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Stability of steady flow through a corrugated pipe

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Abstract

The stability of fully developed pipe flows has been rigorously studied both numerically and physically for smooth walled pipes. However, a similar treatment for rough walled pipes is lacking. In this paper we present a numerical study of the effect of wall corrugation on the linear stability of fully developed pipe flow. Linear stability to axisymmetric and non-axisymmetric disturbances was investigated for corrugated pipe geometries with constant average radius to wavelength ratios of unity. Six non-dimensional corrugation amplitudes were investigated for Reynolds numbers from 1500 to 11800 with computational domains one corrugation wavelength long.

Results of the analysis showed that the presence of the wall corrugations destabilized the flow to non-axisymmetric disturbances while stability to axisymmetric disturbances was maintained for all cases. The critical Reynolds number was found to increase asymptotically toward infinity with diminishing wall corrugation amplitude, consistent with findings for smooth walled pipes, and to decrease to a minimum of 1770 before increasing again as the wall corrugation amplitude was further increased. Some correlation between the base flow topology and the critical Reynolds number was observed. However, further work would be required to draw any firm conclusions about this relationship. For all cases of instability the critical azimuthal wave number, β, was found to be greater than 0. This is in contradiction to an earlier work suggesting that the critical azimuthal disturbance mode decays to $\beta = 0$, or axisymmetric modes, for non-dimensional wall corrugation amplitudes, ε, less than 0.1.

Introduction

Transition to turbulence in smooth walled pipes has been investigated by various means using both physical experiments and numerical methods. Early experimental studies [10] found that fully developed pipe flows became unstable at Reynolds numbers ≈ 2000. However, numerous linear stability studies since then have determined that for perfectly smooth walled pipe flows the laminar profile is linearly stable to axisymmetric and non-axisymmetric disturbances for all Reynolds numbers [11, 8]. The process of transition observed in physical experiment is thus thought to be triggered by non-linear perturbation growth. Numerical investigations [7, 12] and experimental studies [6] have also noted that the flow becomes increasingly sensitive to external dsturbances with the critical disturbance amplitude for transition falling off with $Re^{-\alpha}$, where α has some dependence on the type of disturbance. Generally speaking values in the 1-1.5 range have been found though this problem is far from completely solved.

These new findings for smooth walled pipes present an interesting perspective on previous work on the stability of rough walled pipes. Nikuradse's [9] seminal paper on the laws of flow in rough pipes established that random small scale roughness had no influence on the critical Reynolds number. However, with more recent results for smooth walled pipes suggesting that uncontrolled disturbances inherent to the experimental apparatus could have significant influence on the results, the question of what effect wall roughness has on flow stability remains open.

While there is very limited previous numerical work on rough walled pipes, linear stability analysis for other wall bounded flows, such as parallel plate flows, have found that the presence of roughness elements significantly alters the linear stability of the base flow compared to the smooth wall case [3, 5]. Cotrell [4] aimed to address this by investigating the linear stability of fully developed corrugated pipe flow to axisymmetric disturbances. Results of this study suggested the flow is linearly unstable with the critical Reynolds number decreasing asymptotically as the wall amplitude was increased. Furthermore, results of a cursory investigation of non-axisymmetric disturbances suggested that the critical azimuthal wave number was zero for non-dimensional wall corrugation amplitudes less than 10%.

The purpose of the present work is to extend the body of knowledge on the effect of wall roughness on flow stability by conducting a linear stability analysis for fully developed flow through a corrugated pipe. The work seeks initially to verify the results presented by Cotrell and extend the knowledge by considering more thoroughly the stability of the flow to non-axisymmetric disturbances.

Formulation

The linearised Navier-Stokes equations written for an infinitesimal perturbation $\boldsymbol{u}'$ to a base flow $\boldsymbol{U}$ are:

$$\partial_t \boldsymbol{u}' = -\left[\boldsymbol{U}.\nabla + (\nabla \boldsymbol{U})^T.\right]\boldsymbol{u}' - \nabla \boldsymbol{p}' + Re^{-1}\nabla^2 \boldsymbol{u}' \qquad (1)$$

$$\nabla.\boldsymbol{u}' = 0 \qquad (2)$$

Symbolically we can express the evolution equations for the perturbation velocity as:

