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Reduced order estimation of nonstationary flows with electrical impedance tomography

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Abstract

We consider the simultaneous reconstruction of a nonstationary concentration distribution and the underlying nonstationary flow field. As the observation modality, we employ electrical impedance tomography. Earlier studies have shown that such an estimation scheme is in principle possible since the evolution of an inhomogeneous concentration carries information also on the velocity field. These results have, however, been restricted to either stationary velocity fields or simplified non-physical models. In the general case, the estimation of the velocity field up to the fine details of the flow with diffuse tomography is impossible. In this paper we show, however, that it is possible to estimate a reduced order representation of a physical fluid dynamics model, here the Navier-Stokes model, simultaneously with the concentration. This is accomplished by considering a proper orthogonal decomposition representation for the velocity field, and careful modelling of the uncertainties of the models, in particular, the subspace of the velocity field that is not estimated.

Introduction

Inverse problems are characterized as problems that tolerate measurement and modelling errors poorly, and are thus unstable. With inverse problems in which there are significant measurement errors and/or model uncertainties, the Bayesian framework for inverse problems is often adopted. In this framework, the modelling of the measurement process and the explicit modelling of the primary (interesting) uncertainties as well as the secondary unknowns and uncertainties, are carried out separately [7]. Nonstationary inverse problems are a special class of Bayesian inverse problems in which the primary unknown can be modelled with a stochastic evolution model, typically a stochastic PDE. These problems can be tackled with sequential Bayesian estimation methods and, in special cases, recursive algorithms such as the Kalman filter and its variants.

Problems that are governed by stochastic convection-diffusion (CD) models in fluids have served as standard examples for sources of nonstationary inverse problems [15, 14, 16]. In these problems, the stochasticity of the models is typically due to unknown boundary data for the CD model or the fluid dynamics model, unknown velocity field, and highly approximate reduced order models for the primary unknowns.

Partial solutions to the stochastic CD problems with electrical impedance tomography, and treating the velocity field as a nuisance parameter, have been studied earlier. The state estimation problem governed by a convection-diffusion model under uncertainties in boundary data has been discussed in a series of papers starting with [15]. In [16], the state estimation scheme was verified in a 3D pipeline flow and the problem has been also verified with real data.

The problem of simultaneously estimating partial information on the flow was considered in [17]. In [17], the projection of the velocity field onto a low-dimensional subspace of 1-3 dimensions was considered. Such low-dimensional approximations are in this context often referred to as *reduced order models* [2]. It was found that such a reduced order model could be identified simultaneously with the concentration. The reduced order subspaces were, however, ad hoc choices, and a nonphysical random walk model was used as the model for the evolution of the projection coefficients.

In this paper, we consider the problem of simultaneous estimation of the concentration and a reduced order approximation for the unknown nonstationary velocity field. In particular, we consider Navier-Stokes flows. Moreover, all unavoidable uncertainties, errors that are due using coarse finite element approximations, as well as the non-estimated part of the velocity field, are treated using the nonstationary approximation error approach.

Electrical Impedance Tomography

For physically realizable quasi-stationary EIT measurements, the complete electrode model (CEM) [1] is the best available measurement model. The CEM is of the form:

$$\nabla\cdot(\sigma\nabla u) = 0, \quad \vec{r}\in\Omega \tag{1}$$

$$u+z^{(\ell)}\sigma\frac{\partial u}{\partial\vec{n}} = U^{(\ell)}, \quad \vec{r}\in e^{(\ell)},\ \ell=1,2,\ldots,L \tag{2}$$

$$\sigma\frac{\partial u}{\partial\vec{n}} = 0, \quad \vec{r}\in\partial\Omega\backslash\Gamma_{\Omega,e} \tag{3}$$

$$\int_{e^{(\ell)}}\sigma\frac{\partial u}{\partial\vec{n}}\,\mathrm{d}S = I^{(\ell)}, \quad \vec{r}\in e^{(\ell)},\ \ell=1,2,\ldots,L \tag{4}$$

where $\sigma=\sigma(\vec{r})$ is the conductivity distribution, $u=u(\vec{r})$ is amplitude of the electrical potential in target domain Ω, and L is the number of the electrodes. $U^{(\ell)}$, $I^{(\ell)}$, and $z^{(\ell)}$, respectively, are the electric potential, current and contact impedance corresponding to the ℓ^{th} electrode $e^{(\ell)}$. Further, $\Gamma_{\Omega,e}=\cup_{\ell=1}^{L}e^{(\ell)}$ is the union of boundary patches covered by the electrodes.

