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On the Stability and Optimal Growth of Time-Periodic Pipe Flow

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Abstract

A Chebyshev spectral co-location and divergence-free method is adapted to solve the linearised Navier–Stokes (LNS) equations for time-periodic pipe flow. The method is spectral in the pipe axis and azimuth, allowing for specific axial and azimuthal wave numbers. A Krylov subspace method is used to find the leading (real-part) eigenvalues of the LNS flow evolution over one oscillation period. We demonstrate the relevance to Floquet Theory and proceed to confirm the linear-stability of these flows. The new method is compared against a previous stability study in pipe flows. Finally, the optimal energy growth for periodic pipe flow is formulated in forward and adjoint variables. The growth problem is checked against an alternative time-stepper methodology. This investigation lays the ground-work for a validation study of recently presented three-dimensional linear instabilities in periodic pipe flow.

Introduction

The investigation of time-periodic pipe flows has received renewed interest in the past decade. This may be attributed to an increase in computational power, advances in spectral methodology (to which the geometry is idealised), and the discovery of new autonomous processes in shear flow [5]. Industrially, these types of flows and their transitional properties are of critical interest in cardio-related surgery where artificial components are the largest contributor to blood degradation. Peristaltic pumping of delicate suspensions and non-Newtonian fluids are also of importance, where high shear rates can cause damage to both the fluids and the pumping equipment.

It is known that steady laminar flow in pipes is stable to infinitesimal perturbations [13], while channels are asymptotically unstable at $Re = 5772$ for the axial wave-number 1.02 [12]. Moderately careful experiments in pipe flow (Hagen-Poiseuille) demonstrate a transitional Reynolds number of the order $2000-3000$. Most experimental observations in this area are related to slug or puff structures, which arrive with increasing frequency as system parameters of energy and Reynolds number are monotonically increased. Similarly, piston-driven experiments for oscillatory flows in pipes have revealed a number of transitional stages, each characterised by macro-scale fluid properties. Turbulence associated with these periodic flows comes in bursts often in the deceleration, or reverse-flow component of the cycle is linked to the governing Reynolds number.

The initial numerical understanding for periodic pipe flow was laid-down by Yang and Yih [17]. Addressing the axisymmetric stability problem (2D) it was found that the flow tended towards neutral stability in the asymptote of Reynolds number, and monotonically so for increasing frequency. Later axisymmetric work by [6] using long-wave Orr–Sommerfeld bases confirmed the known linear stability of periodic pipe flow. In 2009 Nebauer and Blackburn [11] (NB09) revisited the problem to extend the linear result to non-axisymmetric (3D) solutions. It was found that an increase in the three-dimensionality (in the azimuthal direction) resulted in an increase in the flow stability, for all Reynolds numbers studied. The solution space extended that of Yang and Yih in both Reynolds number and frequency parameter. This extension confirmed the prediction of Yang and Yih; that the flow is asymptotically stable to all perturbations.

The stability of oscillatory pipe flow is closely related to the stability of oscillatory Stokes layers, and of oscillatory channel flow. Instabilities in these flows were recorded by [3, 4]; in the latter, axisymmetric instability of oscillatory pipe flow was also reported. Further, [15] recently reported non-axisymmetric instabilities of oscillatory pipe flow. These findings have prompted us to implement a different numerical approach to the study of instability in oscillatory pipe flow than was previously used in NB09. The present work focuses on the development and validation of a fully spectral (Chebyshev–Fourier–Fourier) LNS solver and its use in studying Floquet instability of oscillatory pipe flow using time–stepper type methods [16].

This work focuses on the application of a LNS solver. The solver is based on the work of [10] and uses a set of divergence-free (DF) basis as the trial functions in a spectral solution. The time-stepper LNS solver is coupled to a Krylov–Arnoldi iterative method for Floquet stability analysis.

