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# Stabilisation of first-mode disturbances in compressible boundary-layer flows

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### Abstract

The growth of inviscid, subsonic disturbances in an external, compressible boundary-layer flow is linked with the existence of generalised points of inflection in the base flow. An inviscid neutral mode can propagate at a wavespeed equal to the streamwise boundary-layer velocity at the generalised point of inflection, and this neutral mode is adjacent to unstable modes in wavenumber space. This is an extension of the classical Rayleigh inflection point theorem in the incompressible theory. A compressible, flat-plate boundary layer contains a single generalised point of inflection and hence is unstable in the inviscid limit. Under the influence of a favourable pressure gradient and a heated surface, the boundary-layer flow develops a velocity overshoot; the streamwise velocity exhibits a local maximum. This development also modifies the number and location of generalised points of inflection, and hence, the behaviour of inviscid disturbances. In this paper we summarise the linear stability properties of boundary layers with a velocity overshoot and focus on the case where the first-mode disturbance is neutralised.

### Introduction

The development of supersonic and hypersonic transport vehicles requires a strong understanding of, and ability to control, compressible aero-thermodynamic flow phenomena. The efficiency of these vehicles relies on delaying the transition to turbulence in order to minimise drag and maintain stable flight. Turbulence also increases thermal load on the vehicle through aerodynamic heating that may damage surface materials. Current hypersonic vehicles under development are slender and wedge-shaped to minimise Görtler and centrifugal instabilities that otherwise rapidly cause transition [1]. Transition in the flow over these vehicles is triggered by the growth of two-dimensional and oblique normal-mode (Tollmien-Schlichting) disturbances that can be modelled with linear stability theory [2]. The results can be used to find amplification rates along a surface that can be correlated with transition using, for example, the "$e^N$ method" [8]. In the large Reynolds number (inviscid) limit, unstable modes are correlated with generalised points of inflection [3], which are an extension of Rayleigh's inflection point theorem of the incompressible theory. In classical flat-plate boundary layers there is a single generalised point of inflection, however, in some cases (see [6]) there may be zero or multiple generalised points of inflection. In this paper we focus on boundary-layer profiles that contain a local maximum in the streamwise velocity. These boundary layers can arise on a flat plate when the surface is heated and there is a favourable pressure gradient accelerating the flow [9].

### Boundary-Layer Theory

The compressible boundary-layer equations governing flow in the viscous flow close to the surface of an object a large Reynolds-number external flow can admit self-similar solutions that satisfy

$$\left(Cf''\right)' + ff'' + \beta(1+k)\left(g - f'^2\right) = 0, \qquad (1)$$

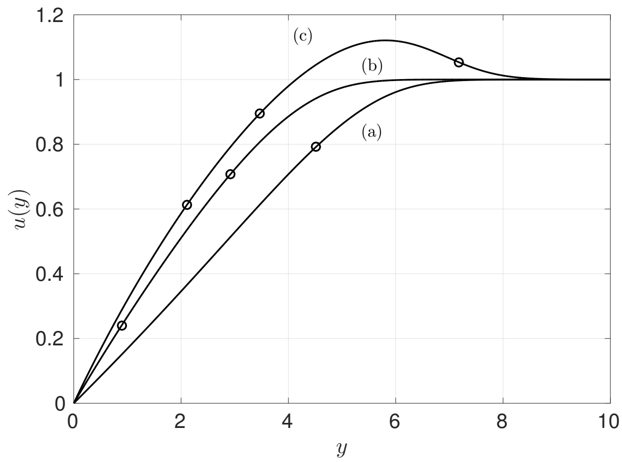


Figure 1: Boundary-layer streamwise velocity profiles and generalised points of inflection (○) for (a) $M = 3$, $g_w = 1.2$, $\beta = 0$; (b) $M = 3$, $g_w = 1.2$, $\beta = 0.05$; (c) $M = 6$, $g_w = 1.6$, $\beta = 0.3$.

