Evolution of zero pressure gradient turbulent boundary layers

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Abstract

The streamwise evolution of a turbulent boundary layer (TBL) developing under constant pressure on a smooth wall is considered. The closure problem is described for the zero pressure gradient (ZPG) flow where the only assumptions made are the use of classical similarity laws, such as Prandtl’s law of the wall and Coles’ law of the wake, together with the mean continuity and mean momentum differential and integral equations. The important parameters are identified and the problem is reduced to one semi-empirical input with the assumption that the Reynolds shear stress can be described by a two-parameter family. Good agreement is shown with the experimental data.

Introduction

Here we compute the evolution of turbulent boundary layers developing in a zero-pressure gradient. Unlike channel or pipe flows where there is a minimum development length for the flow to be regarded as *fully developed*, a boundary layer is continuously evolving and can therefore never be *fully developed*. Comparisons of previous experimental data sets (as discussed in the review of Marusic et al. [4]) have shown significant differences between different studies even though local parameters, such as \( R_b \) (Here, \( R_b \) is the Reynolds number based on free stream velocity \( U_1 \)) and momentum thickness \( \theta_1 \) is matched. A likely cause for these disparities are the different evolution conditions for each boundary layer. Nagib et al. [8] have discussed this issue in the context of a *well-behaved* ZPG boundary layer, but this again relates the specific initial conditions each experimental boundary layer evolves from. In this paper we will consider the issue of evolution and attempt to develop a framework for computing the evolution of ZPG boundary layers starting from arbitrary initial conditions.

Perry et al. [9], initially developed a preliminary mathematical framework for computing the evolution of turbulent boundary layers. It was found that there are 4 parameters that control the streamwise evolution of the boundary layer and the Reynolds shear stress distribution namely \( S \), \( \Pi \), \( \beta \) and \( \zeta \). Here, \( S = U_1/\Pi \) where \( U_1 \) is the local free stream velocity and \( \Pi \) is the friction velocity. \( \Pi \) is the wake parameter, \( \beta = (\delta^*/\eta_0)(dp/dx) \) is the Clauser pressure gradient parameter where \( \delta^* \) is the displacement thickness, \( p \) is the free stream static pressure, \( \eta_0 \) is the wall shear stress, \( \delta \) is the streamwise distance and the non equilibrium number, \( \zeta = S\delta/\Pi dx \) where \( \delta \) is the boundary layer thickness. The definitions of the law of the wall and law of the wake can be used to formulate a mean velocity defect law of the form,

\[
\frac{U_1 - U_2}{U_1} = F[\eta, \Pi],
\]

Perry et al. [9] had shown that integration of the streamwise momentum integral equation of the form,

\[
\frac{d\theta}{dx} + \frac{(H+2)\theta}{U_1} \frac{dU_1}{dx} = \frac{C_f}{2}
\]

(2)

(Here, the shape factor, \( H = \delta^*/\theta \). Where, \( \delta^* \) and \( \theta \) are the displacement thickness and momentum thickness respectively. \( C_f \) is the skin friction co-efficient) coupled with equation (1) leads to an expression for the total shear stress of the form,

\[
\tau_\theta = f_1[\eta, \Pi, S] + g_1[\eta, \Pi, S] \zeta + g_2[\eta, \Pi, S] \beta
\]

(3)

where \( \eta = \delta/\theta \) and \( \zeta \) is the wall-normal distance. Here \( f_1, g_1 \) and \( g_2 \) are known analytical functions where their precise form depends on the wall-wake formulation of Jones [3].

\[
U_1/\Pi = \frac{1}{\kappa} \log \left[ \frac{\tau_\theta}{\nu} \right] + A - \frac{1}{3\kappa} \eta^3 + \frac{\Pi}{\kappa} 2\eta^2 (3-2\eta)
\]

(4)

Here, \( \kappa = 0.41 \) is the Karman constant and \( A \) is the universal smooth wall constant, taken here to be 5.0. There are many wall-wake formulations available in the literature, for example, see [6].

