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Approximate Solution of the Navier–Stokes Equations

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Abstract

This paper analyses the Navier–Stokes equations in three dimensions for an unsteady incompressible viscous fluid in the presence of a body force using, as far as the author is aware, a novel application of homotopy analysis. An explicit approximate solution incorporating arbitrary initial conditions is developed and the relationship between this approximation and the corresponding exact solution is discussed.

An analysis of the existence, uniqueness, smoothness and convergence of both the explicit approximation solution and the corresponding exact solution are also presented. In particular, some conclusions regarding the formation of singularities within finite time periods for solutions to the Navier–Stokes equations (and their non–viscous counterparts) in the three dimensional case are noted.

Finally, the potential utility of the solution strategy employed in this paper in the context of direct numerical simulation of fluid flows is considered briefly.

Introduction

This paper[1] outlines an analysis of the Navier–Stokes equations for an incompressible, unsteady fluid in the presence of a body force with arbitrary initial conditions through an application of homotopy analysis.[2] Other authors have applied the homotopy analysis method to analyse fluid flows (typically in a steady flow regime or in the case of unsteady flows in two dimensions) but not, as far as the author is aware, in the instance of a three dimensional unsteady incompressible flow.

For example, Ragab, et. al., [4] develop approximate solutions to the Navier–Stokes equation in cylindrical coordinates for an unsteady one dimensional motion of a viscous fluid with a fractional time derivative using homotopy analysis. Alizadeh–Pahlavan and Borjian–Boroujeni [1] produce an analytical solution to the problem of a steady laminar boundary layer flow past a flat plate in a viscous fluid. The analysis is based upon the transformation of the Navier–Stokes equations in two dimensions to a non-linear ODE using a similarity transformation. Homotopy analysis is then used to solve this ODE. Xu, et. al., [6] focus on a homotopy analysis of laminar, incompressible, and time-dependent flows in two dimensions of a viscous fluid in a porous channel with orthogonally moving walls for the cases where the motion of the walls is uniform and non–uniform.

The remainder of this paper outlines the method including the characterisation of the solution for each component of the velocity and the pressure within the unsteady fluid flow as a Taylor series along with the development of explicit expressions for the first two terms of the Taylor series for each velocity component and the pressure[3]. Secondly, this paper examines the issues of the existence, uniqueness, smoothness and convergence of the Taylor series solutions. In particular, some conclusions are presented regarding the formation of singularities within finite time periods for solutions to the Navier–Stokes equations (and their non–viscous counterparts) in three dimensions. Finally, this paper notes briefly the potential utility of the method in the context of direct numerical simulation of fluid flows.

Nomenclature

ρ	fluid density
ν	coefficient of kinematic viscosity
x	x space variable where $-\infty < x < \infty$
y	y space variable where $-\infty < y < \infty$
z	z space variable where $-\infty < z < \infty$
t	time variable where $0 \le t < \infty$
$u(x,y,z,t;q)$	
$v(x,y,z,t;q)$	velocity vectors for the fluid
$w(x,y,z,t;q)$	
$p(x,y,z,t;q)$	pressure within the fluid
$g(x,y,z,t)$	body forces acting on the fluid
$\delta(t)$	Dirac delta function
q	homotopy parameter
i	$(-1)^{½}$

Outline of the Mathematical Problem

Consider the following set of "generalised" PDE's and associated boundary and initial conditions for four unknown functions $u(x,y,z,t;q)$, $v(x,y,z,t;q)$, $w(x,y,z,t;q)$ and $p(x,y,z,t;q)$:

$$u_t + q[uu_x + vu_y + wu_z] = -p_x/\rho + \nu[u_{xx} + u_{yy} + u_{zz}] + g_x \quad (1)$$

$$v_t + q[uv_x + vv_y + wv_z] = -p_y/\rho + \nu[v_{xx} + v_{yy} + v_{zz}] + g_y \quad (2)$$

$$w_t + q[uw_x + vw_y + ww_z] = -p_z/\rho + \nu[w_{xx} + w_{yy} + w_{zz}] + g_z \quad (3)$$

$$u_x(x,y,z,t;q) + v_y(x,y,z,t;q) + w_z(x,y,z,t;q) = 0 \quad (4)$$

$u(x,y,z,t;q)$, $v(x,y,z,t;q)$, $w(x,y,z,t;q)$ and $p(x,y,z,t;q)$ bounded as x, y, z and t become large (5)

