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Do we find hurricanes on other planets ?

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Abstract

Vortices with intense rotation occur in nature at very different scales, with the bathtub vortex representing one of the smallest and atmospheric vortices – tornadoes, mesocyclones and cyclones – representing vortices of much greater scales. Large vortices have been also found in the atmospheres of other planets – the famous Jovian Great Read Spot (GRS), whose size exceeds the Earth diameter, has been observed for more than 300 years. Fujita [10] in his classical work on vortices in planetary atmospheres introduced a unified treatment of the vortical motion of different scales starting from a lab vortex (that is referred to here as a bathtub vortex) and finishing with the largest known vortex of GRS. The vortices were classified according to their scales, and vortical motions of this kind are viewed by Fujita as a truly universal feature of the nature. Modern science tends to view these vortices as completely different phenomena and has good reasons for this: the vortices are characterized not only by different scales but also by different levels of buoyancy, turbulence and axial symmetry present in the flow. Thus, although we cannot expect that any common approach can fully characterize the whole structure of these vortices, this does not eliminate the possibility of finding common explanations for certain features of the vortices. The term "bathtub-like vortical flows" is used to characterize axisymmetric vortices with significant intensification of rotation at the center due to converging secondary motion present in the flow.

We first give the overview of the theory which is based on asymptotic analysis of the evolution of vorticity in axisymmetric bathtub-like flows under assumptions of prime influence of the convective terms. If the axial vorticity is sufficiently strong, the bathtub-like flows are expected to be controlled by the compensating regime that prevents further increases of the relative rotation strength. We examine applicability of this theory to certain intermediate regions of large atmospheric vortices (tornadoes and hurricanes) and compare theoretical predictions with atmospheric measurements. We also discuss vortices observed on other planets.

Equations governing axisymmetric vortical flows

The normalized system of equations governing axisymmetric incompressible flows with vorticity can be written in the form [1, 14, 15, 16]

$$L_*^2\frac{\partial^2\Psi}{\partial Z^2}+R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right)=-R\Omega_\theta \tag{1}$$

$$\Omega_\theta=L_*^2\frac{\partial V_r}{\partial Z}-\frac{\partial V_z}{\partial R},\ \frac{D\Omega_\theta/R}{DT}=-2K_*^2\frac{\Gamma\Omega_r}{R^3}+\frac{(...)}{\mathrm{Re}_*} \tag{2}$$

$$\frac{D\Gamma}{DT}=\frac{(...)}{\mathrm{Re}_*},\ \frac{D}{DT}\equiv\mathrm{St}_*\frac{\partial}{\partial T}+V_z\frac{\partial}{\partial Z}+V_r\frac{\partial}{\partial R} \tag{3}$$

$$V_z=\frac{1}{R}\frac{\partial\Psi}{\partial R},\ V_r=\frac{-1}{R}\frac{\partial\Psi}{\partial Z},\ \Omega_r=\frac{-1}{\mathrm{St}_*R}\frac{\partial\Gamma}{\partial Z},\ \Omega_z=\frac{-1}{\mathrm{St}_*R}\frac{\partial\Gamma}{\partial R} \tag{4}$$

where the dimensionless parameters — the Reynolds number, the Strouhal number, the rotation vorticity number and the geometrical parameter — are introduced as

$$\mathrm{Re}_*=\frac{L_*v_*r_*}{\nu},\ \mathrm{St}_*=\frac{r_*^2\omega_*}{\gamma_*},\ K_*=\frac{(\gamma_*\omega_*)^{1/2}}{v_*},\ L_*\equiv\frac{r_*}{z_*} \tag{5}$$

The viscous terms are shown by dots in (1)-(3). The cylindrical coordinates z, r, θ, the velocities v_z, v_r, v_θ the vorticity ω_z, ω_r, ω_θ, the stream function ψ and the circulation $\gamma=v_\theta r$ are normalized according to

$$R=\frac{r}{r_*},\ Z=\frac{z}{r_*/L_*},\ V_r=\frac{v_r}{v_*L_*},\ V_z=\frac{v_z}{v_*},\ \Psi=\frac{\psi}{\psi_*},\ \Omega_\theta=\omega_\theta\frac{r_*}{v_*},$$

$$T=\frac{t}{t_*},\ v_*=\frac{\psi_*}{r_*^2},\ \Gamma=\frac{\gamma}{\gamma_*},\ \Omega_z=\frac{\omega_z}{\omega_*},\ \Omega_r=\frac{\omega_r}{\omega_*L_*} \tag{6}$$

where $t_*=\gamma_*/(\omega_*v_*r_*L_*)$. The characteristic values and parameters based on these values are indicated by asterisk.