$$\partial_t \boldsymbol{u}' = L\boldsymbol{u}' \qquad (3)$$

where L is a linear operator on $\boldsymbol{u}'$. This equation allows solutions of the form:

$$\boldsymbol{u}'(z,r,\theta,t) = \exp\left(\lambda_j t\right)\widetilde{\mathbf{u}}_j(z,r,\theta) \qquad (4)$$

where λ_j and $\widetilde{\mathbf{u}}_j$ are eigenvalues and eigenvectors of the linear operator L. The long term growth or decay of perturbations is thus dependent on the sign of the eigenvalues L. For reasons of numerical tractability, the state transition operator $A(T) = \exp(LT)$ is used to evolve the solution over time interval T [1]. As the eigensystems of A and L are related, the stability of the system can be determined by the magnitude of the dominant eigenvalue of A rather than the sign of the least

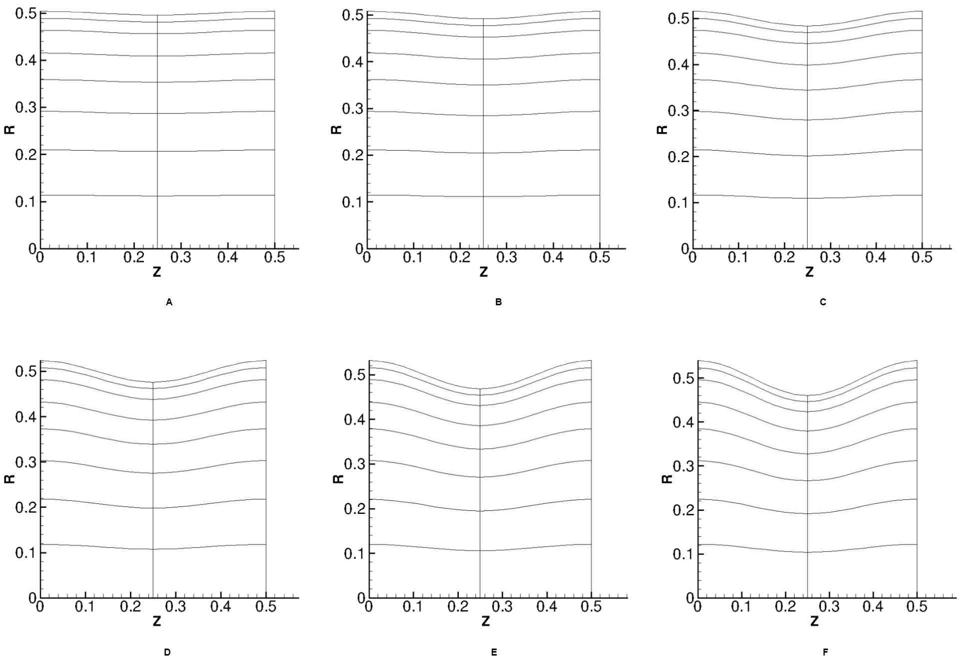

Figure 1: Spectral element meshes used. Non-dimensional corrugation amplitudes are (**a**) a=0.015904 (**b**) a=0.031808, (**c**) a=0.063612, (**d**) a=0.095420, (**e**) a=0.127228, (**f**) a=0.159032

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stable eigenvalue of L. This is numerically easier to implement with any eigenvalue with magnitude greater than unity indicating instability, any eigenvalue with magnitude less than unity indicating stability and any eigenvalue with magnitude equal to unity indicating neutral stability.

Methodology

For the pipe geometry, the radius of the pipe as a function of the axial co-ordinate z is given by $R(z)/D = 0.5 + 0.5h\cos(2\pi z/\Lambda)$ where h is the peak-to-peak corrugation height normalized by mean diameter, D, z is the axial coordinate and Λ is the axial corrugation wavelength, here $\Lambda = 0.5D = \overline{R}$. The average radius of the pipe, $\overline{R}$, was thus set to 0.5 and the ratio $\overline{R}/\Lambda$ was set to 1. Six corrugation amplitudes were investigated with computational domains limited to one wavelength as shown in Table 1 and Figure 1.

a
0.015904
0.031808
0.063612
0.095420
0.127228
0.159032

Table 1: Dimensionless corrugation amplitudes used

The numerical method is summarized briefly here. However, for a full description the reader is directed to [2]. Direct numerical simulation of the incompressible Navier-Stokes equations on the meridional semi-plane meshes shown in figure 1 was used to determine the base flows $\boldsymbol{U}$. Spectral elements were used for the discretization with spatial convergence verified by recording the converged axial velocity after repeated refinement of the grid resolution for the highest corrugation amplitude mesh at the highest nominal Reynolds number. The grid resolution was chosen so that further refinement produced changes in the converged axial velocity less than 0.01%. An axial body force determined approximately from the nominal Reynolds number was used to drive the flow, with the Reynolds number for each case then determined from the bulk flow rate and the average pipe diameter. Imposed boundary conditions were no slip on the pipe wall and axial periodicity.