$$u(x,y,z,0;q) = u_0(x,y,z) \quad (6)$$

$$v(x,y,z,0;q) = v_0(x,y,z) \quad (7)$$

$$w(x,y,z,0;q) = w_0(x,y,z) \quad (8)$$

The solutions to the generalised PDE's subject to the associated boundary and initial conditions (i.e., equations (1) – (8)) are assumed to be both capable of representation as a Taylor series in q about the point $q = 0$ and convergent for $0 \le q \le 1$:

[1] This paper is a revised and condensed version of an earlier set of notes by the author (see "*Navier–Stokes Equations: Discussion Notes*", ISBN: 978-0-9586147-9-5, 29 November 2013).

[2] For a discussion of "homotopy analysis" including worked examples, see Liao [3]. This paper provides a long listing of applications of homotopy analysis to non-linear problems (including fluid flow problems) and sets out a comparison to both perturbation and non-perturbative methods.

[3] While there are many approximate solutions to the Navier–Stokes equations in three dimensions presented in the literature, the application of homotopy analysis presented here offers insight into the question of the existence and uniqueness of solutions to Navier–Stokes equations in three dimensions for an unsteady incompressible viscous fluid.

$$u(x,y,z,t;q) = \sum_{n=0}^{\infty} [d^n u(x,y,z,t;0)/dq^n]q^n/n!$$

$$v(x,y,z,t;q) = \sum_{n=0}^{\infty} [d^n v(x,y,z,t;0)/dq^n]q^n/n!$$

$$w(x,y,z,t;q) = \sum_{n=0}^{\infty} [d^n v(x,y,z,t;0)/dq^n]q^n/n!$$

$$p(x,y,z,t;q) = \sum_{n=0}^{\infty} [d^n p(x,y,z,t;0)/dq^n]q^n/n!$$

When $q = 1$, the above generalised PDE's and associated boundary and initial conditions (i.e., equations (1) – (8)) correspond to the Navier–Stokes equations for an unsteady incompressible fluid in the presence of a body force in three (unbounded) spatial dimensions and the Taylor series for $u(x,y,z,t;1)$, $v(x,y,z,t;1)$, $w(x,y,z,t;1)$ and $p(x,y,z,t;1)$ represent the corresponding solutions to these equations. Accordingly, the practical task is to develop expressions for the coefficients in each of the above Taylor series. This is done by successively differentiating equations (1) – (8) with respect to q, setting q equal to 0 and then solving the resultant "subsidiary" problems.

Expressions for the First Term in Each Taylor Series

The first step in the analysis is to develop explicit expressions for the first term in each Taylor series, namely: $u(x,y,z,t;0)$, $v(x,y,z,t;0)$, $w(x,y,z,t;0)$ and $p(x,y,z,t;0)$. When $q = 0$, equations (1) – (8) are as follows:

$$u_t = -p_x/\rho + \nu[u_{xx} + u_{yy} + u_{zz}] + g_x \qquad (9)$$

$$v_t = -p_y/\rho + \nu[v_{xx} + v_{yy} + v_{zz}] + g_y \qquad (10)$$

$$w_t = -p_z/\rho + \nu[w_{xx} + w_{yy} + w_{zz}] + g_z \qquad (11)$$

$$u_x(x,y,z,t;0) + v_y(x,y,z,t;0) + w_z(x,y,z,t;0) = 0 \qquad (12)$$

$u(x,y,z,t;0)$, $v(x,y,z,t;0)$, $w(x,y,z,t;0)$ and $p(x,y,z,t;0)$ bounded as x, y, z and t become large (13)

$$u(x,y,z,0;0) = u_0(x,y,z) \qquad (14)$$

$$v(x,y,z,0;0) = v_0(x,y,z) \qquad (15)$$

$$w(x,y,z,0;0) = w_0(x,y,z) \qquad (16)$$

Since the coupled terms that appeared in equations (1) – (8) above are no longer present in the above PDE's (i.e., equations (9) – (12)), the general solutions[4] for each of $u(x,y,z,t;0)$, $v(x,y,z,t;0)$ and $w(x,y,z,t;0)$ can be derived separately using Fourier transforms on the x, z, and y variables and a Laplace transform on the t variable. If the function $u(x,y,z,t;0)$ in the transform space is given as $U(\omega,\eta,\varepsilon,s)$, i.e., as follows:[5]