The rotational vorticity parameter $K_*=(S_*/\mathrm{Ro}_*)^{1/2}$ represents a geometrical average of the swirl ratio ($S_*\equiv\gamma_*/(v_*r_*)$) and the inverse Rossby number ($\mathrm{Ro}_*\equiv v_*/(r_*\omega_*)$). The parameter K_* determines the rate of generation of tangential vorticity Ω_θ by equation (2). If K_* is small, the flow on the planes passing through the axis must be close to potential (the term potential is used here to indicate that $\Omega_\theta\approx 0$); while the high values of K_* correspond to the strong vortex approximation [5, 17, 18, 14, 15, 16]. As vorticity is accumulated near the axis, γ_* increases, St_* becomes small and the flow can be treated as quasi-steady (but not fully steady).

Strong vortex approximation

Different aspects of the strong vortex solution for axisymmetric flows were repeatedly considered in publications [5, 17, 18, 14, 15, 16]. This approximation is characterized by strong vorticity in the flow so that $1/K_*^2$ can be assumed to be small. The case of small K_* resulting in $\Omega_\theta\approx 0$ is quite simple and less interesting for bathtub-like vortical flows. The relatively slow rate of evolution that is common for bathtub-like flows can be mathematically expressed by the condition $\mathrm{St}_*\ll 1$. We follow the analysis of refs. [14, 15, 16], which involves higher-order terms related to velocity/vorticity interactions. The quasi-steady version of the strong vortex approximation is obtained by expanding all values (Ψ, V_z, V_r, Γ, $\mathrm{St}_*\Omega_z$, $\mathrm{St}_*\Omega_r$, Ω_θ,) according to

$$(\cdot)=(\cdot)_{00}+K_*^{-2}(\cdot)_{10}+\mathrm{St}_*(\cdot)_{01}+K_*^{-2}\mathrm{St}_*(\cdot)_{11}+...,\tag{7}$$

Several terms in these expansions (specifically Ψ_{01}, Γ_{10} and the corresponding dependent terms V_{z01}, Ω_{z10}, etc.) are not needed and can be set to zero. We note first the following relationships

$$V_{zij}=\frac{1}{R}\frac{\partial\Psi_{ij}}{\partial R},\ V_{rij}=\frac{-1}{R}\frac{\partial\Psi_{ij}}{\partial Z},\ \Omega_{rij}=\frac{-1}{R}\frac{\partial\Gamma_{ij}}{\partial Z},\ \Omega_{zij}=\frac{1}{R}\frac{\partial\Gamma_{ij}}{\partial R},$$

$$R\Omega_{\theta ij}=-R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi_{ij}}{\partial R}\right)-L_*^2\frac{\partial^2\Psi_{ij}}{\partial Z^2},\ V_{r10}\frac{\partial\Gamma_{00}}{\partial R}=0 \tag{8}$$

The leading and following order equations are given by

$$\Gamma_{0i}=\Gamma_{0i}(R,T),\ \Omega_{r0i}=0,\ \Psi_{0i}=F_0(R,T)_i+F_1(R,T)_iZ,$$

$$\Psi_{01} = 0, \ \Gamma_{10} = 0, \ \frac{D_{00}}{DT} \equiv V_{z00}\frac{\partial}{\partial Z} + V_{r00}\frac{\partial}{\partial R} \tag{9}$$

$$V_{r00}\frac{\partial \Gamma_{00}}{\partial R} = \frac{(...)}{\mathrm{Re}_*}, \ \frac{\partial \Gamma_{00}}{\partial T} + V_{r00}\frac{\partial \Gamma_{01}}{\partial R} = \frac{(...)}{\mathrm{Re}_*},$$

$$2\frac{\Gamma_{00}}{R^3}\Omega_{r11} = -\frac{D_{00}\Omega_{\theta 00}/R}{DT} + \frac{(...)}{\mathrm{Re}_*},$$

$$V_{r11}\frac{\partial \Gamma_{00}}{\partial R} + V_{r10}\frac{\partial \Gamma_{01}}{\partial R} = -\frac{D_{00}\Gamma_{11}}{DT} + \frac{(...)}{\mathrm{Re}_*} \tag{10}$$

where $i, j = 0, 1$.

We are interested in the intensification region, which is located between the inner (core) and outer (peripheral) scales of the vortex and characterized by significant intensification of rotational motion as the fluid flows towards the axis of the flow. The convective evolution of vorticity is presumed to be of prime importance for the intensification region. The inviscid approximation of the quasi-steady strong vortex is now considered to obtain a solution for the flow in the intensification region. We put $\mathrm{Re}_*^{-1} = 0$ and simplify equations (9)-(10)

$$\Gamma_{00} = \Gamma_{00}(T), \ \Gamma_{01} = \Gamma_{01}(R,T), \ \Omega_{r00} = \Omega_{r01} = 0, \ \Gamma_{10} = 0,$$

$$\Psi_{00} = F_0(R,T) + F_1(R,T)Z, \ \Psi_{01} = 0 \tag{11}$$

$$\frac{\partial \Gamma_{00}}{\partial T} = -V_{r00}\frac{\partial \Gamma_{01}}{\partial R} = -V_{r00}\Omega_{z01}R,$$

$$2\frac{\Gamma_{00}\Omega_{r11}}{R^3} = -\frac{D_{00}\Omega_{\theta 00}/R}{DT},$$

$$V_{r10}\frac{\partial \Gamma_{01}}{\partial R} = V_{r10}\Omega_{z01}R = -\frac{D_{00}\Gamma_{11}}{DT} \tag{12}$$

