20th Australasian Fluid Mechanics Conference
Perth, Australia
5-8 December 2016

Creeping Convection in a Horizontally Heated Ellipsoid

G. D. McBain

Modelling & Simulation, Memjet Australia Pty Ltd, Macquarie Park, New South Wales 2113, Australia

Abstract

Natural convection in horizontally heated ellipsoidal cavities is considered in the low Grashof number limit, solving the Laplace equation for steady thermal conduction in the unbounded solid exterior, the Oberbeck–Boussinesq equations in the fluid-filled interior, and matching the temperature at the interface.

In the hierarchy of equations governing the asymptotic expansion for small Grashof number, at each order a forced Stokes problem must be solved for the momentum correction. The creeping flow is known for the sphere in closed form in terms of the toroidal and poloidal potentials, spherical coordinates, and spherical harmonics. Rather than attempting to generalize this to ellipsoidal coordinates, it is re-expressed in terms of the primitive pressure-velocity variables as polynomials in the Cartesian coordinates. This form, equivalent in the sphere, suggests solutions for the pressure in an ellipsoid, which can then be found together with the velocity in closed form by the method of undetermined coefficients. Similarly, the perturbations to the temperature satisfy Poisson equations which can be solved by the same method. Polynomial formulæ are given for the creeping flow and the first-order correction to the temperature.

In the limit as one of the axes of the ellipsoid tends to infinity, the three-dimensional solution reduces to a two-dimensional solution for natural convection in a horizontal elliptical cylinder, tranversely horizontally heated. This exact solution is believed to be new too.

Introduction

Natural convection in horizontally heated cavities is a much studied subject [2, ch. 5], often used as a test-case for computational fluid dynamics [8, 7].

Most often, rectangular cavities are studied, either because of relevance to applications like double-glazed windows [1, 14], or simply because that facilitates meshing, particularly structured meshing. Some cavities are round though, as in natural subterranean reservoirs [25] or in closed-porous insulation materials [18]. Revisiting those problems [23, 24] revealed that the sphere actually proved more amenable to analysis than the box, admitting closed-form solutions for the creeping-flow limit.

The present paper shows how the spherical solution may be generalized to the ellipsoidal cavity. Surprisingly, this generalization, expressing the primitive velocity–pressure dependent variables as polynomials of the Cartesian coordinates turns out to be simpler than the previous closed-form solutions for the sphere, using spherical harmonics, spherical coordinates, and toroidal and poloidal potentials.

Mathematical formulation

Consider an ellipsoidal cavity defined by $\Theta < 0$ with

$$\Theta \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1, \tag{1}$$

as shown in figure 1, oriented with one axis vertical with respect to the uniform gravitational field $\mathbf{g} = -g\mathbf{j}$ in a uniform infinite

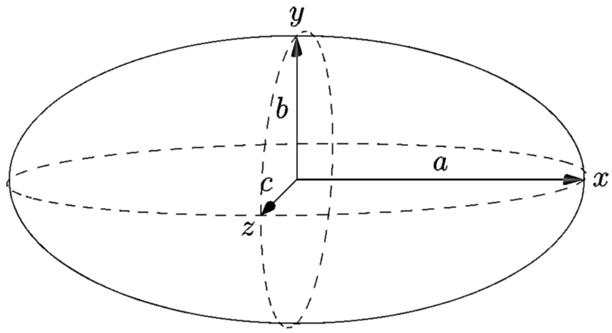

Figure 1: A ellipsoid with arbitrary ratios of axes, defined by the zero level-set of equation (1), aligned with the Cartesian coordinate system. Here a temperature gradient is applied in the x-direction and gravity in the $-y$-direction.

solid subject to an overall horizontal temperature gradient $\boldsymbol{\nabla} T \sim T_{x,\infty}\mathbf{i}$ parallel to another of the cavity's axes.

In the surrounding solid, the steady temperature satisfies Laplace's equation, $\nabla^2 T = 0$; in the cavity, the temperature is coupled to the velocity $\boldsymbol{u}$ and pressure p by the Oberbeck–Boussinesq equations

$$\nabla \cdot \boldsymbol{u} = 0 \tag{2}$$

$$\rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\boldsymbol{\nabla} p + \rho g \beta T \mathbf{j} + \mu \nabla^2 \boldsymbol{u} \tag{3}$$

$$\rho C \boldsymbol{u} \cdot \nabla T = k \nabla^2 T, \tag{4}$$

where ρ, C, β, μ, and k are the density, specific heat, and coefficients of thermal volumetric expansion, viscosity, and thermal conductivity of the fluid. The temperature T is expressed as the excess over that the centre of the cavity (constant, by symmetry, and equal to the value that would prevail without the cavity).