$$U(\omega,\eta,\varepsilon,s) =$$

$$(2\pi)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} u(x,y,z,t;0)e^{i(\omega x + \eta y + \varepsilon z)}e^{-st}\, dtdxdydz \qquad (17)$$

then $V(\omega,\eta,\varepsilon,s)$, $W(\omega,\eta,\varepsilon,s)$, $P(\omega,\eta,\varepsilon,s)$, $G(\omega,\eta,\varepsilon,s)$ and the transforms of the associated initial conditions are defined correspondingly. For example, the PDE for $u(x,y,z,t;0)$ plus the associated initial condition becomes in the transform space:

$$sU(\omega,\eta,\varepsilon,s) - U_0(\omega,\eta,\varepsilon) = (i\omega)P(\omega,\eta,\varepsilon,s)/\rho -$$
$$\nu[\omega^2 + \eta^2 + \varepsilon^2]U(\omega,\eta,\varepsilon,s) - (i\omega)G(\omega,\eta,\varepsilon,s) \qquad (18)$$

where $U(\omega,\eta,\varepsilon,s)$ is the velocity, $u(x,y,z,t;0)$, in the transform space; $P(\omega,\eta,\varepsilon,s)$ is the (unknown at this stage) pressure function in the transform space; $G(\omega,\eta,\varepsilon,s)$ is the body force function in the transform space; and $U_0(\omega,\eta,\varepsilon)$ is the initial condition in the transform space. Rearranging equation (18) yields:

$$U(\omega,\eta,\varepsilon,s) = [U_0(\omega,\eta,\varepsilon) + (i\omega)P(\omega,\eta,\varepsilon,s)/\rho -$$
$$(i\omega)G(\omega,\eta,\varepsilon,s)]/(s + \nu[\omega^2 + \eta^2 + \varepsilon^2]) \qquad (19)$$

Similar expressions for $V(\omega,\eta,\varepsilon,s)$ and $W(\omega,\eta,\varepsilon,s)$ can be derived as well:

$$V(\omega,\eta,\varepsilon,s) = [V_0(\omega,\eta,\varepsilon) + (i\eta)P(\omega,\eta,\varepsilon,s)/\rho -$$
$$(i\eta)G(\omega,\eta,\varepsilon,s)]/(s + \nu[\omega^2 + \eta^2 + \varepsilon^2]) \qquad (20)$$

$$W(\omega,\eta,\varepsilon,s) = [W_0(\omega,\eta,\varepsilon) + (i\varepsilon)P(\omega,\eta,\varepsilon,s)/\rho -$$
$$(i\varepsilon)G(\omega,\eta,\varepsilon,s)]/(s + \nu[\omega^2 + \eta^2 + \varepsilon^2]) \qquad (21)$$

The corresponding continuity equation (i.e., equation (12)), after applying the Fourier and Laplace transforms, can be used in conjunction with the above expressions for $U(\omega,\eta,\varepsilon,s)$, $V(\omega,\eta,\varepsilon,s)$and $W(\omega,\eta,\varepsilon,s)$ (i.e., equations (19) – (21)) to develop an expression for the pressure function in terms of the respective transform variables, ω, η, ε, and s:

$$P(\omega,\eta,\varepsilon,s) = \rho[G(\omega,\eta,\varepsilon,s) + [i\eta V_0(\omega,\eta,\varepsilon) + i\omega U_0(\omega,\eta,\varepsilon) +$$
$$i\varepsilon W_0(\omega,\eta,\varepsilon)]/[\omega^2 + \eta^2 + \varepsilon^2]] \qquad (22)$$