In a bathtub-like vortical flow $F_0 = 0$ since $Z = 0$ is a streamline. Assuming that the stream function can be represented by a power law $F_1 \sim R^\alpha$ with the exponent α unknown a priori, we find the following consistent expressions

$$\Psi_{00} = C_0R^\alpha Z, \ V_{r00} = -C_0R^{\alpha-1}, \ V_{z00} = \alpha C_0 R^{\alpha-2}Z,$$

$$\Omega_{\theta 00} = -\chi C_0 R^{\alpha-3}Z, \ \Omega_{z01} = \frac{1}{R}\frac{\partial \Gamma_{01}}{\partial R} = -\frac{\Gamma'_{00}}{RV_{r0}} = \frac{\Gamma'_{00}}{C_0R^\alpha},$$

$$\Gamma_{01} = -\frac{\Gamma'_{00}}{(\alpha-2)C_0R^{\alpha-2}}, \ \Gamma'_{00} \equiv \frac{\partial \Gamma_{00}}{\partial T}, \ \chi \equiv \alpha(\alpha-2)$$

$$\Omega_{r11} = 2\chi\frac{C_0R^{2\alpha-3}Z}{\Gamma_{00}}, \ \Gamma_{11} = -\chi\frac{C_0^2R^{2\alpha-2}Z^2}{\Gamma_{00}},$$

$$V_{r10} = 2\chi\frac{C_0^4R^{4\alpha-5}Z^2}{\Gamma_{00}\Gamma'_{00}}, \ \Psi_{10} = \frac{2}{3}\chi\frac{C_0^4R^{4\alpha-4}Z^3}{\Gamma_{00}\Gamma'_{00}} \tag{13}$$

The asymptotic correctness of the strong vortex approximation is determined by the following parameter

$$\varsigma \equiv \left|\frac{\Psi_{10}}{\Psi_{00}}\right| = \frac{2}{3}\alpha(2-\alpha)\frac{C_0^3Z^2}{\Gamma_{00}\Gamma'_{00}}R^{3\alpha-4} \tag{14}$$

Large values of ς indicate that the asymptotic expansion corresponding to the strong vortex approximation is no longer valid. If $\alpha < 4/3$ and $\alpha \neq 0$ then $\varsigma \to \infty$ as $R \to 0$. Hence, the strong vortex approximation can not be sustained over a wide range of radii for any α less than 4/3 and greater than 0. The physical explanation for this fact is given in the next section.

Note that $\alpha = 2$ and $\alpha = 0$ represent special cases (potential vortex and vortical sink) where the flow image on *r*-*z*-plane is potential and the correcting terms are nullified $\Omega_{r11} = \Gamma_{11} = V_{r10} = \Psi_{10} = 0$. A large K_* is not needed to sustain the flow in this case. Equations (13) are formally valid for $\alpha = 0$ but the case of $\alpha = 2$, which is more interesting for the present study, needs a special treatment:

$$\Psi_{00} = C_0R^2Z, \ V_{r00} = -C_0R, \ V_{z00} = 2C_0Z,$$

$$\Omega_{z01} = \frac{\Gamma'_{00}}{C_0R^2}, \ \Gamma_{01} = \frac{\Gamma'_{00}}{C_0}\ln(R) \tag{15}$$

The compensating regime

The leading terms of the solution (13) are used their dimensional form for the stream function ψ, velocity components v_z and v_r, tangential vorticity ω_θ and the flow convergence λ

$$\psi = c_0r^\alpha z, \ v_r = -c_0r^{\alpha-1}, \ v_z = \alpha c_0 r^{\alpha-2}z,$$

$$\omega_\theta = \alpha(2-\alpha)c_0r^{\alpha-3}z, \ \lambda \equiv -\frac{1}{r}\frac{\partial v_r r}{\partial r} = c_0\alpha r^{\alpha-2} \tag{16}$$

$$\omega_r = 0, \ \omega_z = \frac{-c_1}{rv_r} = \frac{c_1}{c_0r^\alpha}, \ \gamma = \gamma_0(t) + \gamma_1,$$

$$\gamma_1 \equiv \left\{ \begin{array}{ll} \frac{c_1}{c_0}\ln(r), & \alpha = 2 \\ \frac{c_1}{c_0(2-\alpha)}r^{2-\alpha}, & \alpha < 2 \end{array} \right\} \tag{17}$$

The values of α from the range $1 < \alpha \leq 2$ correspond to a bathtub-like flow. Note that $\omega_\theta = 0$ for $\alpha = 2$, while smaller values of α from the range $1 < \alpha < 2$ correspond to greater ω_θ as illustrated by the following equation obtained from (16)

$$2 - \alpha = \frac{r}{v_z}\omega_\theta = r\frac{\partial \ln(v_z)}{\partial r} \tag{18}$$