The relationship between the noiseless measurements U, the conductivity σ and the current pattern I can be written as $U=\bar{R}(\sigma)I$. In this paper, we consider the additive *measurement noise model*

$$V=\bar{R}(\sigma)I+e \tag{5}$$

For numerical solution, for example, the finite element method would be applied to the variational form of CEM.

In the case of time-depending conductivity, we tag all variables with time

$$V_t=\bar{R}(\sigma_t)I_t+e_t \tag{6}$$

where the subscript t is a discrete time index referring to the time instant of the measurement. The measurements are obtained at discrete times. We denote $R_t(\sigma)=\bar{R}(\sigma)I_t$.

The evolution model will be formulated in terms of concentration $c(\vec{r},t)$ of an electrically conducting substance. The observation model in terms of concentration is the related composite map

$$V_t=(R_t\circ\sigma)(c_t)+e_t=R_t^*(c_t)+e_t \tag{7}$$

Nonstationary Inverse Problems and Approximation Errors

Nonstationary inversion refers to problems in which the unknown is a time-varying quantity and the measurements are obtained sequentially. With a nonstationary inverse problem, the measurements that are obtained at a time, often do not allow for reasonable reconstructions of the unknown target. If a feasible stochastic evolution model for the target, however, can be constructed, the reconstruction problem can be recast as a statistical state estimation problem. For a general description of the state estimation formulation of nonstationary inverse problems, see [7].

Most of the nonstationary inverse problems are related to transport phenomena, such as fluid flows in process industry (process tomography) [15, 5]. Process tomography refers to a variety of imaging techniques used in process industry. Process tomography has been applied to such end uses as process monitoring, control, and design. In particular, process tomography has applied to imaging of mixers [3, 12], separators [6], chemical reactors [18], and industrial pipelines and vessels [13, 19]. Several imaging modalities have been used in process tomography, with the most common modalities having been related to electromagnetic probing, that is capacitance tomography and impedance tomography [20].

Let X_t be the unknown *state variable* and $t \in \mathbb{N}$. Let the sequence of measurements be Y_t. In sequential Bayesian estimation, the task is to model the posterior distribution $\pi(X_t \,|\, D)$ where $D = \{Y_\ell,\ \ell \in I\}$, and where I is a set of time indices. Most commonly, one is to consider the *filtering problem* in which $I = \{1, \ldots, t\}$ and we denote $D_t = \{Y_\ell,\ \ell = 1, \ldots, t\}$.

Let the discrete-time *(state) evolution model* and the *observation model* be

$$\begin{aligned} X_{t+1} &= F_t(X_t, W_t) && (8)\\ Y_t &= G_t(X_t, E_t) && (9) \end{aligned}$$

respectively. In (8-9), F_t and G_t are deterministic models, and W_t and E_t are the *state noise* and *observation noise* processes, respectively. For (8-9) to be a feasible description for the problem, the state and observation noise processes have to be modelled so that they represent *all* uncertainties in these models.

Typical uncertainties that are related to the evolution and/or the observation model, include unknown boundary data [9, 15], uninteresting distributed parameters [8], and the geometry of the domain [11]. Naturally, the formulation of such problems could be done so that (parametrizations) of these uncertainties are taken as unknowns to be estimated simultaneously with the primary interesting unknowns. This approach will, however, often lead to a computationally significantly more complex problem.

For the computationally efficient modelling of auxiliary unknowns, as well as discretization and other model reduction related errors, the *approximation error approach* was proposed in [7] in which, for example, EIT and deconvolution type problems were considered. The approximation error approach has also been successfully applied to optical tomography and model reduction (discretization), anisotropy, handling unknown boundary data, and has also been verified with real EIT data under model reduction, geometry related uncertainties and unknown contact impedances. The extension of the approximation error approach to nonstationary problems has also been developed in which the model reduction, long time stepping and other computational problems were under particular consideration. The computational scheme in this paper conforms to these papers. For references of approximation error approach, see for example [10].