The function $p(x,y,z,t;0)$ (i.e., the inverse transform of equation (22)) is as follows:

$$p(x,y,z,t;0) = \rho g(x,y,z,t) +$$
$$\delta(t)\rho(2\pi)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [i\eta V_0(\omega,\eta,\varepsilon) + i\omega U_0(\omega,\eta,\varepsilon) +$$
$$i\varepsilon W_0(\omega,\eta,\varepsilon)]/[\omega^2 + \eta^2 + \varepsilon^2]]e^{-i[\omega x+\eta y+\varepsilon z]}d\omega d\eta d\varepsilon \qquad (23)$$

where $\delta(t)$ is the Dirac delta function. Equation (22) can be substituted into the expressions for $U(\omega,\eta,\varepsilon,s)$, $V(\omega,\eta,\varepsilon,s)$ and $W(\omega,\eta,\varepsilon,s)$ to yield the following:

$$U(\omega,\eta,\varepsilon,s) = [U_0(\omega,\eta,\varepsilon) - [\eta\omega V_0(\omega,\eta,\varepsilon) + \omega^2 U_0(\omega,\eta,\varepsilon) +$$
$$\varepsilon\omega W_0(\omega,\eta,\varepsilon)]/[\omega^2 + \eta^2 + \varepsilon^2]]/(s + \nu[\omega^2 + \eta^2 + \varepsilon^2]) \qquad (24)$$

$$V(\omega,\eta,\varepsilon,s) = [V_0(\omega,\eta,\varepsilon) - [\eta^2 V_0(\omega,\eta,\varepsilon) + \omega\eta U_0(\omega,\eta,\varepsilon) +$$
$$\varepsilon\eta W_0(\omega,\eta,\varepsilon)]/[\omega^2 + \eta^2 + \varepsilon^2]]/(s + \nu[\omega^2 + \eta^2 + \varepsilon^2]) \qquad (25)$$

$$W(\omega,\eta,\varepsilon,s) = [W_0(\omega,\eta,\varepsilon) - [\eta\varepsilon V_0(\omega,\eta,\varepsilon) + \omega\varepsilon U_0(\omega,\eta,\varepsilon) +$$
$$\varepsilon^2 W_0(\omega,\eta,\varepsilon)]/[\omega^2 + \eta^2 + \varepsilon^2]]/(s + \nu[\omega^2 + \eta^2 + \varepsilon^2]) \qquad (26)$$

The particular solutions for $u(x,y,z,t;0)$, $v(x,y,z,t;0)$ and $w(x,y,z,t;0)$ (i.e., the inverse transforms of equations (24) – (26)) are as follows:

[4] The reference to the "general" solution reflects the fact that the pressure function, $p(x,y,z,t;0)$, is yet to be determined at this stage.

[5] The definition of the Fourier transform and its inverse used in this paper corresponds to the convention used in [5]. The definition of the Laplace transform and its inverse follows the usual conventions. The convolution theorems used at various points in this paper are defined on a basis consistent with the corresponding transform to which they relate.

$$u(x,y,z,t;0) = (2\pi)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [U_0 - [\eta\omega V_0 + \omega^2 U_0 + \varepsilon\omega W_0]/[\omega^2 + \eta^2 + \varepsilon^2]]e^{-i(\omega x + \eta y + \varepsilon z) - \nu[\omega^2 + \eta^2 + \varepsilon^2]t} d\omega d\eta d\varepsilon \tag{27}$$

$$v(x,y,z,t;0) = (2\pi)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [V_0 - [\eta^2 V_0 + \omega\eta U_0 + \varepsilon\eta W_0]/[\omega^2 + \eta^2 + \varepsilon^2]]e^{-i(\omega x + \eta y + \varepsilon z) - \nu[\omega^2 + \eta^2 + \varepsilon^2]t} d\omega d\eta d\varepsilon \tag{28}$$

$$w(x,y,z,t;0) = (2\pi)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [W_0 - [\eta\varepsilon V_0 + \omega\varepsilon U_0 + \varepsilon^2 W_0]/[\omega^2 + \eta^2 + \varepsilon^2]]e^{-i(\omega x + \eta y + \varepsilon z) - \nu[\omega^2 + \eta^2 + \varepsilon^2]t} d\omega d\eta d\varepsilon \tag{29}$$

