CERTAIN HYDRAULIC ASPECTS OF EFFLUENT DISCHARGE INTO TIDAL WATERS

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Abstract

The paper discusses the problem of outflow into tidal water from a tank which is, at the same time, receiving an inflow. The outflow may be permitted at any time within the period of the tide or it may be restricted to a part of the period only; both cases are considered. Although the problem gives rise to a first order differential equation which, in general, cannot be solved in finite terms, it is shown how the variation in the tank level can be examined and depicted.

If the inflow and tank area are both constant and the tide is periodic, with a constant amplitude, the curve of levels in the tank will also become periodic with constant upper and lower limits. These periodic results are important and, from experiment, a field of significant results are presented which enables the required size of tank and outfall to be determined for a given tide and inflow. It is shown how, by transformation equations, results from this field may be used for any tide of similar shape to the experimental one.

When the inflow is variable no convenient solution capable of general application can be offered. In practical applications of the problem there will be diurnal variations in tide amplitude from springs to neaps, as well as a variation of inflow. This more usual and general problem is considered and some solutions obtained by computer, are presented.

A brief reference is made to algebraic methods of solving the differential equation, when the inflow varies, or is constant, and when the tidal curve is approximated to be a straight line or parabola. A set of curves, which are of practical use, are given

for the case of constant inflow when discharge is restricted to a portion of the tide which is falling at a constant rate.

1. Provided the conditions are right, a convenient and economical means of effluent disposal for places near the coast is through an outfall into tidal waters, for purification to take place by dilution and biological change. To obviate difficulties and to preserve the amenities of the coast, certain prescribed conditions have to be complied with. This usually means finding a suitable location for the end of the outfall so that there is no possibility of the sea currents conveying the unpurified effluent back inshore. Under favourable conditions and with a satisfactory location of the outfall, discharge may be permissible at all stages of the tide; this may not always be possible or economic, however, in which case discharge may be limited to a part only of the full tidal period. This means that storage of the effluent is necessary when discharge is prohibited. Storage may also be required, unless pumping is resorted to, when the system conveying the effluent to the outfall is below high tide. It is the purpose of this paper to examine the problem of storage and discharge and, in certain cases, present solutions from which the storage tank and outfall sizes may be obtained.

Throughout the paper it is assumed that the area of the tank, A, is constant and discharge from the outfall is below low water of all tides.

2. Assume a cylindrical tank area A, to receive an effluent flow at a constant rate Q and, at the same time t, there is a discharge rate q through an outfall from the tank into tidal water, with a head h between the tank and tide levels, fig. 1. In an interval of time (t, t+dt) a volume Qdt flows into the tank and qdt flows out. If z is the tank level at time t measured from a datum taken at mean tide level (M. T. L.), then, since the inflow volume equals the increase in tank storage plus the outflow volume, we get Qdt=Adz+qdt. Adopting the usual type of hydraulic formula relating flow and head and putting $q=kh^{\frac{1}{2}}$, where k is a constant depending upon the geometry and friction properties of the pipe, we have

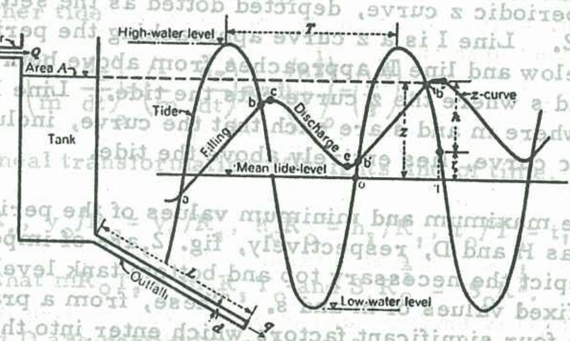
$$Q = A\frac{dz}{dt} + kh^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad 1.$$

Putting m = Q/A, s = Q/k, equation 1 becomes

$$\frac{1}{m}\frac{dz}{dt} + \frac{h^{\frac{1}{2}}}{s} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 2.$$

If y represents the height of the tide measured above M. T. L., then h = z-y; since there can be no discharge when y>z and if there is no back flow from the tide into the tank*,

Fig. 1.