Vortical flows usually have a sufficiently wide range of radii to create conditions for substantial amplification of the axial vorticity. Since different radii r_* can be characterized by different characteristic values of the parameter K_*, we introduce the local value K defined in terms of local parameters by $K = (\gamma\omega_z)^{1/2}/v_z$. In principle, K may exhibit a strong dependence on r as specified by the equation

$$K^2 \equiv \frac{\gamma\omega_z}{v_z^2} = K_*^2\frac{\Gamma\Omega_z}{V_z^2} \sim \frac{\gamma}{r^{3\alpha-4}} \tag{19}$$

The main argument used here is that the local values of parameter K_* can not be very small or very large in a developed bathtub-like vortical flow due to a velocity/vorticity interaction mechanism that compensates for possible increases and decreases of the parameter. The value of α that ensures that the magnitude of the parameters K is independent of r is denoted α^*. In general, the condition $K \sim$ const does not correspond to exact power law but simple estimates of α^* can be obtained under the limiting conditions of $\gamma_0 \gg \gamma_1$ or $\gamma_0 \ll \gamma_1$

$$\alpha^* = \left\{ \begin{array}{ll} 4/3, & \gamma_0 \gg \gamma_1 \\ 3/2, & \gamma_0 \ll \gamma_1 \end{array} \right\} \tag{20}$$

The regime of $K \sim$ const is called compensating since the flow effectively compensates for possible deviations of α from α^*. Indeed, if $\alpha < \alpha^*$ and $K \to 0$ as $r \to 0$, then no sufficient amount of the tangential vorticity can be generated near the axis and the flow must become potential ($\omega_\theta \approx 0$) there. This leads us to a contradiction since the potential flow corresponds to $\alpha = 2$. If α falls below α^*, K decreases towards the axis as determined by (19) resulting in undergeneration of ω_θ and this increases the effective value of α according to (18). If, on the contrary, $K \to \infty$ as $r \to 0$, then, on one hand, equation (19) requires sufficiently large $\alpha > \alpha^*$ while, on the other hand, the large magnitudes of tangential vorticity ω_θ generated by (2) would, according to