Expressions for the Second Term in Each Taylor Series

The development of explicit expressions for the second term in each Taylor series, namely: $u_q(x,y,z,t;0)$, $v_q(x,y,z,t;0)$, $w_q(x,y,z,t;0)$ and $p_q(x,y,z,t;0)$, begins by differentiating equations (1) – (8) once with respect to q and setting q equal to zero to yield the following:

$$u_{tq} + [uu_x + vu_y + wu_z] = -p_{xq}/\rho + \nu[u_{xxq} + u_{yyq} + u_{zzq}] \tag{30}$$

$$v_{tq} + [uv_x + vv_y + wv_z] = -p_{yq}/\rho + \nu[v_{xxq} + v_{yyq} + v_{zzq}] \tag{31}$$

$$w_{tq} + [uw_x + vw_y + ww_z] = -p_{zq}/\rho + \nu[w_{xxq} + w_{yyq} + w_{zzq}] \tag{32}$$

$$u_{xq}(x,y,z,t;0) + v_{yq}(x,y,z,t;0) + w_{zq}(x,y,z,t;0) = 0 \tag{33}$$

$u_q(x,y,z,t;0)$, $v_q(x,y,z,t;0)$, $w_q(x,y,z,t;0)$ and $p_q(x,y,z,t;0)$ bounded as x, y, z, and t become large (34)

$$u_q(x,y,z,0;0) = v_q(x,y,z,0;0) = w_q(x,y,z,0;0) = 0 \tag{35}$$

The terms in each of the above PDE's in square brackets on the left hand side are known (being based upon the results developed in the previous section) and play a role similar to the "body forces" function in the analysis of the first term in each Taylor series above. If the following notational simplification is made:

$$[uu_x + vu_y + wu_z] = -a_0(x,y,z,t) \tag{36}$$

$$[uv_x + vv_y + wv_z] = -b_0(x,y,z,t) \tag{37}$$

$$[uw_x + vw_y + ww_z] = -c_0(x,y,z,t) \tag{38}$$

the PDE's for $u_q(x,y,z,t;0)$, $v_q(x,y,z,t;0)$, and $w_q(x,y,z,t;0)$ can be restated as follows:

$$u_{tq} = -p_{xq}/\rho + \nu[u_{xxq} + u_{yyq} + u_{zzq}] + a_0(x,y,z,t) \tag{39}$$

$$v_{tq} = -p_{yq}/\rho + \nu[v_{xxq} + v_{yyq} + v_{zzq}] + b_0(x,y,z,t) \tag{40}$$

$$w_{tq} = -p_{zq}/\rho + \nu[w_{xxq} + w_{yyq} + w_{zzq}] + c_0(x,y,z,t) \tag{41}$$

As was the case for the coefficients in first term in each Taylor series for $u(x,y,z,t;1)$, $v(x,y,z,t;1)$, and $w(x,y,z,t;1)$, the coupled terms that appear in the original problem are no longer present in the above PDE's (i.e., equations (39) – (41)). This means that solutions for each of $u_q(x,y,z,t;0)$, $v_q(x,y,z,t;0)$, $w_q(x,y,z,t;0)$ and $p_q(x,y,z,t;0)$ can be derived separately by adopting the same approach used on the previous section. The expression for $p_q(x,y,z,t;0)$ is as follows:

$$p_q(x,y,z,t;0) = \rho(2\pi)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [[i\varepsilon\chi_0(\omega,\eta,\varepsilon,t) + i\eta\beta_0(\omega,\eta,\varepsilon,t) + i\omega\alpha_0(\omega,\eta,\varepsilon,t)]/(\eta^2 + \omega^2 + \varepsilon^2)]e^{-i\omega x}e^{-i\eta y}e^{-i\varepsilon z}\, d\omega d\eta d\varepsilon \tag{42}$$

where

$$\alpha_0(\omega,\eta,\varepsilon,t) = (2\pi)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_0(x,y,z,t)e^{i\omega x}e^{i\eta y}e^{i\varepsilon z}dxdydz \tag{43}$$

$$\beta_0(\omega,\eta,\varepsilon,t) = (2\pi)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b_0(x,y,z,t)e^{i\omega x}e^{i\eta y}e^{i\varepsilon z}dxdydz \tag{44}$$

$$\chi_0(\omega,\eta,\varepsilon,t) = (2\pi)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_0(x,y,z,t)e^{i\omega x}e^{i\eta y}e^{i\varepsilon z}dxdydz \tag{45}$$

