Thermal Instability of a Horizontal Fluid Layer Subject to Absorption of Incoming Radiation and Surface Heat Exchange

T. Hattori, J. C. Patterson and C. Lei

School of Civil Engineering
The University of Sydney, Sydney, New South Wales 2006, Australia

Abstract

The paper reports on the investigation of the thermoconvective instability in littoral waters subjected to the daytime heating due to the absorption of solar radiation. A quasi-static approximation is adopted to form an ordinary differential eigenvalue problem for the growth rate of disturbance. The direct absorption of the incoming radiation by a water body governed by Beer’s law forms a vertical temperature stratification exponentially decaying from the water surface. The residual radiation reaching the bottom bathymetry is absorbed and re-emitted as a boundary flux, forming a potentially unstable thermal stratification adjacent to the bottom. The heat exchange between the water surface and the ambient causes surface cooling, which also creates a potentially unstable thermal stratification below the water surface. Consequently, the stably stratified layer in the central part is sandwiched by the two unstably stratified layers adjacent to the bounding surfaces. It is found that the surface and bottom instabilities behave independently and the wave number is larger for the surface instability than that for the bottom instability. Further, increasing the water depth and the surface heat exchange coefficient promotes the transition from the bottom instability dominance to the surface instability dominance.

Introduction

In littoral regions of lakes and reservoirs, buoyancy-driven flows in response to the daytime thermal forcing due to solar radiation have important implications for the water quality and nutrient transport [1, 10]. The direct absorption of the incoming radiation by a water body governed by Beer’s law forms a vertical temperature stratification exponentially decaying from the water surface, whilst the residual radiation reaching the bottom bathymetry is absorbed and re-emitted as a boundary flux [5], forming a potentially unstable thermal stratification adjacent to the bottom surface. The associated thermoconvective instability induces vertical mixing in the form of rising plumes. The spatial variation of the water depth across the shore in littoral regions is typically small and it is reasonable to approximate the flow domain by a horizontal fluid layer. Whilst most of the closely related studies (e.g. [5, 8, 9]) discount the spectral dependence of the attenuation coefficient of water and assume a single bulk value, it has been shown [6] that this simple assumption does not correctly model the absorption of the long wave radiation and consequently leads to qualitatively different results. Further, none of the previous related studies account for the heat exchange between the water surface and the ambient, and the adiabatic surface condition is commonly adopted instead. The surface heat exchange causes surface cooling, which also creates a potentially unstable thermal stratification below the water surface. Consequently, the stably stratified layer in the central part is sandwiched by the two unstably stratified layers adjacent to the bounding surfaces. In this study, the thermoconvective instability of a horizontal fluid layer subjected to the absorption of incoming radiation and the surface heat exchange is investigated via a quasi-static linear stability analysis.

Problem Formulation

The flow under consideration is governed by the incompressible Navier-Stokes equations with the Boussinesq approximation and a volumetric heating source term in the energy equation, representing the direct absorption of the incoming radiation. The quantities in the governing equations are normalised by the following scales:

\[
\begin{align*}
  x_i, h & \sim \eta_0^{-1} \quad \text{(length scale)}, \\
  t & \sim (\kappa h_0^2)^{-1} \quad \text{(time scale)}, \\
  T & \sim \frac{I_0}{\rho_0 C_p \kappa h_0} \quad \text{(temperature scale)}, \\
  p & \sim \frac{g \beta I_0}{C_p \kappa h_0} \quad \text{(pressure scale)}, \\
  u_i & \sim \kappa h_0 \quad \text{(velocity scale)},
\end{align*}
\]

where \( u_i = (u, v, w) \) is the velocity component in the \( x_i (= x, y, z) \) direction, respectively. \( \eta_0 \) is the characteristic attenuation coefficient and the present study uses \( \eta_0 = 10 \text{m}^{-1} \), which is a typical bulk value for the PAR (photosynthetic active range) of the spectrum for lakes [7]. It is noted that \( \eta_0 \) is only used for the normalisation, hence the choice of \( \eta_0 \) does not affect any subsequent results and discussions. \( I_0 \) is the total surface intensity and is divided into three wavebands, based on the three-waveband model [6]. The Beer’s law is therefore written as:

