 \omega_*}{\gamma_*} = \frac{r_*}{t_* v_*} \tag{8}$$

The vortical swirl ratio and swirling Rossby number are linked to the conventional swirl ratio S and the conventional Rossby number Ro

$$S \equiv \frac{\gamma_*/r_*}{v_*}, \ \ \mathrm{Ro} \equiv \frac{v_*}{\omega_* r_*} \tag{9}$$

by the equations

$$K^2 = S^2\,\mathrm{St} = \frac{S}{\mathrm{Ro}} = \frac{1}{\mathrm{Ro}^2\,\mathrm{St}} = \frac{1}{\mathrm{Rs}^2} \tag{10}$$

The conventional asymptotic analysis in a tornado-like flow with strong vorticity is based on assuming large K and small St [10, 8].

Perturbation analysis

Assuming that $X = R - R_0 \ll R_0$, and $Z \sim X$, all variables are perturbed with respect to arbitrary local values at $R = R_0$ and $Z = 0$

$$\Gamma = \Gamma_0(T) + \mathrm{St}\left(\Gamma_1(X) + \varepsilon\Gamma'(X,Z,\tau)\right) + ..., \ \ \Gamma_1 = \Omega_0 R_0 X \tag{11}$$

$$\Psi = \Psi_0(Z) + \varepsilon\Psi'(X,Z,\tau) + ..., \ \ \Psi_0 = V_0 R_0 Z \tag{12}$$

The values $\Omega_{z0} = \Omega_0$, $V_{r0} = -V_0$ and Γ_0 are treated as constants representing the undisturbed flow in the vicinity of the bump at a given external time T (the faster running time τ is estimated as $\tau/T \sim X/R_0$), while $\Omega_{r0} = 0$ and $V_{z0} = 0$ at the leading order. The perturbations are linked by the equations

$$V_z' = \frac{1}{R_0}\frac{\partial\Psi'}{\partial X}, \ \ V_r' = -\frac{1}{R_0}\frac{\partial\Psi'}{\partial Z}, \ \ \Omega_\theta' = \frac{\partial V_r'}{\partial Z} - \frac{\partial V_z'}{\partial X}$$

$$\Omega_r' = -\frac{1}{R_0}\frac{\partial\Gamma'}{\partial Z}, \quad \Omega_z' = \frac{1}{R_0}\frac{\partial\Gamma'}{\partial X}. \tag{13}$$

At the order of $O(\varepsilon)$, equations (1)-(3) become a system

$$\frac{\partial^2\Psi'}{\partial Z^2} + \frac{\partial^2\Psi'}{\partial X^2} = -R_0\Omega_\theta, \tag{14}$$

$$\frac{\partial\Omega_\theta'}{\partial\tau} - V_0\frac{\partial\Omega_\theta'}{\partial X} = -2K^2\frac{\Gamma_0\Omega_r'}{R_0^2} = 2K^2\frac{\Gamma_0}{R_0^3}\frac{\partial\Gamma'}{\partial Z}, \tag{15}$$

$$\frac{\partial\Gamma'}{\partial\tau} - V_0\frac{\partial\Gamma'}{\partial X} = -V_r'\Omega_0 R_0 = \Omega_0\frac{\partial\Psi'}{\partial Z} \tag{16}$$

for three unknown variables Ψ', Ω_θ' and Γ'.

Harmonic wave

The solution is sought in the form of a harmonic wave so that for every disturbance $F'(X,Z,\tau) = \hat{F}\exp(i(\Lambda\tau + \beta_z Z + \beta_r X))$ where $\hat{F}$ is the complex amplitude. The system of equations (14)-(16) is transformed into

$$\left(\beta_z^2 + \beta_r^2\right)\hat{\Psi} = R_0\hat{\Omega}_\theta, \tag{17}$$

$$(\Lambda - V_0\beta_r)\,\hat{\Omega}_\theta = 2K^2\frac{\Gamma_0}{R_0^3}\beta_z\hat{\Gamma}, \tag{18}$$

$$(\Lambda - V_0\beta_r)\,\hat{\Gamma} = \Omega_0\beta_z\hat{\Psi} \tag{19}$$

A non-trivial solution of this system is possible provided

$$\Lambda = V_0\beta_r \pm \frac{\left(2K^2\Gamma_0\Omega_0\right)^{1/2}}{R_0}\cos(\alpha), \ \ \cos^2(\alpha) = \frac{\beta_z^2}{\beta_z^2 + \beta_r^2} \tag{20}$$