The particular solutions for $u_q(x,y,z,t;0)$, $v_q(x,y,z,t;0)$ and $w_q(x,y,z,t;0)$ are as follows:

$$u_q(x,y,z,t;0) = (2\pi)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^t [\alpha_0(\omega,\eta,\varepsilon,u) - [\varepsilon\omega\chi_0(\omega,\eta,\varepsilon,u) + \eta\omega\beta_0(\omega,\eta,\varepsilon,u) + \omega^2\alpha_0(\omega,\eta,\varepsilon,u)]/(\eta^2 + \omega^2 + \varepsilon^2)] e^{-i(\omega x+\eta y+\varepsilon z) - \nu[\omega^2 + \eta^2 + \varepsilon^2][t-u]}\, du d\omega d\eta d\varepsilon \tag{46}$$

$$v_q(x,y,z,t;0) = (2\pi)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^t [\beta_0(\omega,\eta,\varepsilon,u) - [\varepsilon\eta\chi_0(\omega,\eta,\varepsilon,u) + \eta^2\beta_0(\omega,\eta,\varepsilon,u) + \omega\eta\alpha_0(\omega,\eta,\varepsilon,u)]/(\eta^2 + \omega^2 + \varepsilon^2)] e^{-i(\omega x+\eta y+\varepsilon z) - \nu[\omega^2 + \eta^2 + \varepsilon^2][t-u]}\, du d\omega d\eta d\varepsilon \tag{47}$$

$$w_q(x,y,z,t;0) = (2\pi)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^t [\chi_0(\omega,\eta,\varepsilon,u) - [\varepsilon^2\chi_0(\omega,\eta,\varepsilon,u) + \eta\varepsilon\beta_0(\omega,\eta,\varepsilon,u) + \omega\varepsilon\alpha_0(\omega,\eta,\varepsilon,u)]/(\eta^2 + \omega^2 + \varepsilon^2)] e^{-i(\omega x+\eta y+\varepsilon z) - \nu[\omega^2 + \eta^2 + \varepsilon^2][t-u]}\, du d\omega d\eta d\varepsilon \tag{48}$$

Higher Order Terms, Approximate and Exact Solutions

The steps outlined in the section immediately above can be repeated to allow for the derivation of higher order terms within each Taylor series expression for $u(x,y,z,t;1)$, $v(x,y,z,t;1)$, $w(x,y,z,t;1)$ and $p(x,y,z,t;1)$.

If the Taylor series for each of $u(x,y,z,t;1)$, $v(x,y,z,t;1)$, $w(x,y,z,t;1)$ and $p(x,y,z,t;1)$ is truncated at a finite number of terms, the result will be an approximation of the solution to the original boundary problem (as represented by equations (1) – (8) where the homotopy parameter, q, is equal to 1).

This approximation can, in principle, be made arbitrarily more accurate (relative to the exact solution) by the addition of extra terms in each Taylor series.

Existence, Uniqueness, Smoothness and Convergence

The analysis above also casts light on the unresolved question of the existence, uniqueness and smoothness of solutions to the Navier–Stokes equations (and their non-viscous counterparts) in the three dimensional case.

The individual coefficients in the Taylor series for each of $u(x,y,z,t;1)$, $v(x,y,z,t;1)$, $w(x,y,z,t;1)$ and $p(x,y,z,t;1)$ can be viewed as either the solution of a diffusion-like problem with a source taken over the same range of x, y, z and t variables as the original problem or, in the case of the corresponding pressure functions, as a reflection of the continuity equation applicable to it (as there is no evolution equation governing the pressure function in either the original or subsidiary problems).

The existence and uniqueness theorems that apply to the solutions of these subsidiary diffusion-like problems, therefore, dictate whether or not each Taylor series for $u(x,y,z,t;1)$, $v(x,y,z,t;1)$, $w(x,y,z,t;1)$ and $p(x,y,z,t;1)$ exists and is unique provided, of course, that the integral transform methods used to solve the various subsidiary problems can be validly applied (i.e., that the Fourier and Laplace transforms exist).[6]

As far as the question of the smoothness of the solutions is concerned, since the solution to each subsidiary problem is a solution to a diffusion-like problem which is itself taken to be bounded as x, y, z and t become large, the same diffusion-like behaviour will be exhibited by the Taylor series expressions for each of $u(x,y,z,t;1)$, $v(x,y,z,t;1)$, $w(x,y,z,t;1)$ and $p(x,y,z,t;1)$.[7]

Insofar as the convergence of each Taylor series is concerned, some general comments can be made. The only term with a non-zero initial condition is the first term in each series—the higher order terms in each Taylor series for $u(x,y,z,t;1)$, $v(x,y,z,t;1)$ and $w(x,y,z,t;1)$ each have a zero initial condition plus a source term that dissipates over time.