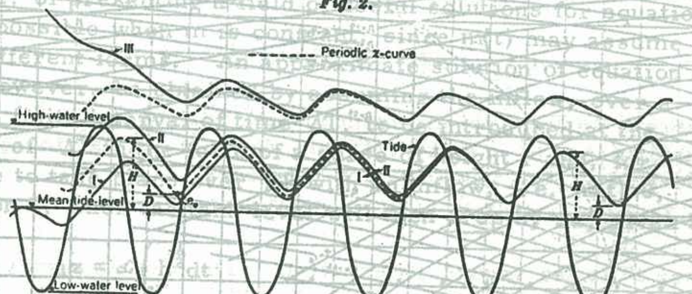


h = 0 when y>z. It is presumed that q will not influence y, in other words, that the discharge into the sea will not affect its level and y will be a function of time t only.

3. Certain features of equation 2 have been examined elsewhere(1), but it will be necessary to discuss them here very briefly. If h=0, dz/dt=m, this gives the filling part of the curve, fig. 1, and the z curve cuts the tide curve at slope m. When h>0, dz/dt=0 for $h=s^2$. From equation 2 dz/dt is univalued and known for any value of h, so the discharge portion of the z curve can be sketched as indicated in fig. 1. A point of contraflexure exists on the z curve, given by dz/dt=dy/dt, as will be seen by differentiating equation 1.

The z curve will not necessarily repeat itself on successive

Fig. 2.



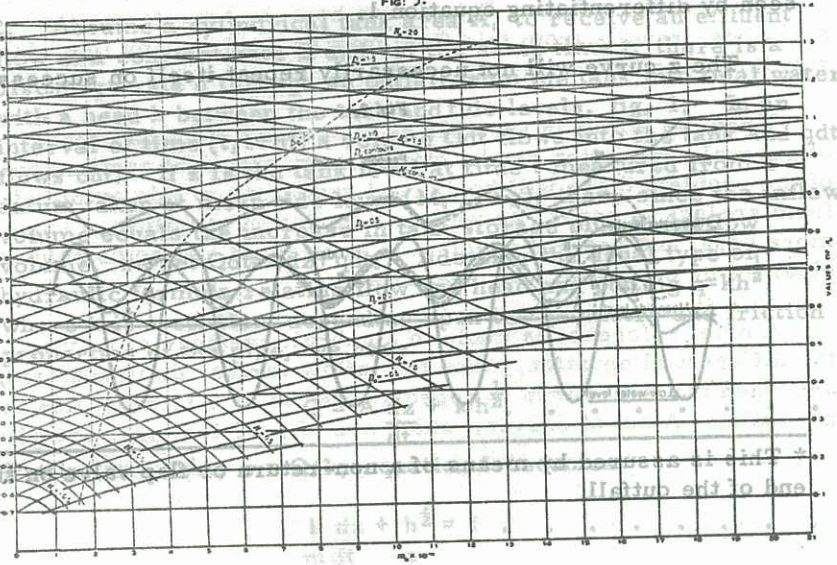
* This is assured by means of a non-return or flap valve on the end of the outfall.

tides, even when T, the tidal period and the half amplitude R, are constant, see fig. 1; it may be shown (1), however, that for m and s fixed equation 2 has one periodic solution which is approached asymptotically by every other solution as $t \rightarrow \infty$, to give a periodic z curve, depicted dotted as the settled z curve in fig. 2. Line I is a z curve approaching the periodic curve from below and line II approaches from above both for values of m and s where the z curve cuts the tide. Line III is a z curve where m and s are such that the curve, including the periodic curve, lies entirely above the tide.

The maximum and minimum values of the periodic z curve, shown as H and D, respectively, fig. 2, are of importance, since they depict the necessary top and bottom tank levels corresponding to fixed values of m and s. These, from a practical aspect, are the four significant factors which enter into the problem and we see that given m and s, H and D may be derived. It may be shown that if any two of the four factors (m, s, H and D) are fixed or known, the remaining two are also fixed.

4. A useful set of results to relate the four factors proved impracticable by theoretical methods and experiments on a model which simulated the conditions of the problem were employed(2). The results were depicted for a given tide with R_o= one foot and T_o= one hour and are shown in figure 3. To differentiate these results for any other tide the corresponding values of the four factors are written H_o, D_o, m_o and s_o. Although the results are shown for one particular tide they may

FIG. 3.



be transformed for use on any other tide.

The terms on the left hand side of equation 2 are dimensionless and, hence, if suffix $_o$ refers to one tide and suffix $_1$ to another tide

$$\left(\frac{1}{m}\frac{dz}{dt}\right)_o = \left(\frac{1}{m}\frac{dz}{dt}\right)_1 \text{ and } \left(\frac{h^{\frac{1}{2}}}{s}\right)_o = \left(\frac{h^{\frac{1}{2}}}{s}\right)_1 .$$

If we adopt a lineal transformation of heights and of time, so that

$$z_o/R_o = z_1/R_1,\ y_o/R_o = y_1/R_1,\ h_oR_o = h_1/R_1,\ t_o/T_o = t_1/T_1,$$

it then follows that $mR_oT_1 = m_oR_1T_o$ and $S_1R_o^{\frac{1}{2}} = S_oR_1^{\frac{1}{2}}$.