Problem Geometry and Parameters

We start with a regular, rigid cylinder of radius R (diameter D) and length l. It is completely filled with a laminar, viscid and incompressible fluid of density ρ and viscosity ν. The governing equation for this system is the Navier–Stokes partial differential system;

$$\partial_t \mathrm{u} = -\mathrm{u}\cdot\nabla\mathrm{u} - \nabla p + \nu\nabla^2\mathrm{u}, \quad \nabla\cdot\mathrm{u} = 0 \tag{1}$$

Here p is the kinematic or modified pressure and $\mathrm{u} = \{u, v, w\}$, the primitive velocities in the radial, azimuthal and axial (r, θ, z) directions. Under a constant pressure gradient and application of the no-slip wall boundary condition, the system (1) has an analytical solution:

$$u(r,t) = \frac{\Pi_0}{4\rho\nu}\left[1-\left(\frac{r}{R}\right)^2\right], \quad \forall\theta, z \quad \partial_r p = 0. \tag{2}$$

The pressure gradient Π_0 is real and ∂_r represents the partial derivative with respect to r. A closed-form solution of (1) for time-periodic pipe flows, under a periodic pressure gradient, can be obtained as analytical Bessel–Fourier solutions, first published by [14]:

$$u_n(r,t) = \Re\left[\frac{K_n iT}{\rho 2\pi n}\left(\frac{J_0\left(i^{3/2} Wo\frac{r}{R}\right)}{J_0\left(i^{3/2} Wo\right)} - 1\right) e^{2\pi i n t/T}\right] \tag{3}$$

where

$$Wo = R\left(\frac{2\pi}{T\nu}\right)^{\frac{1}{2}} \tag{4}$$

is a dimensionless frequency parameter known as the Womersley number, n is a frequency harmonic, J_0 is the zeroth-order complex Bessel function, K_n is an associated complex axial pressure gradient amplitude, and T is the period of the oscillation. In the limit as T grows without bound, this analytical solution asymptotes to the standard parabolic Hagen–Poiseuille solution for the steady laminar flow in a circular pipe, i.e. (2). Alternative formulations of type (4) are available for differing physical investigations. Harmonic piston or wall-driven systems vary slightly in their amplitude terms, or by the subtraction term (in the case of wall-driven flow). However, the underlying structure is a Bessel function quotient of a non-dimensional frequency. [1] demonstrated that such scaling is immaterial in the linear dynamics; save only in the non-dimensional representation of the results.

Through modulation of (4) for any frequency harmonic n we note that any time-periodic base-flow can be prescribed as a Fourier series;

$$\bar{u}(t) = \sum_n \left[a_n \cdot cos(2\pi n/T \cdot t) + b_n sin(2\pi n/T \cdot t)\right], \qquad (5)$$

where $\bar{u}$ is the area-average velocity in z through any cross-section of the pipe, and

$$\bar{u}_p = \max_T \bar{u}(t).$$

Now with a length and velocity scale, the governing Reynolds number is defined as

$$Re = \frac{\bar{u}_p D}{\nu}. \qquad (6)$$

Linear Stability

The stability analysis problem is solved in primitive variables. Starting from (1) it is proposed that $\mathrm{u} = U + \mathrm{u}'$, where U is the base flow whos stability is examined and u' is an infinitesimal perturbation, of the form

$$\mathrm{u}'(r,\theta,z) = e^{i[\alpha z + \beta\theta]}\mathrm{u}'(r), \qquad (7)$$

where α and β are wave numbers in the axial and azimuthal coordinates respectively. Upon substitution and retaining terms linear in u', the linearized Navier–Stokes equations are obtained:

$$\partial_t \mathrm{u}' = -\mathrm{u}' \cdot \nabla U - U \cdot \nabla \mathrm{u}' - \nabla p' + \nu \nabla^2 \mathrm{u}', \quad \nabla \cdot \mathrm{u}' = 0 \qquad (8)$$

All base flows considered in this study are axisymmetric and invariant along the axis of the pipe. Hence, $U = \{u, v, w\}$ with $u = u(r)$, $v = 0$ and $w = 0$. Furthermore, we note that in the present problem, the base flow is T-periodic, i.e. $U(t+T) = U(t)$.