Reduced Order Navier-Stokes Model

For reduced order Navier-Stokes, see for example [2]. Let us write

$$\begin{aligned} \vec{v}(\vec{r},t) &\approx \vec{v}_0(\vec{r}) + \sum_{i=1}^{p} \beta_i(t)\vec{v}_i(\vec{r}) + \sum_{i=p+1}^{N} \beta_i(t)\vec{v}_i(\vec{r}) && (10)\\ &= \vec{v}_0 + \vec{v}^p + \vec{v}^{\tilde{p}} && (11) \end{aligned}$$

In reduced order NS, the velocity field is considered as a stochastic flow and the basis $\{\vec{v}_i\}$ is chosen so that $\mathbb{E}_{\vec{r},t}\|\vec{v} - (\vec{v}_0 + \vec{v}^p)\|^2$ is minimized over all $p+1$-dimensional bases, where the notation $\mathbb{E}_{\vec{r},t}\|\vec{v} - (\vec{v}_0 + \vec{v}^p)\|^2$ refers to the ergodic mean (we have $\Omega \subset \mathbb{R}^2$)

$$\mathbb{E}_{\vec{r},t}\|\vec{v} - (\vec{v}_0 + \vec{v}^p)\|^2 \approx \frac{1}{T}\int_0^T \int_\Omega \|\vec{v}(\vec{r},t) - (\vec{v}_0(\vec{r}) + \vec{v}^p(\vec{r},t))\,\|_{\mathbb{R}^2}^2 \, d\vec{r}\, dt$$

Thus, the POD basis is such that the projection error (as squared norm of the velocity field) is minimized (uniformly) over Ω and over time interval $t \in (0, T)$.

Stochastic Convection-Diffusion Model

The stochastic convection-diffusion equation has earlier been considered in the construction of the evolution model in [15] and we refer for the details to this paper.

The convection-diffusion (CD) equation is of the form

$$\frac{\partial c}{\partial t} = \nabla \cdot \kappa \nabla c - \vec{v} \cdot \nabla c \qquad (12)$$

where $\kappa = \kappa(\vec{r})$ is the diffusion coefficient. In this paper, we consider flows in pipelines and set the following boundary conditions

$$\begin{aligned} c &= c_{\text{in}}, \quad \vec{r} \in \Gamma_{\Omega,\text{in}} && (13)\\ \frac{\partial c}{\partial \vec{n}} &= 0, \quad \vec{r} \in \left(\partial\Omega \backslash \Gamma_{\omega,\text{in}}\right) && (14) \end{aligned}$$

where $c_{\text{in}} = c_{\text{in}}(\vec{r},t)$ is an unknown function, $\Gamma_{\Omega,\text{in}}$ is the input boundary where the flow enters the domain Ω. On the output boundary, the homogeneous Neumann condition is also approximately valid, see [15].

The primary task is usually to estimate the time-varying concentration distribution $c(\vec{r},t)$ inside Ω. Note that if the velocity field $\vec{v}$ and the Dirichlet data on the input boundary were known, we would not need any (EIT or other) measurements to infer the state of the system, that is, the concentration.

Carrying out semidiscrete FEM formulation for the CD model and retaining the explicit dependence on the input boundary condition, and then employing the implicit Runge-Kutta scheme, we arrive at

$$\begin{aligned} c_{t+1} &= c_t + \Delta t \sum_{i=1}^{s} \psi_i \vartheta_i && (15)\\ \vartheta_i &= F\left(c_t + \Delta t \sum_{j=1}^{s} \alpha_{ij}\vartheta_j,\ c_{\text{in},t}, c_{\text{in},t-1}\right) && (16) \end{aligned}$$

where α_{ij} and ψ_i are known constants depending on the integration method we are to use, and s is the number of stages in time integration, and $\vartheta_k = \vartheta_k(c_t, \beta^p, c_{\text{in},t}, c_{\text{in},t-1})$. In particular, we choose the 3-stage, 4th order singly diagonally implicit (SSP-SDIRK34) variant which allows the use of reasonably large time steps without stability problems.