Since H and D are particular values of z, and also since $R_o = 1.0$, $T_o = 1.0$, we get $m = m_oR/T$, $s = s_oR^{\frac{1}{2}}$, $H = H_oR$ and $D = D_oR_1$, where R is in feet and T in hours.

Applications of the use of the field results in fig. 3, and of the above equations of transformation will be found in the paper (2). The results, of course, are transferable only if the tides have the same shape, changing only in R and T.

5. If Q is a variable and a function of t = m(t), (A is a constant), we may write for equation 1,

$$\frac{dz}{dt} + \alpha c h^{\frac{1}{2}} = m(t), \quad . \quad . \quad . \quad . \quad . \quad . \quad 3.$$

where $A\alpha = k$. There is no possibility, by experiment or otherwise, of presenting a field of useful solutions for equation 3, as is possible when m is constant, since m(t) may assume widely different forms. An approximate solution of equation 3 may, however, be obtained by assuming the inflow, over a small but finite interval of time ΔT, to be introduced at the beginning of ΔT in the form of a surge of height r and, for discharge to take place over ΔT with no inflow, i.e. m(t)=0, in equation 3, when the tank level will then drop by Δz, given by

$$\Delta z = \alpha \int_{\Delta T} h^{\frac{1}{2}} dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 4.$$

the height of the surge will be given by

$$r = \int_{\Delta T} m(t)dt = \frac{1}{A}\int_{\Delta T} Q dt$$

and, therefore, A must be known. A solution of equation 4 is required relating z and Δz for any interval ΔT.

The z curve, with its sudden surges r will then take the saw tooth shape shown in fig. 4. The accuracy of the results will depend upon the number of ΔT intervals taken in the period T.

Fig. 4.

If m(t) is periodic and in phase with the tide the z curve will ultimately settle down to become periodic, as for m constant. If, however, m(t) is not in phase, the z curve will reproduce itself only over very many tides.

The method of dealing with Q variable and of producing useful results depends upon knowing the relationship of Δz to z for each interval ΔT in the period T for different values of α. This relationship has been derived in a form which can be of general use for any tide having the shape of a sine curve (3).

6. It is possible to approximate to the tide curve and also to the inflow curve by algebraic expressions relating y and m to t in the form m(t) and f(t), respectively; so if, in equation (1) we put

$$\frac{dz}{dt} = m(t) - \alpha h^{\frac{1}{2}}$$

and $z = h + f(t)$, hence $\frac{dz}{dt} = \frac{dh}{dt} + f'(t)$,

we get $\frac{dh}{dt} = m(t) - f'(t) - \alpha h^{\frac{1}{2}}$,

where f' is the first differential with respect to t. Substituting $h^{\frac{1}{2}} = (m(t)-f'(t))u$, where u takes the sign of $m(t)-f'(t)$, the equation for u becomes

$$2u\frac{du}{dt}\big(m(t) - f'(t)\big) + \big(m'(t)-f''(t)\big)u = 1-u,$$

where f'' is the second differential. This equation may be integrated in finite terms if $m'(t) - f''(t)$ is a constant = a. Therefore, m(t) must be lineal or constant and f(t) parabolic, lineal or constant, since a can be zero.

If, then $m(t) - f'(t) = at + b$, with b, a constant, the equation for u(t) becomes,

$$\frac{dt}{at + b} = \frac{-2u\,du}{2au^2 + u - 1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad 5.$$

this can be integrated by standard methods to give u and t in terms of known functions.

It may be shown (2) that if the tide is represented by a sine curve $y = R \sin 2\pi t/T$, within the range (0, T/4) a close approximation to this curve is a straight line for $0 \leq 12t \leq T$ and an arc of a parabola for $T \leq 12t \leq 3T$. Outside this range the form is reproduced to give the symmetry of the sine curve.

The foregoing algebraic method may thus be used to obtain a z(t) curve within the limits of the approximations mentioned, but it will be seen that, having obtained u(t) and, on substitution h(t), hence z(t), the required relationships will be implicit, with exponential terms generally in a form not well suited to the derivation of z(t) curves, except to check results obtained in other ways. There is, however, one case where useful results may be produced by this algebraic method, as will be shown in the next section.