As in all incompressible flows the pressure is not an independent variable, and as all terms are linear in u', we can write this evolution equation in symbolic form (discarding the prime ($'$) for convenience);

$$\partial_t \mathrm{u} = \mathcal{L}(t)\mathrm{u}, \qquad (9)$$

where $\mathcal{L}$ is a linear operator with T-periodic coefficients through the influence of the base flow. Correspondingly the stability of (9) is a linear temporal Floquet problem [8]. Writing the state evolution of u over one period as

$$\mathrm{u}(t+T) = \mathcal{A}(T)\mathrm{u}(t), \qquad (10)$$

where $\mathcal{A}(T)$ is the system monomodry matrix. We obtain a Floquet eigenproblem:

$$\mathcal{A}(T)\mathrm{u}''_j(t) = \mu_j \mathrm{u}''_j(t). \qquad (11)$$

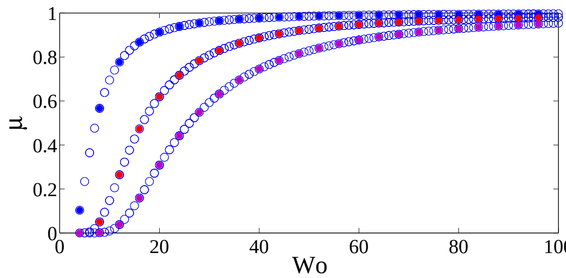

Figure 1: Axially invariant, axisymmetric ($\alpha = 0$, $\beta = 0$) Floquet Multipliers for both NB09 (○) and (16)(●), showing the first 3 modes respectively.

Here $\mathrm{u}''_j(t)$ are phase-specific Floquet modes and μ_j are Floquet multipliers (which generally occur in complex conjugate pairs). Stability of the problem is assessed from the Floquet multipliers: unstable modes have multipliers that lie outside the unit circle in the complex plane (i.e. $|\mu| > 1$), while stable modes lie inside (i.e. $|\mu| < 1$). A key point about the approach (the 'time-stepper' approach of [16]) is that a system monodromy matrix $\mathcal{A}(T)$ is not explicitly constructed; rather, a Krylov method is used that is based on repeated application of the state transition operator whos action is obtained by integrating the linearised Navier–Stokes equations forward in time over interval T.

Numerical Method

We use a numerical method based on the work of [10]. The original divergence free basis methodology is presented in detail by [9]. The underlying principle in this solution space is the implicit assertion that the three-dimensional flow can be constructed from two velocity components, given that the divergence of that field is identically zero. Hence, $\mathrm{v} = \{v_1, v_2\}$, from which a full third component may be calculated.

To develop this concept we note that the operator system (9) can be obtained from the inner product of the operator over a solution space,

$$\langle \partial_t \mathrm{v}, W \rangle_\Omega = \langle \mathcal{L}\mathrm{v}, W \rangle_\Omega, \quad \forall W; \qquad (12)$$

here, $\langle \cdot, \cdot \rangle_\Omega$ is the inner–product over the spatial domain Ω. In the construction of W we note that v is solenoidal (i.e. divergence free) and the test space W should conform for favourable properties. This is of particular interest since $\langle v_1, W \rangle = \langle v_2, W \rangle$ for all solenoidal functions W that vanish over the boundary. From (1) the pressure gradient is imposed by ∇p, which upon projection presents as $\langle \nabla p, W \rangle$. Integration by parts of this expression yields

$$\langle \nabla p, W \rangle = pW - \langle p, \nabla W \rangle. \qquad (13)$$

The product pW of (13) is zero provided $W = 0$ at the boundary. Further more, $\langle p, \nabla W \rangle = 0$ as $\nabla W = 0$. Hence, the pressure has been removed as a variable from the system.