Simultaneous Estimation of Time-varying Conductivity and Nonstationary Flow

The state variable is now $X_t = (\beta_t^p, c_t)$ and we define the deterministic mapping $F: X_t \mapsto X_{t+1}$ that is induced by (11) and (15-16). Note that the deterministic evolution model for concentration depends also on the unknown concentration on the input boundary, that is, $(c_{\mathrm{in},t}, c_{\mathrm{in},t-1})$, which are neither known or listed as variables to be estimated. In the nonstationary approximation error approach, the input boundary data are modelled as one of the stochastic processes. Thus, we write the *stochastic evolution model* in the form

$$X_{t+1} = F(X_t) + W_t + Q_t \tag{17}$$

where the process W_t is induced by the model for the evolution of the input boundary through (15-16), see [15], and Q_t is a process related to other errors and uncertainties, such as pure numerical approximation errors. Note also that the model F does not depend on time explicitly, $F(X_t)$ depends on time only through the time evolution of X_t.

With an additive model for the observation errors E_t in (9), the non-linear observation model is of the form

$$Y_t = V_t = G_t(X_t) + E_t + V_t \tag{18}$$

where $G_t(X_t) = R_t^*(c_t)$, E_t represents the (pure) measurement errors, and V_t is due to the numerical approximation of the forward model $R^*(c_t; I_t)$ as well as the contribution of the noise processes in (17). The equations (17-18) define the state space representation of the problem, in which the variables $X_t = (\beta_t^p, c_t)$ are treated as the state variables to be estimated, and the approximation errors are modelled.

The adoption of a Kalman filter or its approximative nonlinear extended Kalman filter variants calls for feasible modelling of the second order statistics of the associated state and observation noise processes. Usually, ad hoc models are used for the state noise and pure measurement noise only is considered as the observation noise. In the approximation error approach, these process noise models are constructed based either on analytical derivations which are applicable only in linear additive Gaussian noise cases, or based on Monte Carlo simulations over the *modelled distribution of all uncertainties*, see [10]. In this paper, this is carried out via the simulation of a Navier-Stokes flow and the convection-diffusion process with a stochastic model for the input boundary data, as well as simulating the prediction errors of models with different refinement levels.

Let $X_{t|t}$ and $X_{t|t-1}$ denote the EKF filter and predictor estimates, respectively. Further, let $\Gamma_{t|t}$ and $\Gamma_{t|t-1}$ denote the (approximate) EKF filter and predictor covariances, respectively. The recursive EKF algorithm in our case is of the form

$$\begin{aligned}
X_{t|t-1} &= F(X_{t-1|t-1}) + \mathbb{E}Q_t + \mathbb{E}W_t \\
\Gamma_{t|t-1} &= J_{F_t}\Gamma_{t-1|t-1}J_{F_t}^{\mathrm{T}} + J_{F_t}\mathrm{Cov}(X_{t-1}, Q_t) + \Gamma_{W_t} + \\
&\quad \mathrm{Cov}(W_t, Q_t) + \mathrm{Cov}(Q_t, X_{t-1})J_{F_t}^{\mathrm{T}} + \mathrm{Cov}(Q_t, W_t) + \Gamma_{Q_t} \\
K_t &= \left(\Gamma_{t|t-1}J_{R_t^*}^{\mathrm{T}} + \mathrm{Cov}(X_t, V_t)\right)\left(J_{R_t^*}\Gamma_{t|t-1}J_{R_t^*}^{\mathrm{T}} + \Gamma_{E_t}\right. \\
&\quad \left. + \Gamma_{V_t} + J_{R_t^*}\mathrm{Cov}(X_t, V_t) + \mathrm{Cov}(V_t, X_t)J_{R_t^*}^{\mathrm{T}}\right)^{-1} \\
\Gamma_{t|t} &= \left(I - K_t J_{R_t^*}\right)\Gamma_{t|t-1} - K_t\mathrm{Cov}(X_t, V_t) \\
X_{t|t} &= X_{t|t-1} + K_t\left(V_t - R_t^*\left(X_{t|t-1}\right) - \mathbb{E}V_t - \mathbb{E}E_t\right)
\end{aligned}$$

where $\mathrm{Cov}(X,Y)$ is the cross-covariance of X and Y, and J_H refers to the Jacobian mapping of H (and other mappings).