$$\left(\frac{C}{Pr}g'\right)' + \frac{2k}{1+k}\left(1 - \frac{1}{Pr}\right)\left(Cf'f''\right)' + fg' = 0, \qquad (2)$$

where primes denote differentiation to the Mangler-Levy-Lees similarity variable

$$\eta \propto \int \frac{1}{T}\,\mathrm{d}y,$$

$f'$ is the streamwise velocity (nondimensionalised by freestream velocity), $g$ is the enthalpy (nondimensionalised by freestream enthalpy), $C = \mu/T$ is the viscosity-temperature ratio, $Pr$ is the Prandtl number, $\beta$ is the (Falkner-Skan) pressure gradient parameter and $2k = (\gamma - 1)M^2$; where $\gamma$ is the heat-capacity ratio and $M$ is the Mach number [7]. These self-similar solutions are exactly valid in the case when $\beta$ is zero, $Pr$ is unity or in the large-$M$ limit. Solutions are subject to the boundary conditions $f(0) = f'(0) = 0$, $g(0) = g_w$, and $f', g \to 1$ as $\eta \to \infty$. When $\beta$ is positive (representing a favourable pressure gradient) and $g_w > g_{ad}$ (heated wall), where $g_{ad}$ (adiabatic wall enthalpy) is the value for which $g'(0) = 0$, the streamwise velocity profile is not monotonic and "overshoots" the freestream velocity. In this paper we present results for $Pr = 0.7$ and a Sutherland Law viscosity-temperature model. Some typical overshoot boundary-layer velocity profiles are presented in figure 1.

In addition to the presence of the local maximum in the streamwise velocity $f'$, hereafter referred to as $u$, also of interest are the generalised points of inflection, which govern subsonic, inviscid instabilities. These are points where

$$I(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{T}\frac{\mathrm{d}u}{\mathrm{d}y}\right) = 0,$$

and the value of $u$ at such points corresponds to the wavespeed of an inviscid, neutral disturbance that can exist. In figures 2

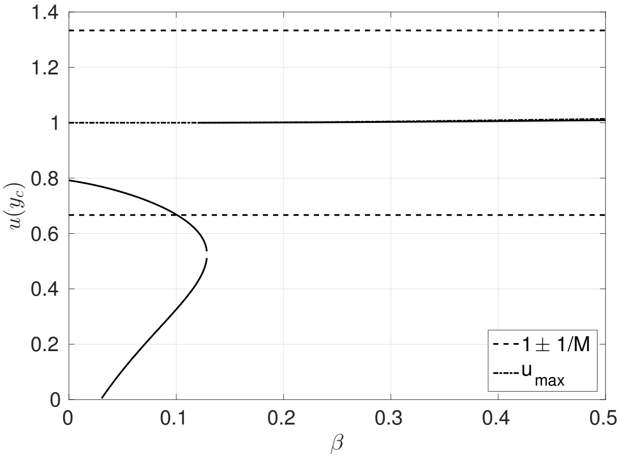


Figure 2: Plots of the inflectional velocities for the case $M = 3$, $g_w = 1.2$ and variable $\beta$.

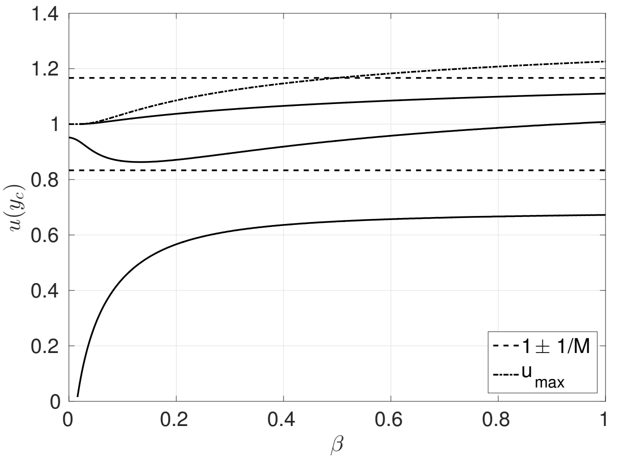


Figure 3: Plots of the inflectional velocities for the case $M = 6$, $g_w = 1.6$ and variable $\beta$.

and 3 the inflectional velocities for two families of overshoot boundary layers are presented. When there is no velocity overshoot at $\beta = 0$, there is a single generalised point of inflection – we will refer to the velocity at this family of generalised points of inflection as $c_s$. As $\beta$ is increased, an additional generalised point of inflection enters the boundary layer from the freestream ($y \to \infty$) and a further one from the wall ($y = 0$). As the maximum streamwise velocity $\bar{u}_{max}$ is very close to unity, so to is the velocity at the freestream-originated generalised point of inflection. In the first case (figure 2) a local minimum in the distribution of $I(y)$ moves above $I(y) = 0$ at approximately $\beta = 0.13$ and two generalised points of inflection no longer exist. In the second case (figure 3) three generalised points of inflection exist for all positive values of $\beta$. As the wall enthalpy $g_w$ is increased the former case evolves into the latter and the Mach number has a similar effect. In this paper the former case is considered to investigate the effect of removing two generalised points of inflection.

