Consider an ellipsoidal cavity defined by
\[ \Theta \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1, \] (1)
as shown in figure 1, oriented with one axis vertical with respect
to the uniform gravitational field \( \mathbf{g} = -g \hat{z} \) in a uniform infinite
solid subject to an overall horizontal temperature gradient \( \nabla T \sim T_y \parallel \) parallel to another of the cavity’s axes.

In the surrounding solid, the steady temperature satisfies
Laplace’s equation, \( \nabla^2 T = 0 \); in the cavity, the temperature
is coupled to the velocity \( \mathbf{u} \) and pressure \( p \) by the Oberbeck–Boussinesq equations
\begin{align*}
\nabla \cdot \mathbf{u} &= 0 \quad (2) \\
\rho u \cdot \nabla u &= -\nabla p + \rho g \beta T \mathbf{j} + \mu \nabla^2 \mathbf{u} \quad (3) \\
\rho C u \cdot \nabla T &= k \nabla^2 T, \quad (4)
\end{align*}
where \( \rho, C, \beta, \mu, \) and \( k \) are the density, specific heat, and coefficients of thermal volumetric expansion, viscosity, and thermal conductivity of the fluid. The temperature \( T \) is expressed as the
excess over that the centre of the cavity (constant, by symmetry, and equal to the value that would prevail without the cavity).

Guided by the work on the spherical special case [23, 24], solutions are sought for the limits in which the solid is a much better conductor than the cavity and in which the cavity is small.

Solution

Thermal conduction in the surrounding solid
Insofar as the solid is a much better thermal conductor than the fluid, it sees the cavity as an insulator and so the outward temperature gradient is normal to the ellipsoid. The temperature, therefore, is analogous to the velocity potential for ideal flow over an ellipsoid with freestream parallel to the \( x \)-axis [10, 16].

This solution has the temperature on the ellipsoid proportional to \( x \), but with a different gradient to that at infinity (higher, as the heat-lines bulge around the adiabatic cavity).

Insofar as the solid is a much better thermal conductor than the fluid, this serves to define a Dirichlet temperature boundary condition on the ellipsoid for the latter, say \( T = x \Delta T/2a \).

Conduction and creeping flow
The limiting solution in the cavity as \( \Delta T \rightarrow 0 \) is stagnant \( (u \sim 0) \)
Two-dimensional convection in an ellipse

Beginning the perturbation process by iteration [12, p. 2] yields the Stokes problem:

\[ \nabla \cdot \mathbf{u}_0 = 0 \quad (5) \]
\[ -\nabla p_0 + \mu \nabla^2 \mathbf{u}_0 = -\rho \beta \Delta T x / 2a. \quad (6) \]

Two-dimensional convection in an ellipse

If \( c \gg a, b \), the ellipsoid defined by equation (1) becomes two-dimensional. Ignoring the \( z \)-component, a stream-function \( \psi(x, y) \) can be introduced such that \( \mathbf{u}_0 = \nabla \times (\psi \mathbf{k}) \). Taking the \( z \)-component of the curl of equation (6) leads to the biharmonic equation \( \mu \nabla^4 \psi = -\rho \beta \Delta T / 2a \) with \( \psi \) and its gradient vanishing on the boundary. As noted by Batchelor [11] for the corresponding problem in a rectangle, this is analogous to the boundary value problem of bending a clamped plate, with \( \psi \) there representing the deflexion. That problem was solved in the ellipse by Bryan [19, 20]; thus

\[ \psi = -\frac{\rho \beta \Delta T a^3 b^3 \Theta^2}{16\mu (3a^2 + 2a^2 b^2 + 3b^4)}. \quad (7) \]

To the best of our knowledge, this has not been previously reported as the solution for creeping convection in the ellipse.

The level-sets of equation (7) are drawn in figure 2.

Note the role of the boundary function \( \Theta \) from equation (1); its square appears for the two boundary conditions on the stream-function, \( \psi = 0 \) and \( \partial \psi / \partial n = 0 \). Similarly, for the Poisson equation in an ellipse for the fully developed velocity in a duct of elliptic section, the solution is proportional to the first power of \( \Theta \) in order to satisfy the single no-slip condition [21, 4, 22].