Guided by the work on the spherical special case [23, 24], solutions are sought for the limits in which the solid is a much better conductor than the cavity and in which the cavity is small.

Solution

Thermal conduction in the surrounding solid

Insofar as the solid is a much better thermal conductor than the fluid, it sees the cavity as an insulator and so the outward temperature gradient is normal to the ellipsoid. The temperature, therefore, is analogous to the velocity potential for ideal flow over an ellipsoid with freestream parallel to the x-axis [10, 16]. This solution has the temperature on the ellipsoid proportional to x, but with a different gradient to that at infinity (higher, as the heat-lines bulge around the adiabatic cavity).

Insofar as the solid is a much better thermal conductor than the fluid, this serves to define a Dirichlet temperature boundary condition on the ellipsoid for the latter, say $T = x\Delta T/2a$.

Conduction and creeping flow

The limiting solution in the cavity as $\Delta T \to 0$ is stagnant ($\boldsymbol{u} \sim \mathbf{0}$)

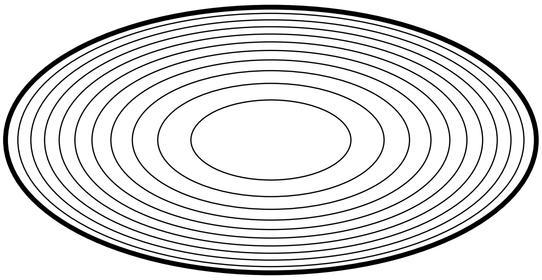

Figure 2: Stream-lines of the creeping flow in the ellipse, derived from the stream-function of equation (7).

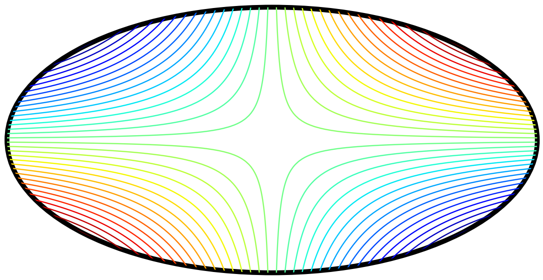

Figure 3: Isobars of the creeping flow in the ellipse from equation (8); the sign is that of xy in each quadrant.

conduction ($T \sim x\Delta T/2a$) [9, § 27].

Beginning the perturbation process by iteration [12, p. 2] yields the Stokes problem:

$$\nabla \cdot \boldsymbol{u}_0 = 0 \tag{5}$$

$$-\boldsymbol{\nabla} p_0 + \mu \nabla^2 \boldsymbol{u}_0 = -\rho g \beta \Delta T x \mathbf{j}/2a. \tag{6}$$

Two-dimensional convection in an ellipse

If $c \gg a, b$, the ellipsoid defined by equation (1) becomes two-dimensional. Ignoring the z-dimension, a stream-function $\psi(x, y)$ can be introduced such that $\boldsymbol{u}_0 = \nabla \times (\psi \mathbf{k})$. Taking the z-component of the curl of equation (6) leads to the biharmonic equation $\mu \nabla^4 \psi = -\rho g \beta \Delta T/2a$ with ψ and its gradient vanishing on the boundary. As noted by Batchelor [1] for the corresponding problem in a rectangle, this is analogous to the boundary value problem of bending a clamped plate, with ψ there representing the deflexion. That problem was solved in the ellipse by Bryan [19, 20]; thus

$$\psi = -\frac{\rho g \beta \Delta T a^3 b^4 \Theta^2}{16\mu\left(3a^4 + 2a^2 b^2 + 3b^4\right)}. \tag{7}$$

To the best of our knowledge, this has not been previously reported as the solution for creeping convection in the ellipse.

The level-sets of equation (7) are drawn in figure 2.