### Linear Stability Theory

The evolution of small-amplitude disturbances to a basic, steady boundary layer may be investigated using linear stability theory. For each quantity in the governing flow equations (continuity, conservation of momentum, conservation of energy and equation of state) a small wavelike perturbation is added, i.e. for $u$,

$$u = \bar{u} + \hat{u}(y) \exp\left[i(\alpha x + \zeta z - \omega t)\right],$$

where $\bar{u}$ is the basic state, $\hat{u}$ is the disturbance amplitude (varying in the normal co-ordinate $y$), $\alpha$ and $\zeta$ are the streamwise ($x$-direction) and spanwise ($z$-direction) wavenumbers and $\omega$ is the frequency. The resulting stability equations are an 8th order system ordinary differential equations that constitute an eigenvalue problem for the complex wave parameters $\alpha$, $\zeta$ and $\omega$. These linearised stability equations are presented elsewhere [5, 7]. Our interest in this paper is the large Reynolds-number (inviscid) limit where the stability equations for a disturbance travelling parallel to the boundary-layer flow may be written as

$$D\hat{v}(\bar{u} - c) = \hat{v}(D\bar{u}) + \hat{p}\left[i\alpha\left(\hat{T} - M^2(\bar{u} - c)^2\right)\right], \qquad (3)$$

$$D\hat{p} = -\hat{v}\left(\frac{i\alpha(\bar{u} - c)}{\hat{T}}\right), \qquad (4)$$

where $D$ denotes differentiation with respect to $y$ and $c = \omega/\alpha$ is the wavespeed. The boundary conditions are $\hat{v}(0) = 0$ and $\hat{v}$ bounded as $y \to \infty$. For a neutral disturbance the wave parameters are real and solutions to equations (3)-(4) are singular at points $y = y_c$ where $\bar{u} = c$. The singularity is regularised if $I(y_c) = 0$ and $1 - 1/M < \bar{u}(y_c) < 1 + 1/M$ and neutral disturbances exist for these wavespeeds that are adjacent to unstable disturbances. There is also an upstream sonic neutral mode at $\alpha = 0$, $c = 1 - 1/M$, responsible for the first-mode disturbance and a downstream sonic neutral mode at $\alpha = 0$, $c = 1 + 1/M$, responsible for the higher-mode, or so-called Mack-mode disturbances. The downstream families begin at $c = 1 + 1/M$ and remain neutral as $\alpha$ increases until the wavespeed decreases to $\bar{u}_{max}$ [11] (unity for non-overshoot boundary layers). These classical inviscid modes are discussed in stronger detail elsewhere (i.e. [3, 5]). For overshoot boundary layers there are a further two inviscid neutral modes; at $\alpha = 0$, $c = \bar{u}_{max}$ and a non-zero wavenumber neutral mode with a non-inflectional wavespeed greater than unity [10]. The inviscid equations may be solved numerically using a shooting method [4]. For damped and neutral ($\Im(c) \leq 0$) disturbances, there is a region near the critical point (at $\bar{u} = \Re(c)$ for damped disturbances) where the inviscid equations are not valid and the integration contour must be indented appropriately into the complex plane; below $y_c$ when $D\bar{u}(y_c) > 0$ (see [4]) and above $y_c$ when $D\bar{u}(y_c) < 0$ (see [11]).

### Parallel Disturbances

In figure 2 the generalised point of inflection that exists in the boundary layer at $\beta = 0$ has an inflectional velocity that decreases below the upstream sonic wavespeed $1 - 1/M$ at approximately $\beta = 0.1$. At $\beta = 0$ this is associated with neutral wavespeed of the first and higher modes. This generalised point of inflection, however, will no longer guarantee a neutral mode when the inflectional velocity is subsonic (i.e. $\bar{u}(y_c) < 1 - 1/M$ or $\bar{u}(y_c) > 1 + 1/M$). A similar event occurs when the boundary layer is cooled and the first mode is completely stabilised [3]. However, the later-discovered second- and higher-mode disturbances are destabilised [4]. In figures 4-7 eigenvalue diagrams are presented for the first-mode and second-mode disturbances of Mach-3 boundary layers with increasing pressure gradient. The growth rates $\Im(\omega)$ are plotted along with the wavespeed $\Re(c)$. The neutralisation of the first-mode disturbance continues to occur at the inflectional wavespeed $c_s$ (and not any other inflectional velocities) up until $\beta = 0.1$; the range of wavespeeds