\[
I(y) = \frac{I_0}{\eta_0^2} \exp \left( \frac{\eta_0}{\eta_0^2} y \right),
\]

where \( \eta_1/\eta_0, \eta_2/\eta_0, \eta_3/\eta_0 = (103.0, 4.1, 0.1) \) and \( (l_1/l_0, l_2/l_0, l_3/l_0) = (0.202, 0.126, 0.66) \), which covers 99 percent of the whole spectrum of solar radiation (with the colour temperature of 5800K) [6]. \( g \) is the gravitational acceleration, \( \beta \) is the thermal expansion coefficient, \( \rho_0 \) is the density, \( C_p \) is the specific heat and \( \kappa \) is the thermal diffusivity. The thermal properties of water are assumed to be constant. The governing equations are therefore given in non-dimensional form as follows:

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -Ra Pr \frac{\partial p}{\partial x_i} + Pr \nabla^2 u_i + Ra Pr \delta_{i2} T, \quad (3)
\]

\[
\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \nabla^2 T + \sum_{n=1}^{3} \frac{\eta_n}{\eta_0} \frac{I_n}{I_0} \exp \left( \frac{\eta_n}{\eta_0^2} \right), \quad (4)
\]

\[
\frac{\partial u_i}{\partial x_i} = 0,
\]

where \( Ra \) is the Rayleigh number \( (= g \beta I_0 / (\rho C_p \kappa h_0^5)) \) and \( Pr \) is the Prandtl number \( (= \nu / \kappa) \) fixed at \( Pr = 7 \) for water, where \( \nu \) is the kinematic viscosity. \( \delta_{i2} \) is the Kronecker delta.

The present study assumes the initial condition of the flow to be stationary and iso-thermal. The radiation is instantaneously...
applied at \( t = 0 \), and maintained thereafter. The fluid layer is bounded by a stress-free top \((y = 0)\) and a rigid bottom \((y = -h)\).

In order to account for the heat exchange between the water surface and the ambient, a heat flux condition as a function of the water surface temperature is applied. Adopting a similar formulation to that used in [4], the top boundary condition is written as:

\[
\frac{\partial T}{\partial y} = -K(T - T_e).
\]

where \( K \) is a non-dimensional heat exchange coefficient and is typically in the range \( 0 < K < 33 \) from field measurements [2, 4], and \( T_e \) is the equilibrium temperature [4]. In the field situations, both \( K \) and \( T_e \) continuously change in response to varying meteorological conditions, however, for our present purposes \( K \) and \( T_e \) are assumed to be constant with time. \( K \) is treated as a controlling parameter and is varied in the range \( 0 < K < 50 \) to cover the above range, and \( T_e \) is taken to be equal to the initial temperature. It is noted that the \( K = 0 \) case is identical to the previous study [6] with the adiabatic surface condition. At the bottom boundary, the re-emission of the residual radiation imposes a fixed heat flux condition given as:

\[
\frac{\partial T}{\partial y} = -\sum_{n=1}^{3} \frac{I_n}{l_0} \exp \left( -\frac{\eta n}{\eta_0} h \right).
\]

The present study considers \( Ra \) in the range \( 10^7 \leq Ra \leq 10^9 \) and \( h \) in the range \( 5 \leq h \leq 50 \), which are physically relevant to the field situations.