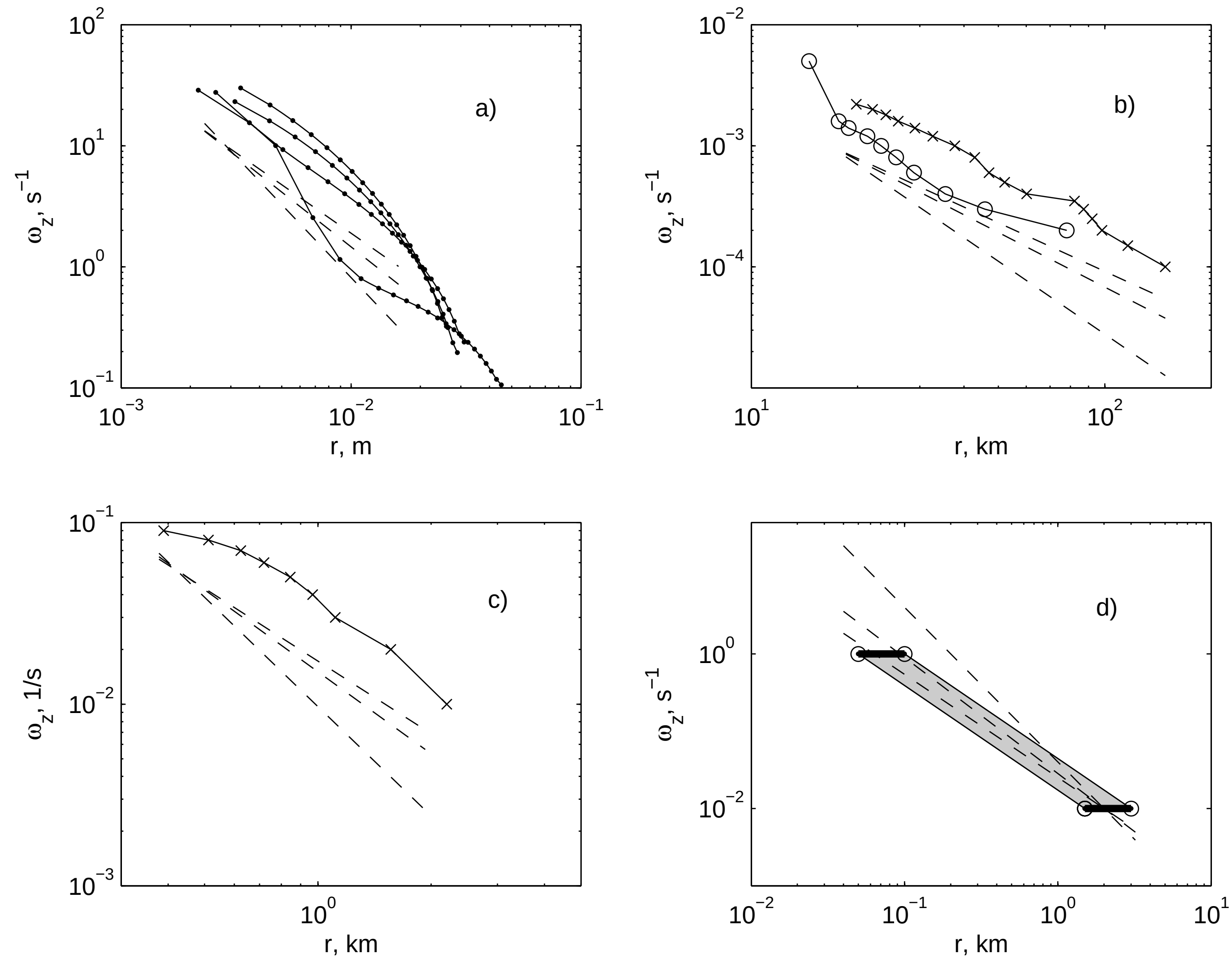

Figure 1: Dependence of axial vorticity on radius for different vortices: a) bathtub vortex [25], b) hurricanes Hilda (x) and Inez (o) [12, 13], c) tornado 4 of McLean storm [8] and d) typical characteristics of large supercell tornados according to [4, 8, 3], outer scales: $\omega_z \sim 0.01\mathrm{s}^{-1}$ at $r \sim$ 1.5-3.5km (scale of 3-7km) and inner (core) scales: $\omega_z \sim 1\mathrm{s}^{-1}$ $r \sim$ 50-100m (scale of 100-200m). In all figures, the dashed lines show three exponents of $\alpha = 2$, $3/2$ and $4/3$ in $\omega_z \sim 1/r^{\alpha}$. The references point to sources of data used in evaluation of the curves.

(18), decrease α below α^*. In addition we note that small values K do not correspond to bathtub-like flow since the case of $\omega_\theta \approx 0$ and $\alpha = 0$ does not have increasing flow convergence towards the vortex center which is observed in developed vortices; while excessively large K near the axis is likely to trigger vortex breakdown and instabilities (as discussed in the following sections). Thus we expect that α falls below 2 to the range of $4/3 \lesssim \alpha \lesssim 3/2$ as vorticity strengthens in bathtub-like vortical flow. This power law is applicable only within the intensification region, where inviscid velocity/vorticity interactions are presumed to be dominant, while the core and the outer regions of the flow may be subject to the dominant influence of (turbulent) viscosity, buoyancy and other case-dependent factors.

In realistic vortices (bathtub vortex, tornadoes, hurricanes, etc) the compensating regime is essentially based on prime importance of the convective evolution of vorticity. This regime is expected to be valid only in asymptotically intermediate region located between the inner and the outer scales. This region is called intensification region due to significant amplification of circular motion occurring there. The inner scale is related to the core (eye) of the flow where the influence of viscosity and buoyancy is most significant. The structure of the vortex at its outer scales is not necessarily symmetric and the outer scale of the compensating regime may of order of few centimeters in case of a bathtub vortex, few kilometers in case of a tornado and as large as 250km in case of a hurricane. The peripheral influence of a strong hurricane extends even further to the radius of $\sim$500km. Examples of vortices different scales are shown in Figure 1. The profiles of axial vorticity evaluated as $\omega_z \sim 1/r^{\alpha}$ according to (17) are compared with the lines of $\alpha = 2$, $3/2$ and $4/3$.

The condition $K \sim$ const and the exponent of $\alpha = 4/3$ were introduced in refs. [15, 16] as a general feature of the intensification region in bathtub-like vortical flows. The fact that the exponent of $\beta = 0.5$ in $v_\theta = \gamma/r \sim 1/r^{\beta}$ (which corresponds to $\alpha = \beta + 1 = 3/2$) represents a reasonable empirical approximation for the measurements of the rotational velocities in hurricanes was known for a long time and is mentioned in many publications (see [22, 11, 7]). Riehl [22] noted that assuming both the moment of the tangential component of the surface stress $r\sigma_\theta$ and the drag coefficient C_D to be independent of r is sufficient (but not necessary) for α to be 1.5. Pearce [21] put forward arguments supporting this assumption. The data reported by Hawkins and Rubsam [12] and by Palmen and Riehl

[20] indicate, however, that $C_D \sim 1/r^{\zeta}$ with ζ ranging between 0.4 and 0.7 while Palmen and Riehl [20] determined that, on average, $r\sigma_\theta \sim 1/r^{0.6}$. In his thermodynamic theory of steady tropical cyclones, Emanuel [6] demonstrated that $\alpha \approx 1.5$ just outside the radius of maximal winds is consistent with typical temperature changes on the sea surface and in the tropopause.

Although tornadoes generates the fastest winds, they are much more susceptible to atmospheric fluctuations than hurricanes. Even large supercell tornadoes are significantly affected by atmospheric irregularities. There are very few direct measurements of wind profiles in tornadoes and measurements may suffer from underresolving the core of tornadoes. Wurman and Gill [26] conducted high resolution measurements of a F4 tornado formed in a supercell storm near Dimmitt (Texas) in 1995 and reported $\beta = 0.6 \pm 0.1$ in $v_\theta \sim 1/r^{\beta}$ (γ_0 is small in the reported profile) that corresponds to $\alpha = \beta + 1 = 1.6 \pm 0.1$. The value of $\alpha^* = 3/2$ is within this range.