This combination of a finite initial condition and both the dissipative behaviour of, and the presence of the $1/n!$ factor in, each term in the Taylor series for each of $u(x,y,z,t;1)$, $v(x,y,z,t;1)$ and $w(x,y,z,t;1)$ indicates that the contribution of the n^{th} term approaches zero in each series as $n \rightarrow \infty$. These considerations imply that the Taylor series for $u(x,y,z,t;1)$, $v(x,y,z,t;1)$, $w(x,y,z,t;1)$ and $p(x,y,z,t;1)$ are suitably well behaved in terms of convergence[8].

Furthermore, at no point does the time variable, t, appear in the form of $1/(t-k)^{\alpha}$, where k is a positive constant and $\alpha \geq 1$, in the series for $u(x,y,z,t;1)$, $v(x,y,z,t;1)$, $w(x,y,z,t;1)$ and $p(x,y,z,t;1)$ for the zero viscosity case[9]. This observation implies that there is no finite time blow up in the zero viscosity case[10] for an incompressible unsteady flow and, since viscosity has the effect of dampening the velocity components of the fluid flow relative to the corresponding zero viscosity case, it also implies that no finite time blow up occurs in the case where the viscosity is non-zero for an incompressible unsteady flow either.

Implications for Direct Numerical Simulation

As noted above, the individual coefficients in the Taylor series for each of $u(x,y,z,t;1)$, $v(x,y,z,t;1)$ and $w(x,y,z,t;1)$ can be viewed as the solution of a diffusion-like problem with a source taken over the same range of x, y, z and t variables as the original problem (as noted earlier, the coefficients for the Taylor series for $p(x,y,z,t;1)$ are "derivative" due to the lack of an evolution equation governing the pressure function in either the original or subsidiary problems).

This suggests that direct numerical simulation of the solution to the problem considered here can be analysed by decomposing equations (1) – (8) into a series of diffusion-like problems with a source as has been done here.

Conclusions

This paper analyses the Navier–Stokes equations in the three dimensional case for an unsteady incompressible viscous fluid in the presence of a body force using, as far as the author is aware, a novel application of homotopy analysis. In particular, an explicit approximate solution to the Navier–Stokes equations is developed and the relationship between this approximation and the corresponding exact solution is presented.

The existence, uniqueness, smoothness and convergence of the explicit approximate series solutions and for the corresponding exact solution are also discussed. In particular, conclusions regarding the formation of singularities within finite time periods for solutions to the Navier–Stokes equations (and their non–viscous counterparts) in the three dimensional case are noted.

In addition, the potential utility of the solution strategy employed in this paper in the context of direct numerical simulation of fluid flows is considered briefly.

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[6] It is possible, in principle, that there exist valid solutions for which the corresponding Fourier and Laplace Transforms as defined here do not exist.

[7] In addition, the $e^{-\nu[\omega^2+\eta^2+\varepsilon^2]t}$ term within the solution to each subsidiary problem ensures that as the time variable, t, becomes large, each subsidiary solution tends to zero for all x, y and z and so too will the expressions for each of $u(x,y,z,t;1)$, $v(x,y,z,t;1)$ and $w(x,y,z,t;1)$.

[8] Depending on the particular problem, it is sometimes necessary to introduce a scaling parameter, the convergence control parameter, into the generalised set of PDE's and associated boundary and initial conditions to ensure that the Taylor series converges. See Liao [3] for further details on this point.

[9] In fact, the time variable, t, appears only as positive whole numbered powers of t in the Taylor series for $u(x,y,z,t;1)$, $v(x,y,z,t;1)$ and $w(x,y,z,t;1)$ in the zero viscosity case for an unsteady incompressible fluid flow. The requirement that $\alpha \geq 1$ is a necessary condition for there to be a singularity in the non-viscous flow case. See Gibbon [2] for details.

[10] Provided, of course, the body force does not itself contain a singularity that emerges within a finite time period.