### Linear Stability Analysis

#### Base Flow

The base flow has zero velocity components and is solely governed by a one-dimensional inhomogeneous heat equation directly formed from the energy equation (4):

\[
\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial y^2} + \sum_{n=1}^{3} \frac{I_n}{l_0} \exp \left( -\frac{\eta n}{\eta_0} y \right),
\]

which is subjected to the above boundary conditions. The analytical conduction solution is obtained using the method of eigenfunction expansion as:

\[
T(y,t) = \gamma_i (y+h)^2 + \sum_{n=1}^{3} \frac{I_n}{l_0} \left( \gamma_n (y+h)^2 - \exp \left( -\frac{\eta n}{\eta_0} h \right) (y+h) \right)
\]

where \( \gamma_i \) is a non-dimensional heat exchange coefficient and \( \gamma_n \) is typically in the range \( 2 < \gamma_n < 3 \).

The base flow has zero velocity components and is solely governed by the energy equation (4):

\[
\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial y^2} + \sum_{n=1}^{3} \frac{I_n}{l_0} \exp \left( -\frac{\eta n}{\eta_0} y \right),
\]

which is subjected to the above boundary conditions. The analytical conduction solution is obtained using the method of
Introducing a stream function $\psi'$ with $u' = -\partial \psi'/\partial y$ and $v' = \partial \psi'/\partial x$, equation (11) reduces to:

$$\begin{align*}
\frac{d}{d\tau} \left( \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} \right) \psi' &= Pr \left( \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} \right)^2 \psi' + RaPr \frac{\partial T'}{\partial x},
\frac{\partial T'}{\partial \tau} + \frac{\partial \psi'}{\partial x} \frac{\partial T'}{\partial y} &= \left( \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} \right) T'.
\end{align*}$$

The base temperature is assumed to be quasi-static, i.e. the evolution of the base temperature does not affect its instantaneous stability. This means that the evolution time for the perturbation $\tau$ is shorter than the evolution time for the base temperature $t$, and $t$ is treated as a controlling parameter. The accuracy of the analysis therefore improves as the growth rate of the disturbance increases [3]. For the considered parameter ranges, $\sigma > 0$ as shown later, hence it is considered that the approximation does not adversely affect the accuracy of the analysis. A direct stability analysis based on direct numerical simulation of the flow is currently underway for further justification for the present analysis. Based on the quasi-static approximation, $\psi'$ and $T'$ are expanded in the following form:

$$\begin{align*}
\psi' &= ik\tilde{\psi}\exp(ikx + \sigma \tau),
T' &= \tilde{T}\exp(ikx + \sigma \tau),
\end{align*}$$

where $k$ is the wave number in the $x$ direction; $\sigma$ is the growth rate; and $\tilde{\psi}$ and $\tilde{T}$ are eigenfunctions. Substituting equation (13) into (12) yields:

$$\begin{align*}
\frac{d^2 \tilde{\psi}}{d\tau^2} - 2k^2 \frac{d^2 \tilde{\psi}}{d\tau^2} + k^4 \tilde{\psi} + Ra\tilde{T} &= \frac{\sigma}{Pr} \left( \frac{d^2 \tilde{\psi}}{d\tau^2} - k^2 \tilde{\psi} \right),
\frac{d^2 \tilde{T}}{d\tau^2} - k^2 \tilde{T} + k^2 \frac{d \tilde{T}}{d\tau} &= \sigma \tilde{T}.
\end{align*}$$

The associated boundary conditions are written as:

$$\begin{align*}
\tilde{\psi} &= \frac{d^2 \tilde{\psi}}{d\tau^2} = \frac{d \tilde{T}}{d\tau} + K\tilde{T} = 0 \quad (y = 0),
\tilde{\psi} &= \frac{d \tilde{\psi}}{d\tau} = \frac{d \tilde{T}}{d\tau} = 0 \quad (y = -h).
\end{align*}$$

Equations (14) and (15) form an ordinary differential eigenvalue problem. For given values of $Ra$, $Pr$ (recall that $Pr$ is fixed at $Pr = 7$), $t$, $h$ and $K$, $\sigma$ is obtained as a function of $k$. The eigenvalue problem is solved using the finite difference method. A uniformly distributed grid with the grid size of 0.01 is adopted. Further increasing the grid resolution has a negligible effect on the solution accuracy for the range of parameter values considered.