The most comprehensive analysis of the exponent in vortical flows by Mallen et. al. [19] reported averages for axisymmetric tangential velocity and axial vorticity distribution in tropical storms involving 251 (!) different cases. The best approximation for the exponent is $\alpha = 1.37$ was determined as the average over all storms with standard deviation of 0.14 while the averages of 1.31 and 1.48 were suggested for the weakest and the strongest storms. These values are quite close to $4/3$ and $3/2$ advocated here. While the fact that the exponent of $3/2$ can be used to approximate the tangential velocity in hurricanes was known from wind measurements in hurricanes for a long time [22, 11, 7], the exponent of $4/3$ was theoretically introduced for bathtub-like flows [15, 16] prior to the major work of Mallen et. al. [19] which determined that 1.31 is the average value of the exponent α for weak hurricanes.

Unsteady inviscid evolution of the vorticity

Obtaining solution for unsteady vortical flow with strong vorticity/velocity interactions is not simple. We can, however, investigate the unsteady convection of the initially uniform axial vorticity $\omega_z = \omega_0 = \text{const}$ by a given velocity field with the stream function given by $\psi = c_0 r^{\alpha} z$ as in (16). The flow is presumed inviscid. The Lagrangian trajectories $r_t = r_t(t)$ and $z_t = z_t(t)$ with the initial conditions $r_t(t_0) = r_0$ and $z_t(t_0) = z_0$ are evaluated by integration of $dr_t/dt = v_r$ and $dz_t/dr_t = v_z/v_r$:

$$\frac{\omega_{zt}}{\omega_0} = \frac{z_t}{z_0} = \left(\frac{r_t}{r_0}\right)^{-\alpha}, \ \phi(r_0) - \phi(r_t) = \tau \equiv \int_{t_0}^{t} c_0(t)dt, \quad (21)$$

where $\phi(r) =$ for $\alpha = 2$ and $\phi(r) =$ for $\alpha < 2$.

$$\phi(r) \equiv \left\{ \begin{array}{cc} \ln(r), & \alpha = 2 \\ r^{2-\alpha}/(2-\alpha), & 0 \le \alpha < 2 \end{array} \right\} \quad (22)$$

In evaluation of the Lagrangian value of axial vorticity $\omega_{zt} = \omega_{zt}(t)$ from the initial condition $\omega_{zt}(t_0) = \omega_0$ we use the fact that the vortical lines are frozen into inviscid flows. Substitution of the ratio r_t/r_0 evaluated from second equation results in

$$\frac{\omega_z}{\omega_0} = \left\{ \begin{array}{cc} \exp(2\tau), & \alpha = 2 \\ \left(1 + (2-\alpha)\tau r^{\alpha-2}\right)^{\alpha/(2-\alpha)}, & 0 \le \alpha < 2 \end{array} \right\} \quad (23)$$

$$\gamma = \left\{ \begin{array}{cc} \frac{\omega_0}{2} r^2 \exp(2\tau), & \alpha = 2 \\ \frac{\omega_0}{2} \left((2-\alpha)\tau + r^{2-\alpha}\right)^{2/(2-\alpha)}, & 0 \le \alpha < 2 \end{array} \right\} \quad (24)$$

There is an essential difference between these equations: the second equation in (23) with $\alpha \neq 2$ does approach the quasi-steady solution (17) so that $\omega_z \sim r^{-\alpha}$ for sufficiently large $t - t_0$ or sufficiently small r while, in the first equation of (23) with $\alpha = 2$, the vorticity remains $\omega_z = \omega_z(t)$ and does not become quasi-steady at any time. For the case of $0 \le \alpha < 2$, the quasi-steady (long-term) asymptotics for ω_z given by

$$\frac{\omega_z}{\omega_0} = ((2-\alpha)\tau)^{\alpha/(2-\alpha)} r^{-\alpha} + ..., \quad 0 \le \alpha < 2 \quad (25)$$

is induced by the vertical shear and is essentially independent of the initial conditions.