This solution of the two-dimensional problem by means of the stream-function appears to be a dead end, not leading to the ellipsoid. Its generalization to a three-dimensional vector-potential is tractable in the sphere using the toroidal and poloidal potentials [23], but adapting those to the ellipsoid seems forbidding; the study of ellipsoidal harmonic functions is more complicated by far than the corresponding study of spherical harmonic functions [6, p. xi].

Creeping pressure in the ellipse and sphere

After computing the velocity from the stream-function of equation (7), the corresponding pressure \( p_0 \) can be computed by integrating equation (6):

\[ p_0 = \frac{\rho \beta \Delta T a (3a^2 + b^2)xy}{2 (3a^2 + 2a^2 b^2 + 3b^4)}. \quad (8) \]

The pressure from equation (8) is drawn in figure 3.

The Stokes pressure \( p_0 \) was given in the special three-dimensional case of the sphere [23] as also proportional to \( xy \). That the simple symmetry of the function \( xy \) describes the creeping pressure in both the ellipse and the sphere motivates trying it for the general ellipsoid in a method of undetermined coefficients.

In a sense, in the use here of Cartesian coordinates, we are taking the opposite approach to Lamé, for whom in general ‘the study of a physical problem lead to that of a system of curvilinear coordinates’ [11].

Creeping flow in the ellipsoid

For a given pressure, the Stokes equation (6) reduces to a vector Poisson equation for the velocity or a scalar Poisson equation for each component; say \( p_0 = P_0 x y \) for some constant \( P_0 \), then

\[ -\mu \nabla^2 u_{xy} = P_0 \quad (9) \]
\[ -\mu \nabla^2 v_0 = \left( P_0 - \frac{\rho \beta \Delta T}{2a} \right) x \quad (10) \]
\[ -\mu \nabla^2 w_0 = 0 \quad (11) \]

with \( u_0 = v_0 = w_0 = 0 \) on \( \Theta = 0 \). Beside the trivial \( w_0 = 0 \), the first two Poisson equations (9)–(10) recall the two-dimensional Poisson equation for the fully developed natural convection in a tall horizontally heated vertical cavity of elliptic section [22], suggesting solutions proportional to \( y \Theta \) and \( x \Theta \), respectively, with the boundary function \( \Theta \) as a factor enforcing the no-slip condition and the coefficients of proportionality depending affinely on the undetermined \( P_0 \). The latter is determined by requiring that the velocity be divergence-free. The result is:

\[ u_0 = \frac{\rho \beta \Delta T a b^2 c^2 (a^2 y-i-b^2) \Theta}{4\mu (a^2 b^2 + 3a^2 c^2 + a^2 b^4 + 2a^2 b^2 c^2 + 3b^4 c^2)} \]
\[ p_0 = \frac{\rho \beta \Delta T a (2a^2 b^2 + 3a^2 c^2 + 2a^2 b^2 c^2 + 3b^4 c^2)xy}{2 (a^2 b^2 + 3a^2 c^2 + a^2 b^4 + 2a^2 b^2 c^2 + 3b^4 c^2)} \quad (12) \]

Equation (8) is the limit of equation (13) as \( c \to \infty \); similarly, a function \( \psi \) can be produced such that \( \nabla \times (\psi \mathbf{k}) \) gives \( \mathbf{u} \) from equation (12) in each plane of constant \( \zeta \):

\[ \psi = \frac{\rho \beta \Delta T a b^4 c^3 \Theta^2}{16\mu (a^2 b^2 + 3a^2 c^2 + a^2 b^4 + 2a^2 b^2 c^2 + 3b^4 c^2)}. \quad (14) \]

Again, this reduces to equation (7) as \( c \to \infty \). Further, it acts as a stream-function in each plane of constant \( \zeta \). The stream-lines are all ellipses normal to and centred on the \( z \)-axis and geometrically similar to the section through the ellipsoidal boundary of the \( xy \)-plane. It is that the stream-lines are geometrically similar.
Thus:
The right-hand side is of this form. Further, symmetries their laplacians; cf. [20, § 8.1.1]. Only polynomials in $\mathbf{x}$ vanishing on the ellipsoid, i.e. having $\Theta$ of undetermined coefficients, we build up a library of functions of (9), but slightly more complicated. To facilitate the method (19), from equation (19); $T_1$ is positive in the top half.