The Fastest Growing Mode of Instability

In figure 2, $\sigma$ is plotted as a function of $k$ for $Ra = 10^8$ and $h = 30$ with $K = 50$ at $t = 0.04, 0.06, 0.08, 0.1, 0.2, 0.4$ and $0.6$. $\times$ indicates the fastest growing mode for a given $t$.

![Figure 2: The growth rate $\sigma$ over the wave number $k$ for $Ra = 10^8$ and $h = 30$ with $K = 50$ at $t = 0.04, 0.06, 0.08, 0.1, 0.2, 0.4$ and $0.6$. $\times$ indicates the fastest growing mode for a given $t$.](image)

The associated boundary conditions are written as:

$$\begin{align*}
\tilde{\psi} &= \frac{d^2 \tilde{\psi}}{d\tau^2} = \frac{d \tilde{T}}{d\tau} + K\tilde{T} = 0 \quad (y = 0),
\tilde{\psi} &= \frac{d \tilde{\psi}}{d\tau} = \frac{d \tilde{T}}{d\tau} = 0 \quad (y = -h).
\end{align*}$$

Equations (14) and (15) form an ordinary differential eigenvalue problem. For given values of $Ra$, $Pr$ (recall that $Pr$ is fixed at $Pr = 7$), $t$, $h$ and $K$, $\sigma$ is obtained as a function of $k$. The eigenvalue problem is solved using the finite difference method. A uniformly distributed grid with the grid size of 0.01 is adopted. Further increasing the grid resolution has a negligible effect on the solution accuracy for the range of parameter values considered.

The Fastest Growing Mode of Instability

In figure 2, $\sigma$ is plotted as a function of $k$ for $Ra = 10^8$ and $h = 30$ with $K = 50$ at $t = 0.04, 0.06, 0.08, 0.1, 0.2, 0.4$ and $0.6$. $\times$ indicates the fastest growing mode for a given $t$.

![Figure 3: (a) The fastest growing mode $k_c$ and (b) the corresponding growth rate $\sigma_c$ plotted over $t$ for different $K$ for $Ra = 10^8$ and $h = 30$ (black: $K = 0$ and $K = 1$, red: $K = 5$, green: $K = 10$, blue: $K = 30$ and magenta: $K = 50$).](image)

The associated boundary conditions are written as:

$$\begin{align*}
\tilde{\psi} &= \frac{d^2 \tilde{\psi}}{d\tau^2} = \frac{d \tilde{T}}{d\tau} + K\tilde{T} = 0 \quad (y = 0),
\tilde{\psi} &= \frac{d \tilde{\psi}}{d\tau} = \frac{d \tilde{T}}{d\tau} = 0 \quad (y = -h).
\end{align*}$$

Equations (14) and (15) form an ordinary differential eigenvalue problem. For given values of $Ra$, $Pr$ (recall that $Pr$ is fixed at $Pr = 7$), $t$, $h$ and $K$, $\sigma$ is obtained as a function of $k$. The eigenvalue problem is solved using the finite difference method. A uniformly distributed grid with the grid size of 0.01 is adopted. Further increasing the grid resolution has a negligible effect on the solution accuracy for the range of parameter values considered.
In this study, the thermoconvective instability of a horizontal fluid layer subjected to the absorption of incoming radiation and the surface heat exchange is investigated via a quasi-static linear stability analysis. The surface and bottom instabilities are found to behave independently and the wave number is larger for the surface instability than that for the bottom instability. It is shown that increasing the water depth and the surface heat exchange coefficient promotes the transition from the bottom instability dominance to the surface instability dominance. It has been confirmed that the revealed dependencies of the stability characteristics on the controlling parameters are consistent over the considered parameter ranges.

Acknowledgements

This research was financially supported by the Australian Research Council (DP120104849) and The University of Sydney.

References