Vortex breakdown

According to Benjamin [2] vortex breakdown is controlled by the equation

$$R\frac{\partial}{\partial R}\left(\frac{1}{r}\frac{\partial \Psi_r'}{\partial R}\right) + \left(k^2 + \frac{2K_0^2(R)}{R^2} + \Phi(R)\right)\Psi_r' = 0 \quad (26)$$

which is written here in its dimensionless form and

$$K_0^2 \equiv \frac{\gamma_0 \omega_{z0}}{v_{z0}^2}, \ \ \Phi(R) = \frac{R}{V_{z0}} \frac{\partial \Omega_{\theta 0} R^{-1}}{\partial R} \approx \frac{(2-\alpha)(\alpha-4)}{R^2}$$

while $\Psi' = \Psi_r'(R)\exp(kz)$ represents a small disturbance of the stream function $\Psi = \Psi_0 + \Psi'$ and the subscript "0" denotes in (26) the values related to the undisturbed flow. Equation (26) is a linearization of the Long-Squire equation which is used to characterize axisymmetric inviscid steady vortical flows and can be seen as substitution of a non-linear expression for Ω_θ expressed in terms of Ψ into (1) [2]. Although the vortical flow considered here is unsteady, equation (26) can be approximately applied in a region which is close to the flow axis [15]. If $K(R) \sim R^{\beta}$, the solution of (26) with $k = 0$ can be expressed in terms of the Bessel function [15]. Benjamin [2] noted that existence of non-positive eigenvalues $k^2 \le 0$ corresponds to vortex breakdown. Mathematically, negative values of k^2 appear if K_0^2 becomes sufficiently large. Thus the breakdown can be expected to occur in a vortex which is characterized by sufficiently large values of K in a region close to the axis. It should be noted that, due to the influence of viscosity in the core of the flow, the rotation there is similar to solid-body rotation so that $\gamma_* \sim \omega_* r_*^2$. Hence $(K_*^2)_{\text{core}} \sim K_*^2/\text{St}_*$ is larger in the core than in the surrounding flow. This indicates that vortex breakdowns are more likely to occur within the core (as observed in some most intensive tornadoes). Benjamin [2] argued that if Long-Squire equation is non-linear (this is true with exception of few special cases such as solid-body rotation) then it allows for an alternative solution that the flow will take after its breakdown. In the case of linear Long-Squire equation, the flow does not have any other solution to switch to, becomes unstable and losses its symmetry.

Vortices on the other planets

The intensification region, which is primarily responsible for rapid rotation in the central sections of bathtub-like vortices, does not have its own characteristic scale but is limited by the inner and outer scales which depend on the factors other than inertial evolution of vorticity (turbulent or laminar viscosity, buoyancy, the scales of surrounding flow etc). Thus, in principle, we may observe vortices that are larger than the terrestrial cyclones as long as atmospheric conditions on other planets allow for the outer scale to be noticeably greater than 1000km. However, no distinct extraterrestrial hurricane has yet been found [9]. With significant heat released from the their interiors, Jupiter and Saturn have very active atmospheres but most of their vortices including the Great Red Spot (GRS) are anticyclonic. The anticyclonic vortices resemble a vortex with solid-body rotation more than a bathtub vortex since bathtub-like vortical flows (including hurricanes and tornadoes) demonstrate a significant intensification of rotation near the axis of the flow. GRS is a strong and persistent vortex but not a hurricane. Cyclonic vortices are also present on Jupiter but mostly in form of filament vortices that have rather irregular shapes – these vortices also are not characterized by a symmetric structure with a high rotational activity at the centre. During their encounter with Jupiter, Voyagers 1 and 2 detected a very interesting phenomenon of vortex evolution and breakdown [23]. The Voyagers observed a cyclonic vortex of a regular oval shape. The scale of the vortex was around 10 000 km that is more or less consistent with the relative scale of terrestrial hurricanes. The vortex existed for some time until a bright spot appeared in its centre. This is indicative of a high cyclonic activity in the core regions that is likely to be induced by the formation of a distinct convergence point in the vortex. The bright spot then bifurcated and became S-shaped. In few revolutions of the planet, the vortex was transformed into a filament vortex. It seems that the Voyagers detected a failed attempt to form a hurricane on Jupiter. It is difficult to overestimate the importance of detecting the phenomenon of vortex breakdown that may shed some light on why we are so unlucky to have hurricanes on Earth (and not on Jupiter). This issue is examined in the next section.

During the Voyager mission [24], no vortices have been found in the atmosphere of Uranus but several large vortices were detected on Neptune. The Great Dark Spot (GDS) of Neptune is similar to GRS in its relative size, position and rotation (anticyclonic). The most suitable candidate for a hurricane was Dark Spot 2 (DS2) which had an intense cloudy activity at the center of the vortex and its relative size would be comparable to that of terrestrial hurricanes. Unfortunately it was impossible to determine the direction of rotation for DS2 (cyclonic or anticyclonic) [24]. Although it is not unreasonable to assume that DS2 is a cyclonic vortex with significant intensification of rotation at the center, this assumption will remain speculative, at least until the next space mission can reach Neptune. Even if DS2 was a cyclone, its structure appears to be quite different from that of terrestrial hurricanes.

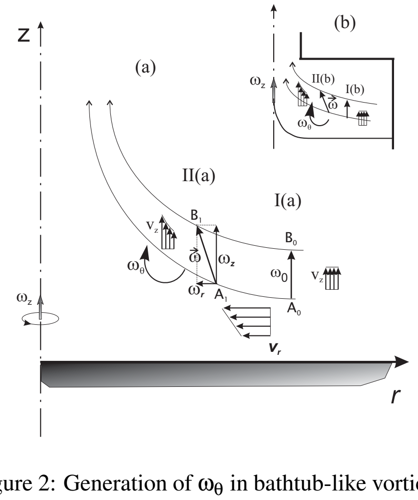

Figure 2: Generation of ω_θ in bathtub-like vortical flows stimulates the updraft and converging motion: (a) vorticity evolution in hurricanes and tornadoes (b) vorticity evolution in a bathtub vortex (shown upside down)

How the vortices form?