The disturbances under consideration do not have unstable modes assuming that $\Gamma_0\Omega_0 \geq 0$ everywhere in the flow. The physical meaning of the angle α is shown in Figure 3.

The physics of the waves

In this section, we return to dimensional variables. The dimensional form of the system (14)-(16) can be rewritten as a single equation for $\psi' = \Psi'\psi_*$, which takes the form of

$$\left(\frac{1}{v_0}\frac{\partial}{\partial t} - \frac{\partial}{\partial r}\right)^2\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right)\psi' = -\frac{1}{\delta_{\mathrm{cr}}^2}\frac{\partial^2}{\partial z^2}\psi' \tag{21}$$

where

$$\delta_{\mathrm{cr}}^2 = 2\frac{r_0^2}{K_0^2}, \ \ K_0^2 = \frac{\gamma_0\omega_0}{v_0^2}, \tag{22}$$

$v_0 = v_* V_0$, $r_0 = r_* R_0$, $\gamma_0 = \gamma_*\Gamma_0$ and $\omega_0 = \omega_{z0} = \omega_*\Omega_0$. The equation, previously obtained for stationary disturbances [7], can be recovered from (21) by discarding $\partial/\partial t$. As expected, the limit of $\delta_{\mathrm{cr}} \to 0$ (i.e. $K_0 \to \infty$) corresponds to the strong vortex approximation [8], which in this case is given by $\partial^2\psi'/\partial z^2 = 0$ at the leading order.

The dimensional form of the harmonic wave is given by the exponent $\exp(i(\lambda t + k_z z + k_r r))$. The dispersion relation (20) becomes

$$\lambda = v_0 k_r \pm \Delta\lambda, \ \ \Delta\lambda = 2\omega_{\mathrm{eff}}\cos(\alpha), \ \ \omega_{\mathrm{eff}} = \frac{(2\gamma_0\omega_0)^{1/2}}{2r_0} \tag{23}$$

These waves represent a generalisation of the inertial waves in fluid flows with a uniform (i.e. solid-body) rotation [9] to arbitrary vortical motions. If fluid rotates uniformly with angular velocity ω_{sb} so that $\gamma = \omega_{\mathrm{sb}} r^2$ and $\omega_0 = 2\omega_{\mathrm{sb}}$, then $\omega_{\mathrm{eff}} = \omega_{\mathrm{sb}}$ and the derived equation coincides with that given by Landau and Lifshits [9]. These waves cannot exist in a potential vortex where $\omega_0 = 0$ even if $\gamma_0 > 0$ is very large but even a small vorticity present in the flow may be sufficient to reach significant ω_{eff} and observe the waves. Note that the waves are unstable when $\omega_z\gamma < 0$ (in this case ω_z must change its sign in the flow). Development of instabilities in the regions where $\omega_z\gamma < 0$ is expected (see proposition 1 in [6]) and related to the Rayleigh instability condition.

Conclusions

Tornado-like vortices are persistent due to several stabilising mechanisms. At large scales, the compensating mechanisms acts to constrain the tangential vorticity, which is significant in developed tornado-like flows, and preserve the axisymmetric structure of the flow. At small scales, the unsteady waves, which are analysed in the present work and found to be similar to the inertial waves propagating in uniformly rotating fluid, can be observed. These waves are stable too provided $\omega_z\gamma \geq 0$ and this condition is typically satisfied in the tornado-like flows. We also note the conceptual similarity of turbulence and tornado-like flows, which both have the outer and the inner scales and the inertial/intensification range in between these scales.

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