7. When discharge from a tank is restricted to a part of T, this will be allowed, as a rule on the outgoing tide only and also when the tidal current moves quickly, so that the effluent may be carried well offshore; discharge into "slack" water, at high and low tide, is avoided as far as possible. A part of the tide fulfilling these requirements, assuming a sine curve*, will then be within the limits $5T \leq 12t \leq 7T$ (from the discussion of the previous section). We may then write

$$T = Tf(t) = 3R(T - 2t) \text{ and } Tf'(t) = -6R.$$

If Q is a constant, (A is assumed constant) m is constant, a is then zero and equation 5 becomes

$$\frac{dt}{c} = \frac{2u\,du}{1-u}, \quad . \quad . \quad . \quad . \quad 6.$$

with $cT = mT + 6R$.

* Even when not conforming to a sine curve, and for the given limits of t, most tides fall approximately at a uniform and at the quickest rate, thus giving the most desirable period and conditions for discharge.

Discharge will extend over T/6 in the interval $5T \leq 12t \leq 7T$, as shown in fig. 5.

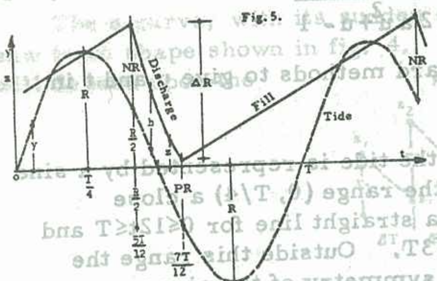


Fig. 5.

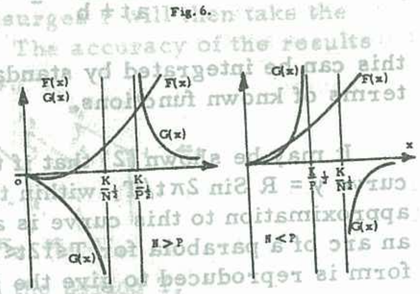


Fig. 6.

Let the top and bottom levels in the tank be NR and PR, respectively, both measured above the tide. Since the duration of discharge is T/6, filling at a uniform rate will then extend over 5 T/6. If the cycle of filling and emptying is repeated on each tide, we see that

$$5QT = 6A\Delta R \text{ and } N + 1 = P + \Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad 7.$$

where R is the depth of the tank, as will be seen from fig. 5.

From the preceding information and integrating equation 6 between the limits $NR = c^2u^2$ and $PR = c^2u^2$ for the interval T/6, we get

$$\frac{x^2(1 + N - P)}{10} - x(N^{\frac{1}{2}} - P^{\frac{1}{2}}) = K \operatorname{Log}_e \frac{K - xN^{\frac{1}{2}}}{K - xP^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad 8.$$

where $(1 + N - P)K = 6 + N - P$, and

$$xQ = \alpha A R^{\frac{1}{2}} = kR^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad 9.$$

Writing equation 8 as F(x) = G(x), the curves of these two functions depend upon the relative values of N and P, as will be seen from figures 6.

If N > P, F(x) will have two roots, it may be shown that G(x) and F(x) meet at x = 0, but do not intersect anywhere else near the origin. There can be no real value of x in the region $KN^{\frac{1}{2}} \leq xN^{\frac{1}{2}}P^{\frac{1}{2}} \leq KN^{\frac{1}{2}}$ and the solution of (8), other than x = 0, will be

$$xP^{\frac{1}{2}} > K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 10.$$

Similarly, for N < P, it can be shown that the solution will be given by

$$xP^{\frac{1}{2}} < K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 11.$$

If N = P, dz/dt will have the same value at the beginning and end of the interval T/6, hence for z to be continuous, $d^2z/dt^2 = 0$ at points within T/6. For m constant and y lineal it will be found on differentiating equation 1, that d^2z/dt^2 is zero when $dh/dt = 0$, that is when h a constant and the discharge will then be constant given by $q = k(RN)^{\frac{1}{2}}$. Since the whole inflow TQ must be discharged in the time T/6, we then get $6TQ = qT$, or $6Q = k(RN)^{\frac{1}{2}}$; with $k = \alpha A$ and using equation 9,

$$xN^{\frac{1}{2}} = xP^{\frac{1}{2}} = 6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 12.$$

Starting with the information of equations 10 and 11, solutions of equation 8, relating H, P and x, were obtained by computer, using Newton's method of successive approximations. A useful check on the results is directly provided by equation 12. From equation 9 it will be seen that $xs = R^{\frac{1}{2}}$ and if results are required for R = 1.0, we have $xs_o = 1$. A series of values relating H_o, D_o and s_o, were obtained from equation 8 and shown as H_o contours with D_o and s_o as axes, fig. 7. It will be seen that $2H_o - 2N = 1$ and $2P - 2D_o = 1$. The factors used in fig. 7 are now consistent with those of fig. 3.