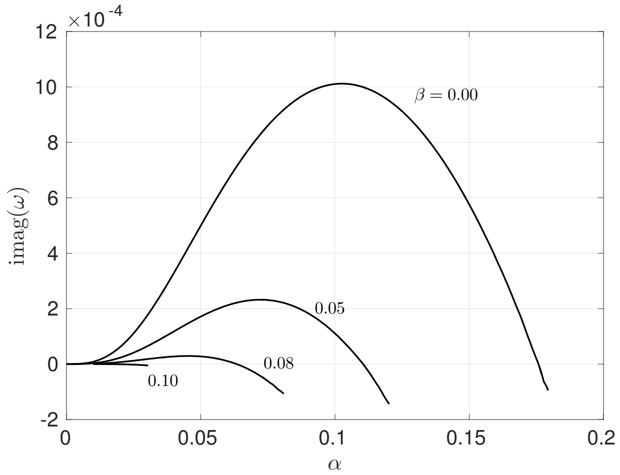


Figure 4: First-mode growth rates for boundary layers with $M = 3$, $g_w = 1.2$ and variable β.

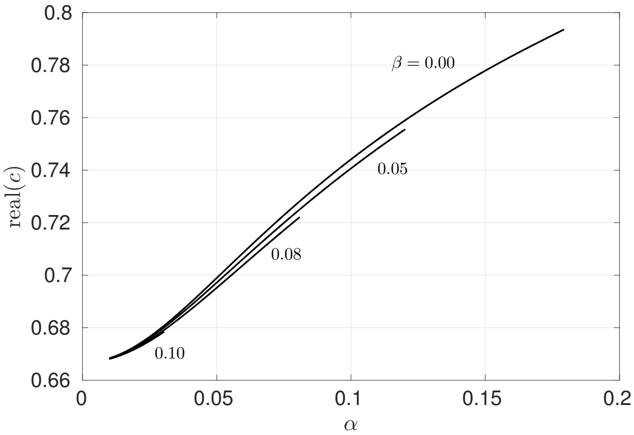


Figure 5: First-mode disturbance wavespeeds for boundary layers with $M = 3$, $g_w = 1.2$ and variable β.

for which the disturbance is unstable reduces with increasing β along with the maximum growth rate. At $\beta = 0.1$ and higher there are no numerically determinable unstable modes adjacent to the upstream sonic mode. The second mode (and higher modes not significant at this Mach number) is affected in a different manner; the decreasing value of $c_s$ (and slightly increasing $\bar{u}_{max}$) increases the range of wavespeeds for which this disturbance is unstable. However, the maximum growth rate does decrease. Up until $\beta = 0.1$ the second mode is neutralised at the inflectional wavespeed $c_s$ at an increasing wavenumber α and for $\beta > 0.1$ the second mode is unstable – with a monotonically decreasing growth rate – to arbitrarily large wavenumbers. The inviscid numerical eigenvalues presented have been confirmed through careful calculations using the viscous stability equations at large Reynolds number.

**Oblique Disturbances**

For flat-plate boundary layers, first-mode disturbances are most unstable when the disturbance travels at an oblique angle to the boundary-layer flow, whereas second- and higher-mode disturbances are most unstable when the disturbance is parallel to

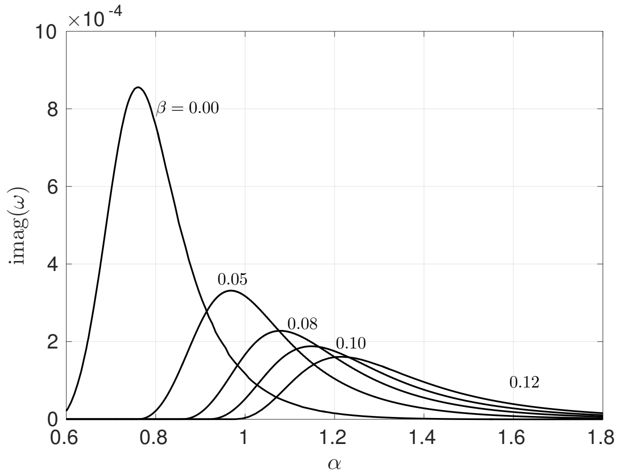


Figure 6: Second-mode disturbance growth rates for boundary layers with $M = 3$, $g_w = 1.2$ and variable β.