Note the role of the boundary function Θ from equation (1): its square appears for the two boundary conditions on the stream-function, $\psi = 0$ and $\partial\psi/\partial n = 0$. Similarly, for the Poisson equation in an ellipse for the fully developed velocity in a duct of elliptic section, the solution is proportional to the first power of Θ in order to satisfy the single no-slip condition [21, 4, 22].

This solution of the two-dimensional problem by means of the stream-function appears to be a dead end, not leading to the ellipsoid. Its generalization to a three-dimensional vector-potential is tractable in the sphere using the toroidal and poloidal potentials [23], but adapting those to the ellipsoid seems forbidding; the study of ellipsoidal harmonic functions is 'more complicated by far than the corresponding study of' spherical harmonic functions [6, p. xi].

Creeping pressure in the ellipse and sphere

After computing the velocity from the stream-function of equation (7), the corresponding pressure p_0 can be computed by integrating equation (6):

$$p_0 = \frac{\rho g \beta \Delta T a (3a^2 + b^2) xy}{2\left(3a^4 + 2a^2 b^2 + 3b^4\right)}. \tag{8}$$

The pressure from equation (8) is drawn in figure 3.

The Stokes pressure p_0 was given in the special three-dimensional case of the sphere [23] as also proportional to xy. That the simple symmetry of the function xy describes the creeping pressure in both the ellipse and the sphere motivates trying it for the general ellipsoid in a method of undetermined coefficients.

In a sense, in the use here of Cartesian coordinates, we are taking the opposite approach to Lamé, for whom in general 'the study of a physical problem lead to that of a system of curvilinear coordinates' [11].

Creeping flow in the ellipsoid

For a given pressure, the Stokes equation (6) reduces to a vector Poisson equation for the velocity or a scalar Poisson equation for each component; say $p_0 = P_0 xy$ for some constant P_0, then

$$\mu \nabla^2 u_0 = P_0 y \tag{9}$$

$$\mu \nabla^2 v_0 = \left(P_0 - \frac{\rho g \beta \Delta T}{2a}\right) x \tag{10}$$

$$\mu \nabla^2 w_0 = 0 \tag{11}$$

with $u_0 = v_0 = w_0 = 0$ on $\Theta = 0$. Beside the trivial $w_0 = 0$, the first two Poisson equations (9)–(10) recall the two-dimensional Poisson equation for the fully developed natural convection in a tall horizontally heated vertical cavity of elliptic section [22], suggesting solutions proportional to $y\Theta$ and $x\Theta$, respectively, with the boundary function Θ as a factor enforcing the no-slip condition and the coefficients of proportionality depending affinely on the undetermined P_0. The latter is determined by requiring that the velocity be divergence-free. The result is:

$$\boldsymbol{u}_0 = \frac{\rho g \beta \Delta T a b^2 c^2 \left(a^2 y \mathbf{i} - b^2 x \mathbf{j}\right) \Theta}{4\mu\left(a^4 b^2 + 3a^4 c^2 + a^2 b^4 + 2a^2 b^2 c^2 + 3b^4 c^2\right)} \tag{12}$$

$$p_0 = \frac{\rho g \beta \Delta T a \left(a^2 b^2 + 3a^2 c^2 + b^2 c^2\right) xy}{2\left(a^4 b^2 + 3a^4 c^2 + a^2 b^4 + 2a^2 b^2 c^2 + 3b^4 c^2\right)} \tag{13}$$

Equation (8) is the limit of equation (13) as $c \to \infty$; similarly, a function ψ can be produced such that $\nabla \times (\psi \mathbf{k})$ gives $\boldsymbol{u}$ from equation (12) in each plane of constant z:

$$\psi = \frac{\rho g \beta \Delta T a^3 b^4 c^2 \Theta^2}{16\mu(a^4 b^2 + 3a^4 c^2 + a^2 b^4 + 2a^2 b^2 c^2 + 3b^4 c^2)}. \tag{14}$$

Again, this reduces to equation (7) as $c \to \infty$. Further, it acts as a stream-function in each plane of constant z. The stream-lines are all ellipses normal to and centred on the z-axis and geometrically similar to the section through the ellipsoidal boundary of the xy-plane. It is that the stream-lines are geometrically similar rather than confocal ellipses that reduces the usefulness for this problem of Lamé's [17] ellipsoidal coordinates.

As $b \to \infty$, equation (12) reduces to the unidirectional vertical solution in the horizontally heated infinite vertical duct [22]; if, further, $c \to \infty$, the classical cubic profile between two parallel vertical walls [27] is regained.