8. Some examples are given to illustrate the method of obtaining solutions with the aid of fig. 7. All units are in feet and seconds; results are given to slide rule accuracy

(i) Suppose Q = 3.0, R = 6.0, T = 12 hrs. 24 mins., H = 6.60, D = 0.0, L = length of outfall = 900; find the diameter d of the outfall and A, the area of the tank, using the D'arcy equation for pipe flow $fLq^2 = \pi^2 d^5 h$, assuming f = 0.01, we have $H_o = 1.10$, $D_o = 0.00$ and from fig. 7, $s_o = 0.122$, (x = 8.2).
From equation 9, $24.6 = k6^{\frac{1}{2}}$
we see, in the D'arcy equation that $fLk^2 = \pi^2 d^5$; on substitution it will be found that

$$d = 2.47$$

The depth of the tank ΔR is 6.60 and from equation 7, A = 16900.

(ii) If it is decided to use a pipe with d = 2.75, keeping all other factors, except D and A, the same as in (i), what will the area and depth of the tank be?

For d = 2.75, k = 13.10, x = 10.75, s_o = 0.0933, from fig. 7, D_o = -.378.
From equation 7, the depth of the tank is 8.868 and the area is 12,900.

(iii) It is useful to examine the changes in the required value of k (and hence of d, since for a given outfall, d is proportional to k) as the tide changes in amplitude. Suppose the values of R change from 6 to 3, in one foot intervals.
If P = 0.4 and N = 0.8 when R = 6.0, the P, N, H_o, D_o values corresponding to R, together with the corresponding values of x are tabulated below; x is obtained from $s_o x = 1$, and then, since k is proportional to x, we have the relative value of k, assuming k = 1.0 for R = 6.0, remembering that $Qx = kR^{\frac{1}{2}}$.

R	N	H_o	P	D_o	100 s_o	k
6	0.80	1.3	0.40	-0.10	12.25	1.0
5	1.06	1.56	0.38	-0.12	13.10	0.98
4	1.45	1.95	0.35	-0.15	14.27	0.95
3	2.10	2.60	0.30	-0.20	15.30	0.88

It follows from these results for k that a pipe designed for the 6 ft. tide will be satisfactory on any of the other tides.

9. Still assuming A constant, the problem in its broadest aspect will embrace m a variable, R varying periodically from spring to neap tide, and y a function of t = f(t). It has been shown how these factors may be dealt with and, in given circumstances, how solutions may be obtained, sometimes directly from curves, otherwise by calculation aided by curves. Taking all these variables together it is possible to derive results by computer even when, as in most practical cases, they may not be expressible as known functions of t.

For example, let m(t) be represented by the graph shown in fig. 8*. It is presumed this curve will reproduce itself every day. The diurnal variation in the unit tide is

* This represents the variation in sewage flow for a large town over 24 hours, if the mean is unity, the maximum is 1.66 and minimum 0.24.

R = 1 + 0.2 Sin ($\pi t/13T$), so that R varies from 1.2 to 0.8 in 13/2 tides. Each tide is sinusoidal and y = R Sin 2$\pi t/T$, where T is now taken as 12.5 hours, hence the tides will be out of step with the periodicity of m(t) by one hour each day.
The equation for z now becomes