Formation of a bathtub-like vortex starts from existence of a converging flow in a region that has relatively weak uniform (i.e. solid-body-like) rotation. During the initial period of vortex formation, the axial vorticity level is, generally, low and K is uniformly small; hence this vortex can be treated as weak and $\alpha = 2$ in the vortex. At this stage, evolution of the vortex can be characterized by equations (21)-(25) with $\alpha = 2$. As rotation is amplified K also increases: $K \sim r^2 \exp(c_0(t - t_0))$ (here $c_0 = \text{const}$ is assumed for simplicity). At certain moment $K \sim 1$ is achieved at the outer rim of the region under consideration, which continues to shrink towards its center. Assuming ω_r is exactly zero in the flow, equation (2) does not generate any ω_θ even if K becomes large. The vortex continues to evolve according to (21)-(25) with $\alpha = 2$ until exponentially increasing K becomes large near the center and the vortex looses its symmetry and stability as discussed in the paragraph after equation (26).

If some radial vorticity ω_r of "correct" sign (illustrated in Figure 2) is present in the flow, then equation (2) does generate vorticity ω_θ directed as shown in Figure 2. In tornadoes and hurricanes ω_r is induced by the flow shear induced by the boundary layer (Figure 2a) while in a bathtub vortex ω_r appears due to a similar shear which is simply induced by inviscid and potential flow over the drain (Figure 2b). A more detailed physical explanation for the mechanism of generating ω_θ is given in Refs. [16, 14]. The generated vorticity changes the nature of the flow enforcing, as discussed previously, $\alpha < 2$ in the region where K is sufficiently large. This amplifies the updraft near the axis, stimulates the flow convergence λ, terminates the exponential increase of K (see equations (21)-(25) with $\alpha < 2$) and stabilizes the vortex. Even if the vortex is disturbed by the surrounding flow, equation (23) indicates that the steady-state distribution $\omega_z \sim 1/r^\alpha$ is continuously reproduced by the vertical shear provided $\alpha < 2$. According to (20), α is expected to eventually fall to its equilibrium value of α^* which lays within the range of $4/3 \lesssim \alpha \lesssim 3/2$. The vortex formation is completed when the region of $\alpha < 2$ reaches the center of the flow. The presence of circumferential vorticity ω_θ stabilizes the convergence point and makes the vortex resilient with respect to external disturbances. The compensating regime, however, does not prevent continuing relatively slow growth of K due to increase γ_0 in (17). This growth may eventually cause the vortex breakdown. It seems that, in stable vortices, this growth is effectively restricted by losses of angular momentum.

In terrestrial hurricanes, existence of the atmospheric boundary layer ensures that $\omega_r \neq 0$ and generates vorticity ω_θ enforcing $\alpha < 2$. This moderates any further increases in K and stabilizes the vortex. The Jovian atmosphere is bottomless. This allows for a cyclonic vortex with $\alpha = 2$ to grow beyond the point of its stability due to rapid and unconstrained increase of parameter K. This vortex is likely to rapidly disintegrate but even if the vortex survives the exponential growth by switching to the compensating regime, the insufficient loss of angular momentum will eventually brake the vortex due to slow but persistent increase in γ_0. It seems that the Voyagers have detected the hurricane breakdown phenomenon on Jupiter when a strong and powerful hurricane was quickly destroyed by instabilities. Could this strategy be used to destroy the terrestrial hurricanes? The atmosphere of Neptune has a more significant fraction of methane and this may result in larger density gradients towards the center of the planet (if not in a phase transition at the bottom of the atmosphere). It could be the case that these density gradients have a stabilizing effect on cyclonic vortices.

Conclusions

Bathtub-like vortical flows (including hurricane and tornadoes), which are characterized by significant intensification of rotation near the center due to convergence of the flow, are considered.

The region in these flows, which is intermediate between the core and peripheral scales, is called intensification region and inviscid axisymmetric interactions of velocity and vorticity are presumed to be of prime importance in this region. Assuming that this assumption is correct, the developed vortical flow can not have too large or too small values of the parameter K within the intensification region at least because the contrary assumption leads to physical contradictions. Moderation of the parameter K requires that the exponent α must fall below its potential value of 2 in the intensification region if the vortex is to remain stable. The condition $K \sim$ const does not result in a strict power law and the exponents are likely to fluctuate around their equilibrium values due to various disturbances present in the flow. Nevertheless, the range of exponents $4/3 \lesssim \alpha \lesssim 3/2$ obtained from this condition seems to be in a very good agreement with available data. The breakdowns of cyclonic vortices on Jupiter are also explained well by their inability to lower the value of α below 2.

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