In constructing $\mathcal{L}$ we start by defining the matrix evolution problem in dual and solenoidal vector space,

$$\boldsymbol{B}\mathrm{u}_t = \boldsymbol{A}\mathrm{u}, \quad \boldsymbol{B} = \langle W, \mathrm{v} \rangle, \quad \boldsymbol{A} = \langle W, \mathcal{L}\mathrm{v} \rangle \qquad (14)$$

hence,

$$\mathcal{L} = \boldsymbol{B}^{-1}\boldsymbol{A}\mathrm{u}. \qquad (15)$$

By replacing the base-flow condition of $\mathcal{L}$ with one which is time-periodic, the only term of (8) affected is the linear

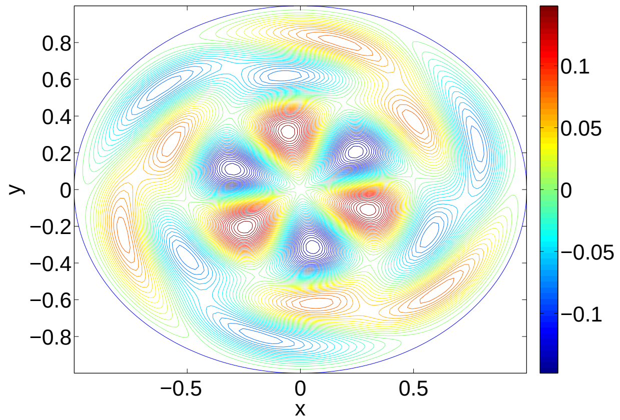

Figure 2: Contours of axial-velocity for the sub-dominent eigenmode of the non-axisymmetric ($\alpha = 0.3,\ \beta = 3$) mode. Energy concentration towards the centre-line and near-wall regions is clear. While the asymptotic influence of this mode is minimal, its transient effect is considerable.

advection-diffusion term. Hence the system's time dependence is limited to the advancing inner-products,

$$\langle \mathrm{v}_\tau, W \rangle = \langle \mathcal{L}(t)\mathrm{v}_0, W \rangle ,$$

for any initial solenoidal state v_0 and final state v_τ. The weak-form of (8) can then be expressed as an integral (over time, T):

$$\mathrm{u}(T) = \int_0^T \boldsymbol{B}^{-1} \left[W \left(\nabla^2 + \nabla U(t) \cdot \mathrm{u}(t) + U(t) \cdot \nabla \mathrm{u}(t) \right) \right] \mathrm{u}(t)\ dt. \tag{16}$$

To perform this integration we use a semi-implicit 2nd order stiff solver, stemming from the numerical differentiation formulas as implemented in Matlab's `ode15s`. Hence, the spatial-base flow retains it's spectral properties with the complexity of the matrix being linked to the inversion of $\boldsymbol{B}$. In this expression the operator matrices $\boldsymbol{B}$, W and ∇^2 are constant. Hence, the preconditioning of $\boldsymbol{B}^{-1}$ is constructed only once. For the LNS evolution, we test the implementation of (16) against the least-stable eigenvalues of NB09, corresponding to $\alpha = 0$, $\beta = 0$ – which is also *Re*-independent. Excellent agreement is seen, as in figure 1. The structure of leading modes in the non-axisymmetric eigenspace have considerable energy concentrated in the centre-line and near-wall regions, for example in figure 2. While these modes are clearly sub-dominant, their structure is of interest when considering the next section, transient analysis.