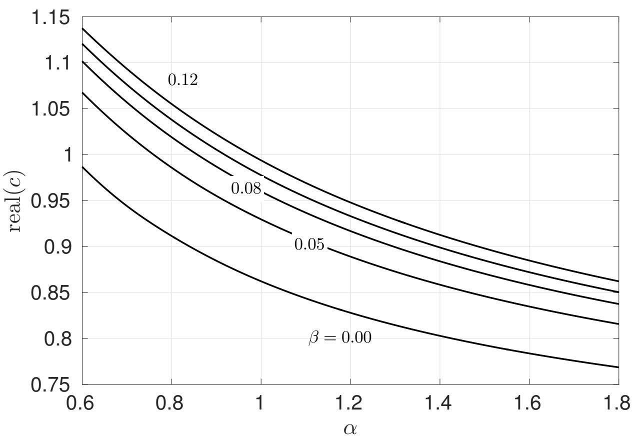


Figure 7: Second-mode disturbance wavespeeds for boundary layers with $M = 3$, $g_w = 1.2$ and variable β.

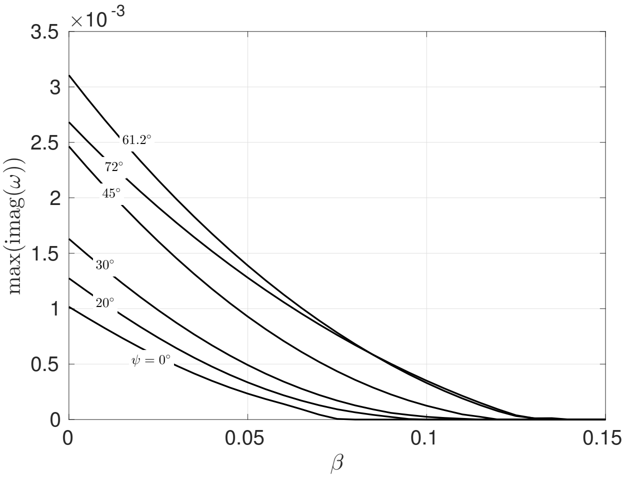


Figure 8: Maximum growth rates of first-mode disturbances for $M = 3$, $g_w = 1.2$ versus $\beta$ for various wave angles $\psi$ (degrees)

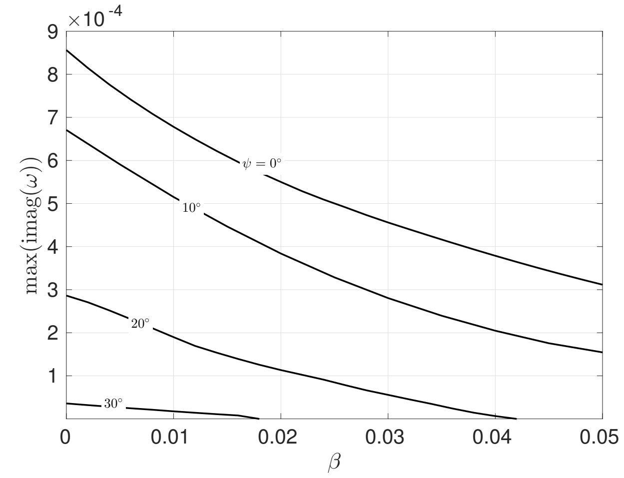


Figure 9: Maximum growth rates of second-mode disturbances for $M = 3$, $g_w = 1.2$ versus $\beta$ for various wave angles $\psi$ (degrees)

the boundary-layer flow [6]. In figures 8 and 9 we present the maximum growth rates of disturbances, first mode and second mode respectively, travelling at an angle $\psi$ to the boundary-layer flow. The boundary-layer profiles considered are identical to those in the previous section. For all wave angles both the first-mode and second-mode disturbances are damped by increasing pressure gradients. Oblique first-mode disturbances are no longer neutralised when $c_s$ reduces below $1-1/M$, however there are no unstable solutions when the generalised point of inflection disappears at approximately $\beta = 0.13$. As with the flat-plate boundary layer, parallel second-mode disturbances have the largest growth rates at all values of the pressure gradient investigated.

## Concluding Remarks

The inviscid, temporal stability results presented demonstrate that in moderate Mach-number boundary layers, heating the surface under the influence of a favourable pressure gradient can serve to dampen and eventually prevent otherwise dominant unstable first-mode disturbances, even for sufficiently oblique disturbances. The maximum growth rate of second-mode disturbances are also reduced by the modifications to the boundary layer. Further analysis of the viscous stability is required to estimate the effect of these changes on the overall transition process.

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