First-order correction to the temperature
The next step of iteration on the temperature equation (4) suggests an expansion in Grashof number $\mathrm{Gr} \equiv 8 \pi^2 \beta \Delta T a^3 / \mu^2$:

$$T \sim T_0 + \text{Gr} \mathbf{Pr} T_1 + O \left( \mathrm{Gr}^2 \right),$$  (15)

where $\Pr \equiv \mu C/k$ is the Prandtl number, with the new term satisfying

$$\nabla^2 T_1 = \frac{\rho}{\mu \text{Gr}} \mathbf{w}_0 \cdot \nabla T_0,$$  (16)

and vanishing on the boundary.

The right-hand side is proportional to $y \Theta$. This is like equation (9), but slightly more complicated. To facilitate the method of undetermined coefficients, we build up a library of functions vanishing on the ellipsoid, i.e. having $\Theta$ as a factor, and calculate their laplacians; cf. [20, § 8.1.1]. Only polynomials in $x$, $y$, and $z$ need be considered, since the laplacian operator is closed in this set, and the right-hand side is of this form. Further, symmetry requires that $T_1$ should be even in $x$ and odd in $y$. Thus:

$$\nabla^2 (x^2 y \Theta) = 2y \left( \frac{5}{c^2 x^2} + \frac{3}{b^2} + \frac{1}{c^2 z^2} \right) x^2 + \Theta;$$  (17)

e.g., etc.; this suggests the form

$$T_1 = \left( c_1 x^2 + c_2 y^2 + c_3 z^2 + c_0 \right) y \Theta \Delta T / a$$  (19)

where $c_1, c_2, c_3$ and $c_0$ are constants depending only on the semi-axes $a, b, c$. These can be determined from a linear system by matching the coefficients of $x, y x^2, y z^2$, and $y^3$:

$$c_1 = b^2 c^2 (a^2 b^2 + 7a^2 c^2 + b^2 c^2) / (5a^2 b^2 + 3a^2 c^2 + b^2 c^2) / \mathbf{d}_1 \mathbf{d}_2$$

$$c_2 = a^2 c^2 (a^2 b^2 + 3a^2 c^2 + 5b^2 c^2) / (5a^2 b^2 + 3a^2 c^2 + b^2 c^2) / \mathbf{d}_1 \mathbf{d}_2$$

$$c_3 = -a^2 b^2 (a^2 b^2 + 3a^2 c^2 + 5b^2 c^2) / (a^2 b^2 + 7a^2 c^2 + b^2 c^2) / \mathbf{d}_1 \mathbf{d}_2$$

$$c_0 = -a^2 b^2 c^2 (a^2 b^2 + 3a^2 c^2 + 5b^2 c^2) \times$$

$$\left( (a^2 b^2 + 7a^2 c^2 + b^2 c^2) / (5a^2 b^2 + 3a^2 c^2 + b^2 c^2) \right) \times$$

$$\left( a^2 b^2 + 3a^2 c^2 + b^2 c^2 \right) / \mathbf{d}_1 \mathbf{d}_2,$$  (23)

where

$$\mathbf{d}_1 = 768 (a^2 b^2 + 13a^2 c^2 + 35a^2 b^2 c^2 + 15a^2 b^2 c^2 + 4a^2 b^2 c^2 + 62a^2 b^2 c^2 + 35a^2 b^2 c^2 + 7a^2 b^2 c^2 + 2a^2 b^2 c^2 + 13a^2 b^2 c^2 + 5b^2 c^2)$$

$$\mathbf{d}_2 = a^4 b^2 + 3a^4 c^2 + a^2 b^4 + 2a^2 b^2 c^2 + 3b^4 c^2.$$  (24)

$T_1$ is displayed in figure 4.
Figure 5: Approximate temperature fields $T_0 + \text{Gr Pr} T_1$ in the plane $z = 0$ for Gr Pr as labelled.

**Useful software**

Verifying that the lengthy expressions above satisfied the partial differential equations and boundary conditions was greatly facilitated by SymPy [15]. Matplotlib [13] and Asymptote [5] were used to plot the figures.

**References**