Optimal Growth

A residual effect of the off-diagonal terms of the advection-diffusion term of (8) is the non-normality of the resulting operator. The non-linear interaction of these terms serve to produce solutions that, for short time-scales, can grow algebraicly. The energy growth is governed in the asymptote of time by the leading operator eigenmodes. However, in the transient time-frame from an initial perturbation to the final modal saturation the interaction of even decaying modes can produce a net growth in energy. We seek an initial condition $\mathrm{u}(0)$, which under operation of (9) produces the greatest energy growth over a finite time τ. As we are dealing with periodic base-flows, we introduce $\phi = \tau/T$ as a measure of the phase-point in the flow. The energy growth is then the norm (inner product) of the velocity field at any time ϕ, normalised by the initial energy of the system:

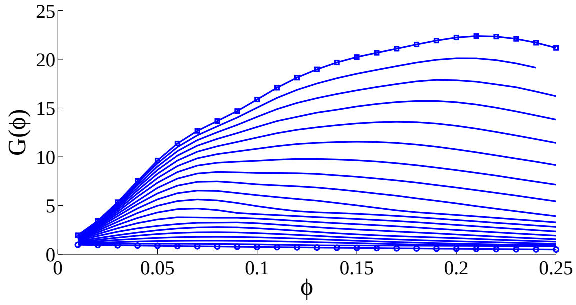

Figure 3: Optimal Growth: The leading singular value for $Wo = 5$ ($\circ$) through to $Wo = 100$ ($\square$), intermediate Wo values are monotonically distributed between the delineated curves. Calculations are for a period of oscillation, or base-period, of $T = 10$. The individual Wo curves are for the axisymmetric case ($\beta = 0$), and over a composite of axial wave numbers, whichever is largest for the phase-point (ϕ).

$$G(\phi) = \frac{\langle \mathrm{u}(\phi), \mathrm{u}(\phi) \rangle}{\langle \mathrm{u}(0), \mathrm{u}(0) \rangle} = \frac{\langle \mathcal{A}\mathrm{u}(0), \mathcal{A}\mathrm{u}(0) \rangle}{\langle \mathrm{u}(0), \mathrm{u}(0) \rangle} = \frac{\langle \mathrm{u}(0), \mathcal{A}^*\mathcal{A}\mathrm{u}(0) \rangle}{\langle \mathrm{u}(0), \mathrm{u}(0) \rangle}, \tag{17}$$

where $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$. Using $\mathcal{A}$ to evolve the system over $0 \to \tau$, the final energy state is $\langle \mathcal{A}\mathrm{u}(0), \mathcal{A}\mathrm{u}(0) \rangle$. It follows from algebra that $\langle \mathrm{u}(0), \mathcal{A}^*\mathcal{A}\mathrm{u}(0) \rangle$. This has the effect of locating the vector $\mathrm{u}(0)$ which is most amplified by $\mathcal{A}$, which is determined by the eigenvectors of $\mathcal{A}^*\mathcal{A}$. This is analogous with linear algebra where the eigensystem of $A^T A$ is related to the singular value decomposition of A. The remaining problem is the correct formulation of the operator adjoint, $\mathcal{A}^*$, which is distinct from the tranpose of $\mathcal{A}$. Having eliminated the pressure using (12), we condense the adjoint Navier–Stokes equations to terms used in the construction of an operator:

$$\partial_t \boldsymbol{u}^* = \nabla U \cdot \boldsymbol{u}^* - U \cdot \nabla \boldsymbol{u}^* - \nu \nabla^2 \boldsymbol{u}^* \tag{18}$$

It follows that the adjoint can be symbolically represented as

$$\partial_t \boldsymbol{u}^* = \mathcal{L}^*(t) \boldsymbol{u}^*. \tag{19}$$

Similarly to (10), the adjoint system is a state transition operator with the physical property of evolving the solution backwards through time. Hence, the effect of $\mathcal{A}^*$ is affected through the divergence-free time-stepper matrix formulation:

$$\boldsymbol{u}^*(0) = \int_{\phi T}^0 \boldsymbol{B}^{-1} \left[W \left(-\nabla^2 + \nabla U(t) \cdot \boldsymbol{u}^*(t) - U(t) \cdot \nabla \boldsymbol{u}^*(t) \right) \right] \boldsymbol{u}^*(t)\ dt. \tag{20}$$