Stability to axisymmetric disturbances was determined by the successful convergence of the base flow solutions. As the numerical method used to determine the base flow was a time-stepping method, any axisymmetric instability would result in an inability to reach a steady state solution. Non-axisymmtric disturbances are restricted to solutions periodic about the pipe axis allowing decomposition of the azimuthal solutions into Fourier modes. Due to the linearity of the problem, superposition allows the stability of the base flow to each Fourier mode to be analysed separately. For each Fourier mode, Equation 4 thus becomes:

$$\boldsymbol{u}'(z,r,\theta,t) = \exp\left(\lambda_j t\right)\hat{\mathbf{u}}_j(z,r)\exp(im\theta), \quad (5)$$

where m is the azimuthal wave number. In this work the base flows were analysed for stability to the first 10 Fourier modes. To determine stability, dominant eigenvalues of A are sought by projection of the linear operator onto a low dimensional Krylov subspace followed by the application of an Arnoldi type iteration method. Convergence of the dominant eigenvalue was assumed once the residual fell below 1×10^{-6}. This is a general description of the technique. For a more detailed description the reader is directed to [1].

Results

Base flows were successfully converged to steady state for all bulk flow Reynolds numbers and pipe geometries suggesting that the flow is linearly stable to all axisymmetric disturbances.

Two base flow topologies were observed dependent on the bulk flow Reynolds number and wall corrugation amplitude. In the first type, shown in figure 2, the flow was predominantly in the direction of axial body forcing with attachment along the wall maintained for the entire domain. In the second type, shown in figure 3, the flow was observed to detach and reattach over the corrugations with a recirculation zone forming in the widest parts of the pipe.

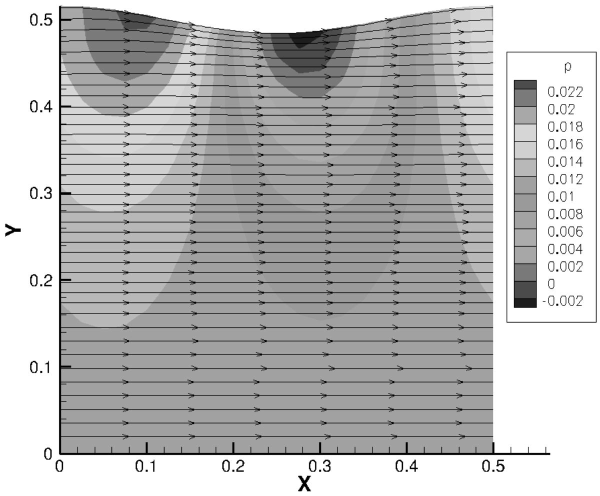

Figure 2: Laminar base flow attached to the wall for the entire computational domain. Dimensionless wall corrugation amplitude is 0.063612, bulk flow Reynolds number is 1790, contours of pressure shown with streamlines.

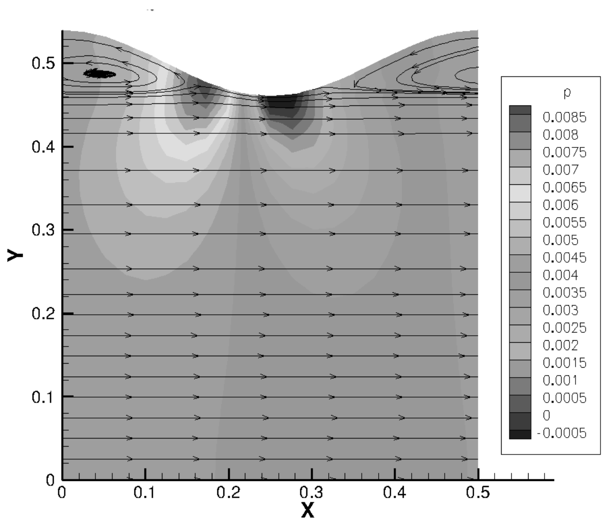

Figure 3: Base flow dettached from the wall with a recirculation zone between corrugations, dimensionless wall corrugation amplitude is 0.159032, bulk flow Reynolds number is 7990, contours of pressure shown with streamlines.