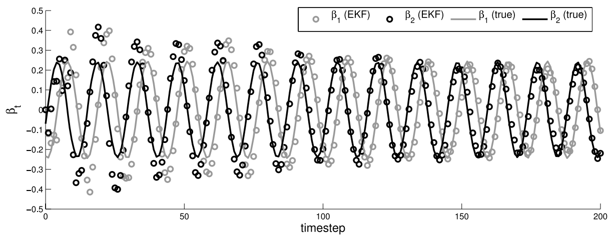

Figure 1: Estimated and true coefficients $\beta_{1,t}$ and $\beta_{2,t}$ describing the velocity field.

In practice, the cross-covariances have to be set to zero matrices. This is due to the problem that the filter and predictor covariances lose their non-negative definite property for numerical reasons, thus making the algorithm unstable [4].

A Numerical Example

We consider the von Karman street, which involves flow past a cylinder with Reynolds numbers between about 80 and 170 [21]. For the estimation of (the principal part of) the velocity field, we estimate two components only, that is, we set $p = 2$ and estimate β_t^2. This choice leaves about 4% of the variance of the flow for the remainder.

We set the Reynolds number to Re = 100. As the measurement protocol, an alternating sequence of two current patterns only (across the pipeline) is employed.

In Figure 1, the $p = 2$ state estimates for the coefficient evolutions $\beta^2(t)$ are shown. The corresponding state estimates for the concentration are shown in Fig. 2. The concentration estimates are relatively good from quite early on. The estimates for β^p seem to recover from the incorrect initial conditions and covariances by about the 100$^{\text{th}}$ time step.

Conclusions

We have considered the simultaneous estimation of a time-varying concentration distribution and a reduced order model for an associated nonstationary Navier-Stokes flow. Based on the numerical studies, the estimation scheme is in principle feasible. The approach is based on the modelling of the related uncertainties in the Bayesian nonstationary inversion framework. In particular, the computational feasibility of the proposed scheme is based on the nonstationary approximation error approach, in which approximate marginalization over the uncertainties and errors is carried out.

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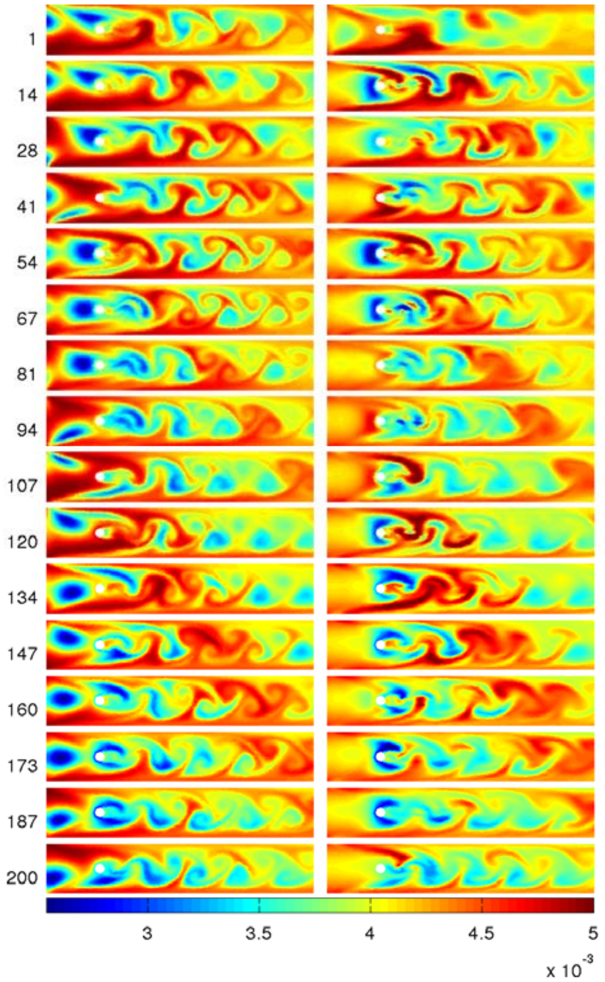

Figure 2: Left column: True concentration distribution. Right column: Estimated concentration distribution. Each row corresponds to a different instant of time.

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