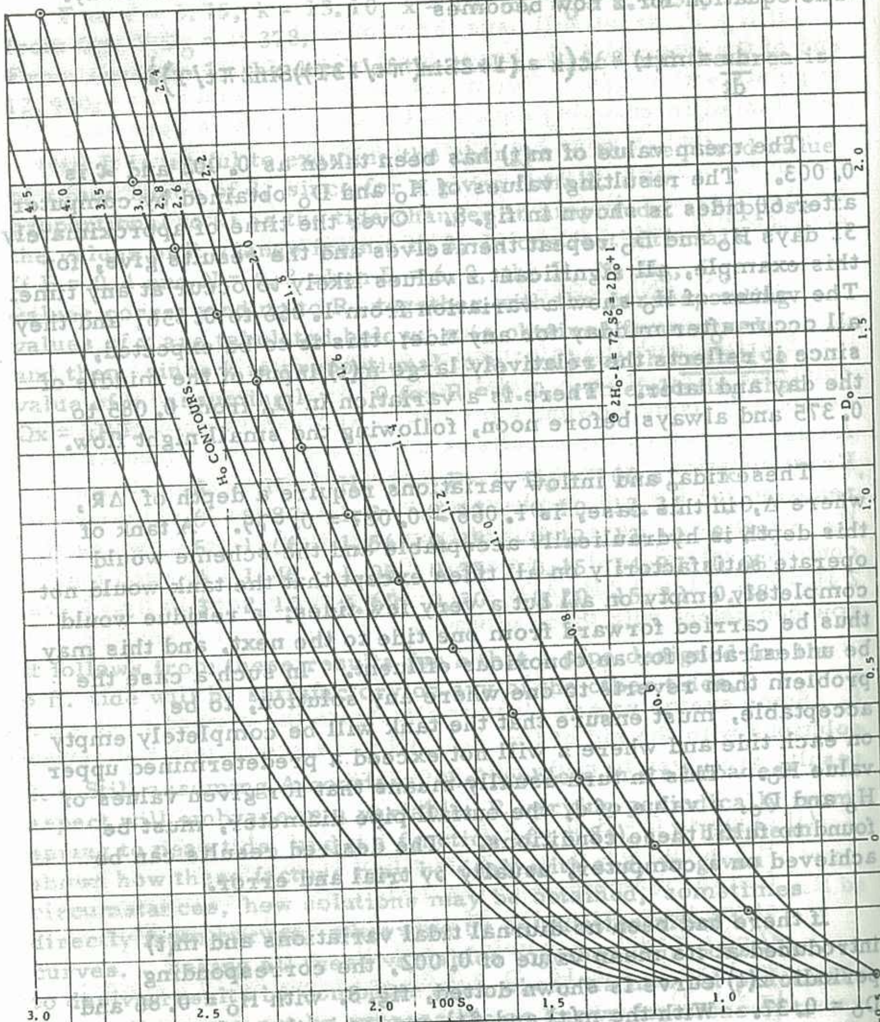
$$\frac{dz}{dt} = m(t) - \alpha\left(z - (1+2\mathrm{Sin}(\pi t/13T))\mathrm{Sin}2\pi t/T\right)^{\frac{1}{2}}$$

The mean value of m(t) has been taken as 0.002 and α is 0.003. The resulting values of H_o and D_o obtained by computer after 60 tides is shown in fig. 8. Over the time of approximately 31 days H_o and D_o repeat themselves and the results give, for this example, all significant z values likely to occur at any time. The values of H_o show a variation from 1.066 to 0.856, and they all occur after midday for any tide; this is to be expected, since it reflects the relatively large m(t) input of the middle of the day and later. There is a variation in D_o from 0.085 to 0.375 and always before noon, following the small night flow.

These tidal and inflow variations require a depth of ΔR, where Δ, in this case, is 1.066 - 0.087 = 0.979. A tank of this depth is hydraulically acceptable and the scheme would operate satisfactorily on all tides except that the tank would not completely empty on all but a very few tides; a residue would thus be carried forward from one tide to the next, and this may be undesirable for an obnoxious effluent. In such a case the problem then reverts to one where any solution, to be acceptable, must ensure that the tank will be completely empty on each tide and where z will not exceed a predetermined upper value H_o. This in turn usually means that for given values of H_o and D_o, a value of d, the outfall pipe diameter, must be found to fulfil these conditions. The desired results can be achieved on a computer, usually by trial and error.

If there had been no diurnal tidal variations and m(t) introduced at its mean value of 0.002, the corresponding periodic z(t) curve is shown dotted, fig. 8. with H_o = 0.86 and D_o = 0.37. With the m(t) and f(t) curves in the relative time positions of fig. 8, and starting with z = 0.486 the resultant z(t) curve is shown chain dotted. These two z curves are given to show the marked effect of the variations in m(t); the effect of a change in R over one tide is small.

Fig. 7



10. Grateful acknowledgments are given to the Council of the Institution of Civil Engineers (London) for permission to reproduce figures 1, 2 and 3 and to Mr. G. Webb, Officer-in-Charge of the University of Natal Computer Laboratory, for his help and advice and for programming some of the problems of the paper. Also thanks are expressed to the University of Natal for the use of their I. B. M. 1620 computer.

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Fig. 8.