First-order correction to the temperature

The next step of iteration on the temperature equation (4) suggests an expansion in Grashof number $\mathrm{Gr} \equiv 8\rho^2 g\beta\Delta T a^3/\mu^2$:

$$T \sim T_0 + \mathrm{Gr}\,\mathrm{Pr}\,T_1 + O\left(\mathrm{Gr}^2\right), \tag{15}$$

where $\mathrm{Pr} \equiv \mu C/k$ is the Prandtl number, with the new term satisfying

$$\nabla^2 T_1 = \frac{\rho}{\mu\,\mathrm{Gr}}\,\boldsymbol{u}_0 \cdot \nabla T_0, \tag{16}$$

and vanishing on the boundary.

The right-hand side is proportional to $y\Theta$. This is like equation (9), but slightly more complicated. To facilitate the method of undetermined coefficients, we build up a library of functions vanishing on the ellipsoid, i.e. having Θ as a factor, and calculate their laplacians; cf. [20, § 8.1.1]. Only polynomials in x, y, and z need be considered, since the laplacian operator is closed in this set, and the right-hand side is of this form. Further, symmetry requires that T_1 should be even in x and z and odd in y. Thus:

$$\nabla^2(y\Theta) = \frac{2}{a^2} + \frac{6}{b^2} + \frac{2}{c^2}, \tag{17}$$

a multiple of which will match a constant on the right-hand side;

$$\nabla^2(x^2 y\Theta) = 2y\left\{\left(\frac{5}{a^2} + \frac{3}{b^2} + \frac{1}{c^2}\right)x^2 + \Theta\right\}; \tag{18}$$

etc., etc.; this suggests the form

$$T_1 = \left(c_x x^2 + c_y y^2 + c_z z^2 + c_0\right) y\Theta\Delta T/a \tag{19}$$

where c_x, c_y, c_z and c_0 are constants depending only on the semi-axes a, b, c. These can be determined from a linear system by matching the coefficients of y, yx^2, yz^2, and y^3:

$$c_x = b^2c^2(a^2b^2 + 7a^2c^2 + b^2c^2)(5a^2b^2 + 3a^2c^2 + b^2c^2)/d_1 d_2 \tag{20}$$

$$c_y = a^2c^2(a^2b^2 + 3a^2c^2 + 5b^2c^2)(5a^2b^2 + 3a^2c^2 + b^2c^2)/d_1 d_2 \tag{21}$$

$$c_z = a^2b^2(a^2b^2 + 3a^2c^2 + 5b^2c^2)(a^2b^2 + 7a^2c^2 + b^2c^2)/d_1 d_2 \tag{22}$$

$$\begin{aligned} c_0 = &-a^2b^2c^2(a^2b^2 + 3a^2c^2 + 5b^2c^2)\times \\ &(a^2b^2 + 7a^2c^2 + b^2c^2)(5a^2b^2 + 3a^2c^2 + b^2c^2)\div \\ &(a^2b^2 + 3a^2c^2 + b^2c^2)d_1 d_2, \end{aligned} \tag{23}$$

where

$$\begin{aligned} d_1 = &768(a^6b^6 + 13a^6b^4c^2 + 35a^6b^2c^4 + 15a^6c^6 + \\ &7a^4b^6c^2 + 62a^4b^4c^4 + 35a^4b^2c^6 + 7a^2b^6c^4 + \\ &13a^2b^4c^6 + b^6c^6) \end{aligned} \tag{24}$$

$$d_2 = a^4b^2 + 3a^4c^2 + a^2b^4 + 2a^2b^2c^2 + 3b^4c^2. \tag{25}$$

T_1 is displayed in figure 4.

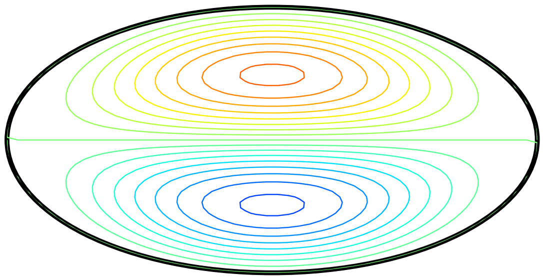

Figure 4: The first-order correction to the temperature in the plane of symmetry $z = 0$, from equation (19); T_1 is positive in the top half.