Evolution and Closure Equations

Calculating the streamwise evolution of the ZPG turbulent boundary layer is possible by solving a coupled set of ODE’s which come from the momentum integral equation, law of the wall, law of the wake and the definitions of \( \zeta \) and \( \beta \),

\[
\frac{dS}{dR_c} = \frac{\chi[R_c, K]}{SE[\Pi]exp[KS]}
\]

(5)

\[
\frac{d\Pi}{dR_c} = \frac{\zeta[R_c, K]}{S^2E[\Pi]exp[KS]}
\]

(6)

Here,

\[
R = \frac{S}{\kappa S^2C_1 - \kappa SC_2 + C_2} + \frac{\beta(2SC_1 - C_2)}{C_2(C_2 - SC_1)}
\]

(7)

\[
\chi[R_c, K] = U_1(U_c)/U_0
\]

(8)

\[
E[\Pi] = exp[-KA - 2\Pi + 1/3]
\]

(9)

\[
\beta = -C_1S^2E[\Pi]exp[KS]K
\]

(10)

where \( U_0 \) is the reference free stream velocity (free stream velocity on top of the trip, i.e. \( U_0 = U_1[0] \)), \( R_e \) is the Reynolds number at different stream wise stations.
Evolution equations were restricted in the initial work of Perry et al. [9] and the only problem that could be solved was the so called quasi equilibrium² flow cases where it could be assumed that \( \zeta \) was sufficiently small to neglect its effect even though \( \Pi \) is permitted to vary with \( x \). Thus the problem reduces to considering the relation, 
\[
C[\Pi, \beta, S] = 0 \tag{8}
\]
where for a given \( \Pi \) it is assumed that the velocity defect distribution is fixed and the shear stress distribution is fixed (approximately). Hence from data, if we know \( \beta \) at a given \( S \) for a fixed \( \Pi \) (i.e. for one experimental data point), then for this fixed \( \Pi \) we can find \( \beta \) versus \( S \) for all \( S \) using equation (3) to ensure that \( \tau /\tau_0 \) profiles are matched (approximately for all \( S \)). It is found that for \( S \) sufficiently large \( \beta = \beta_a \) (the asymptotic value of \( \beta \)) and \( C \) is no longer a function of \( S \). If this procedure is repeated for different values of \( \Pi \), a one-to-one relationship between \( \beta_a \) and \( \Pi \) can be found which is based on experiments. This formulation is consistent with the universal relation for eddy viscosity i.e. \( \partial/\partial \zeta = \phi(\Pi, \Pi) \). Unfortunately such formulations are known to break down in non-equilibrium flows, i.e. flows with significant \( \zeta \) contribution - see Marusic & Perry [5] for an example.

In the case of zero pressure gradient boundary layers, 
\[
\begin{align*}
U_0 - U_1 & = \chi[\beta, K] = 1 \\
K & = \frac{\nu}{U_1^2} \frac{dU_1}{dx}
\end{align*} \tag{9}
\]
and the acceleration parameter, \( K = 0 \), because the free stream velocity \( (U_1) \) is invariant with respect to \( x \) (i.e. \( dU_1/dx = 0 \)). Therefore, in summary, the streamwise evolution of ZPG flow can be computed using equations (5), (6) and (7). Here we have to solve three equations to find four unknowns namely \( S, \Pi, \zeta \) and \( \beta \). This is not possible unless we have a fourth equation comprising all the unknown parameters and therefore the fourth (i.e., closure) equation has to be derived empirically, which is of the following form, 
\[
F[\Pi, S, \beta, \zeta] = 0. \tag{10}
\]
For ZPG turbulent boundary layer, Clauser’s pressure gradient parameter, \( \beta = 0 \) which reduces (10) to the following form, 
\[
F[\Pi, S, \zeta] = 0. \tag{11}
\]
Now, a two parameter family of shear stress profiles of the form, 
\[
\frac{\tau}{\tau_0} = f[\eta, \Pi, \zeta, a] \tag{12}
\]
is suggested and when used in conjunction with (3) some information can be obtained regarding formulation (10) as follows. Consider the \((S, \zeta)\)-plane. If such a plane contains an experimental data point, then \( S, \Pi \) and \( \zeta \) are known for that data point and so also is \( \tau/\tau_0 \) versus \( \eta \) from (3). Trace out a curve for increasing \( S \) of fixed shear stress profile shape on the \((S, \zeta)\)-plane. By taking \( S \to \infty \) we obtain asymptotic values of \( \zeta \) (i.e., \( \zeta_a \)).

²An equilibrium boundary layer, according to the definitions of Townsend [12, 13] and Rotta [11] requires all mean-relative motions and energy containing components of turbulence (for example, Reynolds shear stress and the turbulent intensities) to have distributions that become invariant with streamwise development when scaled with local length and velocity scales - see Narasimha & Prabhu [7].

Going to \( S \to \infty \) is simply a convenient curve-fitting procedure and cannot be approached experimentally.