As a further check of axisymmetric stability two geometries were analysed for axisymmetric stability using the same eigenvalue algorithm used for non-axisymmetric stability. Figure 4 shows the growth of the perturbations with increasing bulk flow Reynolds number. These results show that the flow remains

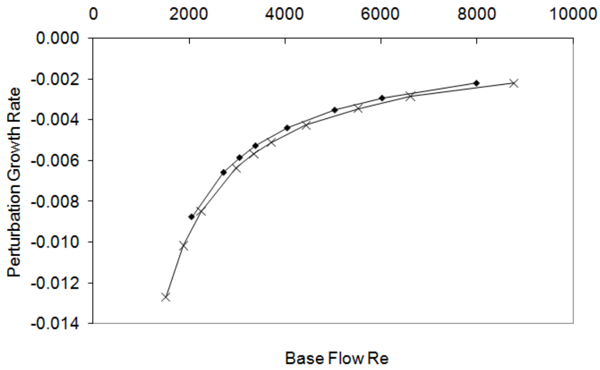

Figure 4: Growth rate of axisymmetric perturbations against bulk flow Reynolds number for dimensionless corrugation amplitudes a=0.159032 (dots) and a=0.127228 (crosses).

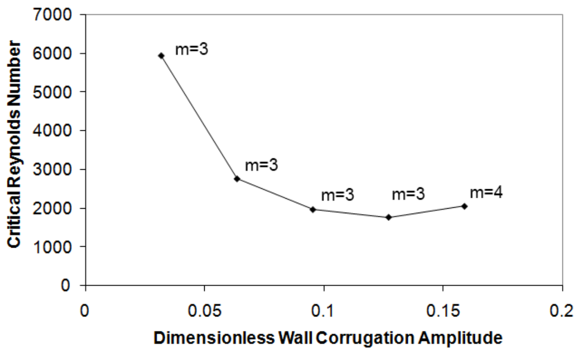

Figure 5: Critical Reynolds number against dimensionless wall corrugation amplitude for non-axisymmetric disturbances

stable but becomes less so as the bulk flow Reynolds number increases. For both cases, while the flow became less stable, the growth of the perturbations remained negative and appeared to be asymptoting toward zero or a negative growth rate, suggesting the flow was stable for all Reynolds numbers.

These findings contradict the results of [4] who suggested that flow through a corrugated pipe was unstable to axisymmetric disturbances and that this was the primary mode of instability in the limit of small corrugation amplitudes. Without a more rigorous understanding of the methodology used by Cotrell however, this discrepancy is left as a point of disagreement.

Results of the non-axisymmetric stability analysis found the flow to be linearly unstable for most of the geometries considered. Figure 5 shows the variation of the critical Reynolds number with the wall corrugation amplitude as determined by this study. The results indicate that as the corrugation amplitude becomes increasingly small the critical Reynolds number becomes increasingly large, possibly asymptoting to infinity as the amplitude approaches zero. The asymptotic nature of the curve agrees with earlier findings of complete linear stability of smooth walled pipes to non-axisymmetric disturbances.

At the other end of the spectrum, as the wall corrugation amplitude was increased the flow became increasingly unstable, with the critical Reynolds number attaining a minimum value of 1770, before increasing with further increases in the wall corrugation amplitude. Due to the limited number of wall corrugation amplitudes investigated and the linear interpolation between data points this minimum critical Reynolds number is of limited accuracy but is a reasonable initial estimate.

The Fourier modes associated with the critical Reynolds numbers were also found to be contradictory to the findings of [4]. Figure 5 also shows the wave numbers of the least stable Fourier modes for each of the wall corrugation amplitudes. Findings in [4] suggest that the critical wave number for disturbances decreases with wall corrugation amplitude, eventually reaching zero (axisymmetric disturbances) for non-dimensional corrugation amplitudes less than 0.1. In our study however, with corrugation amplitudes less than 0.1, the critical azimuthal wave number was 3 for the three cases that showed instability as shown in figure 5.