The temperature to first-order

For low finite Grashof numbers, the temperature field can be approximated by truncating the asymptotic expansion (15) to $T_0 + \mathrm{Gr}\,\mathrm{Pr}T_1$, as shown in figure 5. These are yet to be compared with numerical solutions of the full nonlinear equations as done for the sphere [24], but experience there suggests that these should be reasonably accurate to the level shown.

Dicussion & conclusions

The expressions for the pressure, velocity, and temperature given above for the general ellipsoid simplify considerably when two or three of the axes have equal length (so that the ellipsoid becomes a prolate or oblate spheroid or sphere); such simplifications are straightforward and omitted here.

The creeping flow solution in the ellipse was found by an educated guess at the form of the pressure field based on those in the sphere and the ellipse, the former having been found in previous work and the latter deduced from an analogy with a problem in the theory of elastic plates. The zeroth- and first-order temperature fields were found fairly easily as polynomial solutions of the Poisson equation with polynomial right-hand sides; this should apply for higher-order temperature fields too, since the right-hand sides are generated from lower-order solutions and should remain polynomial, of steadily increasing degree.

The next step, however, is the first-order correction to the velocity and pressure; this involves a forced Stokes problem. The solution is expected to be polynomial, but no immediate method of solution is at hand—the poloidal–toroidal decomposition not having been generalized from the sphere. The 'method' of undetermined coefficients will work if provided with sufficient candidates; perhaps a systematic approach could be guided by the enumeration of polynomial solenoidal vector fields in the ellipsoid used to study inviscid rotating flows; see Vantieghem [26] and the references therein.

The zeroth-order exact solution is noteworthy in providing a nontrivial test-case in a simply defined and completely bounded geometry for steady three-dimensional computational fluid dynamics codes. In particular, it is free of the corner-singularities which complicate lid-driven cavities [3].

Topologically, the three-dimensional creeping flow of equation (12) differs little from the two-dimensional flow in the ellipse as drawn in figure 2; however, if gravity or the heating of the surrounding solid were not parallel to one of the axes of the ellipsoid, much more complicated kinematics would result. The creeping flow would be the sum of two or three contributions like equation (12) for the components of the conduction temperature field along the various axes; this combination is unlikely to have closed stream-lines.

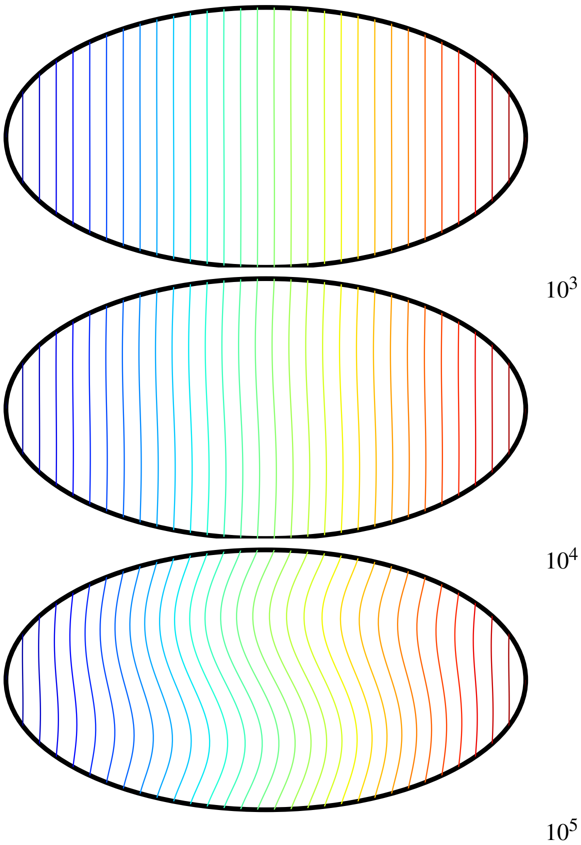

Figure 5: Approximate temperature fields $T_0 + \mathrm{Gr\,Pr}T_1$ in the plane $z = 0$ for Gr Pr as labelled.

Useful software

Verifying that the lengthy expressions above satisfied the partial differential equations and boundary conditions was greatly facilitated by *SymPy* [15]. *Matplotlib* [13] and *Asymptote* [5] were used to plot the figures.

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