This process of keeping the profile shape fixed will be referred to as profile matching which was given by Perry et al [10], i.e.,
\[
\frac{d}{d\eta} \left[ \int_{0}^{\eta} \left( \frac{\tau}{\tau_0} - \frac{\zeta}{\tau_0} \right)^2 d\eta \right] = 0 \tag{13}
\]
where \( \tau/\tau_0 \) and \( \zeta/\tau_0 \) are the shear stress distribution at any point on the \((S, \zeta)\)-plane for a fixed \( \Pi \) and at a known datum point respectively. Taking (13) to \( S \to \infty \) means that \( \partial/\partial \zeta = \partial/\partial \zeta_a \) as the derivative and we can show generally that
\[
A[\Pi, S] + B[\Pi, S] \zeta = C[\Pi] \zeta_a \tag{14}
\]
where \( A, B \) and \( C \) are known analytical functions. Their precise form depends on the law of wall-wake formulation (4). If this process is repeated for different \( \Pi \) values then we obtain a relationship between \( \Pi \) and \( \zeta_a \) and we thus have a known function \( \Psi \).
\[
\Psi[\Pi, \zeta_a] = F(\Pi, S \to \infty, \zeta_a) = 0 \tag{15}
\]
which is the asymptotic closure equation.

**Experimental Setup**

In order to evaluate the evolution calculation procedure, a series of mean velocity profile measurements were made along the length of the flat plate and the mean flow parameters such as \( \Pi, \delta, S, \zeta \) and \( U \) were obtained by fitting the equation (4).

The experiments were performed in an open blower wind tunnel at The University of Melbourne. The wind tunnel has a settling chamber containing honeycomb and screens. It has a contraction area with a ratio of 8.9:1 that leads into an initial inlet section area of 940mm × 375mm. The roof of the wind tunnel is fully adjustable to change the pressure gradient accordingly, and the tunnel has a working section length of 6700 mm. For this

![Figure 1: Co-efficient of pressure (Cp) along the wind tunnel.](image-url)
Table 1: Mean flow parameters for 4 out of 16 stations are shown. Flow parameters such as $\Pi$, $\delta$ and $U$ are obtained by fitting (4) on the mean velocity profile. $l$ is the non dimensionalised wire length. $R_{\delta}$, $R_{\theta}$ and $R_x$ are Reynolds numbers based on displacement thickness, momentum thickness and streamwise distance respectively.

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Results and Discussion

In this experimental study, a flow case with a free stream velocity of nominally 10 m/s was investigated with a total of 16 mean-flow stations. All of the measurements are performed using single hot-wire constant temperature anemometry (CTA). The sensor used was platinum-wollaston wire of 5 $\mu$m diameter with an overheat ratio of 1.8. Mean flow parameters for 4 out of 16 stations are shown in Table (1).

Figure 2: Evolution of wake parameter ($\Pi$) for different Non dimensionalised free stream velocity ($S$). Box (2) symbols represent the experimental data points, the solid line corresponds to evolution calculations with same initial conditions as the experimental data and dotted lines correspond to evolution calculations with different initial conditions.

The shear-stress profile matching technique was used to find the corresponding $\zeta_\theta$ values for each experimental station. In order to calculate the evolution of this flow a functional relationship for (15) needs to be established. Using the experimental data of this study a second order polynomial is proposed for (15)

$$\zeta_\theta = -4.18\Pi^2 + 5.79\Pi - 1.81 \quad (17)$$

By using equations (14) and (17), (5) and (6) can now be solved. Figures (2), (3) and (4) show typical calculations for the evolution of ZPG turbulent boundary layers from different initial conditions. From figures (2) and (3), we can see that the ZPG turbulent boundary layer is not an equilibrium layer, at least for finite $S$, since $\Pi$ varies with $R_x$ rapidly. Coles [2] notes that for $R_\theta$ less than about 2500, $\Pi$ drops significantly for decreasing $R_\theta$. This effect has been incorporated into the analysis, though this should have a negligible effect on flow at high $R_x$. Figures (3) and (4) show the evolution of parameters $\Pi$ and $S$ versus $R_x$. Figure (5) shows the evolution of Reynolds number based on momentum thickness versus Reynolds based on $x$. It is obvious that $R_\theta$ grows linearly with respect to stream wise Reynolds number ($R_x$). To help determine the range in which (17) might be valid, a series of initial conditions were tried. As can be seen in the figures (2) and (3) solution trajectories become constant at about $\Pi = 0.9$ as expected. (It is noted that the value of $\Pi$ depends on the wake formulation used. In this study equation (4) was used. If we use the Coles’ [1] wake function, solution trajectories become constant at about $\Pi = 0.55$.)
Conclusion

A framework is described that allows us to study streamwise evolution of boundary layers evolving from specific upstream conditions. The framework involves using the law of the wall, the law of the wake, together with the momentum integral and differential equations. The evolution calculation scheme is compared to an experimental ZPG boundary layer evolving over a 6m plate and good agreement is observed.

References