By concatenating the monomodry operators from an initial condition $\mathrm{u}(0)$, forward in time in (16) and then in 'reverse' by (20) we terminate with the vector $\boldsymbol{u}^*(0)$ which is the effect of $\mathcal{A}^*\mathcal{A}$ operating on $\mathrm{u}(0)$. The eigenspace of $\mathcal{A}^*\mathcal{A}$ is found through the same Krylov–Arnoldi process of the Floquet stability analysis. The eigenvalues are the singular vectors of the operator $\mathcal{A}$, determined at phase-points ϕ. Sufficiently large singular values can be associated with large algebraic growth in the fluid energy, and ultimately to the bypass transitional mechanics. This can be true even for globally stable systems. We test the implementation of (16)→(20) in figure 3 for axisymmetric and axially-invariant modal structures. Again, the we use the Gauss–Lobatto–Legendre (GLL) spectral-element (SE) time-stepper numerical method of [2] as a comparison. In solving the eigenproblem using the SE (label GLL in figure) method

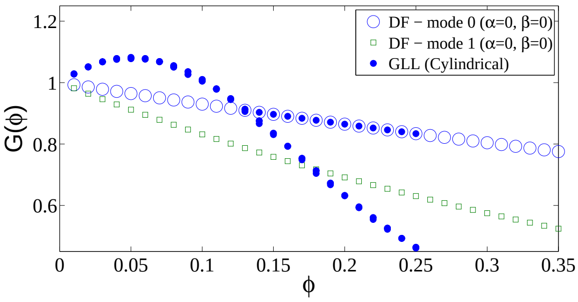

Figure 4: Optinal Growth: The leading two singular values for $Wo = 10$, $Re = 650$ for $\beta = 0$. The GLL (spectral-element) methodology is unable to restrict the axial wave number. Showing the leading two singular values causes the 'bump' in the results. These singular values are a collection of axial wave numbers, in no particular order, $\alpha = 5, 7, 9$ and 11.

we note that the two-dimensional element plane does not allow spectral decomposition, and hence, the control on the axial wave-number (α) is limited. Due to computational cost of repeated forward-backward evolutions, we have converged only the first two singular values. Further convergence in the stack would uncover the 'missing' $\alpha = 0$ singular values, calculated by the DF methodology. The very good alignment of those axially-invariant modes identified was considered sufficient to validate the operator construction.

A key (apparently new) result is the *independence* of Reynolds number in the axially-invariant and axisymmetric case ($\alpha = 0, \beta = 0$), while retaining dependency over Wo and ϕ. This mode does not contain any energy growth (all Wo yet studied have $G_{\max} \leq 1$), in the limit of t these modes contain the most amplification (be it less than unity). This is the first evidence that the prediction of Hall [7] may be correct, and the possible linear instabilities located by [4] may be closed in Reynolds Number.

The initial transient, amplifying processes are concentrated in high axial and azimuthal wave numbers. Whether such amplification is likely to bootstrap a bypass transition process or not is outside the scope of this paper. The higher modal structures seem to be transient, demonstrated in figure 3, showing the monotonic decrease in energy once they reach a local maxima. This is consistent with the current Floquet results, where all high order modes are decaying.

There is a tenancy for all axial and azimuthal structures to decay in energy below their associated one-dimensional structure. The survival of these transient phenomena may be linked to the Wo and Re parameters. However, current evidence suggests that all transients does subside, and the axially-invariant axisymmetric cases carry the long-term stability properties of time-periodic pipe flow.

Conclusions

A divergence-free numerical method has been adapted for use in time-periodic pipe flows. It has successfully been applied to axisymmetric axially-invariant linear Floquet stability analysis and transient growth. The formulation is spectral in both the pipe axis and azimuth with Chebyshev structure in the radial direction.

Using this new tool we are now able to investigate recently published results on linear stability in time-periodic pipe flows.

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