A preliminary investigation of the influence of the computational domain on the results of both the axisymmetric and non-axisymmetric stability analysis was done by extending the domain to two wavelengths and repeating the analysis for a dimensionless corrugation amplitude of 0.159032. Results of this analysis found the extension of the computational domain had no significant effect on the results of either stability analysis. The maximum difference in the critical Reynolds number predicted for the two computational domains was 0.5%.

The results of this study also suggest some linkage between the separation and reattachment of the base flow and an increase in flow instability. For the three largest wall amplitude cases a base flow with separation and reattachment over the corrugations was observed for all Reynolds numbers and the resulting critical Reynolds numbers were all found to be around 2000. For the next lowest wall corrugation amplitude, separation of the base flow is not observed initially but does occur for higher Reynolds numbers. In this case the critical Reynolds number is found to have increased significantly to 2750. Continuing this trend, for the lowest wall corrugation amplitudes no separation of the base flow was observed and the critical Reynolds number was again found to have increased dramatically to 5945, a 115% increase over the previous corrugation amplitude. For the lowest wall corrugation amplitude no critical Reynolds number was found, most likely due to it being outside the range of Reynolds numbers considered.

Conclusions

In summary the results of this study suggest that axisymmetric corrugations can have a significant effect on the stability of fully developed pipe flows. Whilst the flow retained stability to axisymmetric disturbances for all geometries and Reynolds numbers considered, instability to non-axisymmetric disturbances was found for all but the lowest corrugation amplitude case. Consistent with findings for smooth walled pipes, the critical Reynolds number was found to increase asymptotically toward infinity as the corrugation amplitude became vanishingly small. As the corrugation amplitude was increased the critical Reynolds number attained a minimum value before beginning to increase again. In our study this minimum value was found to be 1770 for a corrugation amplitude of 0.127228 units. Some correlation between the structure of the base flow, specifically the onset of separation and reattachment in the bulge region, and the critical Reynolds number was observed. However, further work would be needed to confirm this link and provide an explanation for it.

References

[1] Barkley, D., Blackburn, H.M. and Sherwin, S.J., Direct Optimal Growth Analysis for Timesteppers, *Int. J. Numer. Meth. Fluids*, **57**, 2008, 1435–1458.

[2] Blackburn, H.M. and Sherwin, S.J. Formulation of a Galerkin Spectral Element-Fourier method for Three-Dimensional Incompressible Flows in Cylindrical Geometries, *J. Comput. Phys.*, **197**, 2004, 759–778.

[3] Cabal, A., Szumbarski, J. and Flowryan, J.M., Stability of Flow in a Wavy Channel , *J. Fluid Mech.*, **457**, 2002, 191–212.

[4] Cotrell, D.L, McFadden, G.B. and Alder, B.J. Instability in Pipe Flow, *Proc. Natl. Acad. Sci.*, **105**, 2008, 428–430.

[5] Flowryan, J.M., Instability of Flow in a Rough Channel, *Physics of Fluids*, **17**, 0044101.

[6] Hof, B., Juel, A. and Mullin, T., Scaling of the Turbulence Transition Threshold in a Pipe, *Phys. Rev. Lett.*, **91**, 2003, 244502.

[7] Mellibovsky, F. and Mesegeur, A., Critical Threshold in Pipe Flow Transition, *Phil. Trans. R. Soc. A.*, **367**, 2009, 545–560.

[8] Meseguer, A. and Trefethen, L.N., Linearized Pipe Flow to Reynolds Number 10,000,000, *J. Comp. Phys.*, **186**, 2003, 178–197.

[9] Nikuradse, J., Strömungsgesetze in Rauhen Rohren, *Forsch. Arb. Ing.-Wes.*, **361**, reprinted as NACA TM-1292, 1950.

[10] Reynolds, O., An Experimental Investigation of the Circumstances Which Determine Whether the Motion of Water Shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels, *Philos. Trans. R. Soc. London.*, **174**, 1883, 935-982.

[11] Salwen, H., Cotton, F.W., and Grosch, C.E., Linear Stability of Poiseuille Flow in a Circular Pipe, *J. Fluid. Mech.*, **98**, 1980, 273–284.

[12] Viswanath, D., The Critical Layer in Pipe Flow at High Reynolds Number, *Phil. Trans. R. Soc. A.*, **367**, 561–576.