Universal regimes of a free turbulent jet
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Abstract

We apply the $K$-$\varepsilon$ model to analyse the expansion of a free turbulent jet. Due to nonlinearity of turbulent diffusion the model leads to spatially confined solutions (solutions with finite support). We seek the turbulent energy, dissipation and momentum as power series in spatial coordinate across the jet with time-dependent coefficients. The coefficients obey a dynamical system containing slow and fast variables. We show, with the help of numerical analysis, that there exists an attractor for trajectories of the dynamical system, based on a few slow variables.

Introduction

The literature on turbulent jets is extensive and spans over decades [1, 9, 3]. As in other areas of mechanics, special interest present attracting regimes, however, this aspect of the jet dynamics has not received as close attention as, for example, analysis of the structure of pulsations. A self-similar attracting regime of an instantaneous jet, based on the relatively rough dynamics has not received as close attention as, for example, the initial velocity across the jet, $U$, as the velocity scale; the initial width of the jet, $2h$, as the length scale; $U^2$ as the turbulent energy scale; $U^2/\kappa$ as the dissipation rate scale; and $h/U$ as the time scale.

The initial conditions for $K$, $\varepsilon$ and $u$ across the jet are supposed to have dome-like shapes with finite support. We assume that they are symmetric with respect to the middle plane. On the turbulent front, $x=h(t)$, the functions $K(x,t)$, $\varepsilon(x,t)$ and $u(x,t)$ are equal to zero and remain zero beyond the front, for $x>h(t)$.

Turbulent jet as dynamical system

We look for solutions of (1) in the form of power series

$$K = A(t) [1 - B_2(t)x^2 - B_4(t)x^4 - B_6(t)x^6 - ...] ,$$

$$\varepsilon = P(t) [1 - R_2(t)x^2 - R_4(t)x^4 - R_6(t)x^6 - ...] ,$$

$$u = M(t) [1 - N_2(t)x^2 - N_4(t)x^4 - N_6(t)x^6 - ...] .$$

Here $A$, $P$ and $M$ are the amplitudes, expectedly maximum values of the functions reached in the middle of the jet, $x=0$. The structure functions in the square brackets describe the dome-like profiles descending from the maxima down to zero at $x=h(t)$.

Substituting (2) into the dynamic equations (1) and collecting terms with same powers of $x$ gives the system of ODEs

$$A = -\alpha_1 \frac{2A^2B_2}{P} - \alpha_3 P ,$$

$$P = -\beta_1 2A^2 R_2 - \beta_3 P^2 ,$$

$$M = -\chi A^2 MN_2 ,$$

$$B_2 = -\alpha_1 \frac{10A^2 B_2}{P} + \alpha_1 \frac{P B_J}{A} + \alpha_2 \frac{6A^2 B_2 R_2}{P} + \alpha_1 \frac{4AM^2 N_2}{P} + \alpha_3 \frac{PR_2}{A} ,$$

$$R_2 = -\beta_1 \frac{12A^2 R_4}{P} + \beta_1 \frac{8AM^2 N_2^2}{P} - \beta_3 \frac{PR_2}{A} ,$$

$$N_2 = -\chi \frac{12A^2 B_2 N_4}{P} + \chi \frac{2A^2 N_2^2}{P} + \chi \frac{6A^2 N_2 R_2}{P} + \chi \frac{PR_2}{P}.$$
\[ B_4 = -\alpha_1 \frac{58A^2B_2R_4}{p} + \frac{\alpha_1 PB_4}{A} + \frac{10A^2B_2^4}{p} \]
\[ -\frac{20A^2B_2R_2}{p} + \frac{\alpha_1 10A^2B_2R_2}{p} + \frac{10A^2B_2R_4}{p} \]
\[ + \frac{\alpha_1 10A^2B_2R_2}{p} + \frac{30A^2B_2}{p} + \frac{8A^2B_2M^2N_2}{p} \]
\[ -\frac{\alpha_1 10A^2B_2R_2}{p} + \frac{\alpha_1 16A^2M^2N_2}{p} - \frac{\alpha_1 PR_4}{A}. \]
\[ R_4 = -\beta_1 \frac{40A^2B_2R_4}{p} + \frac{\beta_1 2A^2R_2}{p} - \frac{\beta_3 PR_4}{A}, \]
\[ + \frac{\beta_1 10A^2B_2R_2}{p} - \frac{\beta_1 20A^2B_4R_2}{p} \]
\[ + \frac{\beta_1 10A^2B_2R_2}{p} + \frac{\beta_1 30A^2R_2}{p} + \frac{4AB_2M^2N_2}{p} \]
\[ + \frac{\beta_1 10A^2B_2R_2}{p} + \frac{20A^2B_2R_4}{p} - \frac{\beta_3 2B_2R_2}{A}, \]
\[ N_4 = -\frac{\chi 10A^2B_2N_4}{p} + \frac{\chi 2A^2N_2N_4}{p} + \frac{\chi 10A^2B_2^4}{p} \]
\[ -\frac{\chi 20A^2B_2N_2R_4}{p} + \frac{\chi 20A^2B_2R_2}{p} + \frac{\chi 10A^2B_2R_4}{p} \]
\[ + \frac{\chi 20A^2B_2N_2R_4}{p} + \frac{\chi 20A^2B_2R_2}{p} + \frac{\chi 30A^2R_2}{p} \]
\[
\cdots
\]

Among possible closure assumptions we choose that that satisfies the physical requirement that the fronts of the turbulent energy, dissipation rate and momentum coincide at all times. By the physics of diffusion, if the fronts are different initially, they should quickly catch up with each other. Consider a notional situation when the momentum front is initially behind the energy and dissipation-rate front (we suppose that these two coincide). Then the turbulent diffusion will instantaneously transfer the momentum forward up to the energy/dissipation-rate front position. Conversely, if the momentum front is initially ahead of the energy/dissipation-rate front, it will stay motionless for some time since there is no turbulence in the vicinity. The momentum front will move only when the energy/dissipation front catches up, after which the fronts will move together.

Thus, we require that \( K, \varepsilon \) and \( u \) turn into zero at the same location \( x = h(t) \). Taking into account the terms up to the 4th order in (2) we have
\[
1 - B_2h^2 - B_2h^4 = 0, \]
\[
1 - R_2h^2 - R_2h^4 = 0, \]
\[
1 - N_2h^2 - N_2h^4 = 0. \]

The front equations (5) are complemented by the truncated dy-namic equations (3),
\[ A = -\alpha_1 \frac{2A^2B_2}{p} - \alpha_3 P, \]
\[ P = -\beta_1 2A^2R_2 - \beta_3 \frac{P^2}{A}, \]
\[ M = -2\chi MN_2, \]
\[ B_2 = -\alpha_1 \frac{10A^2B_2}{p} + \frac{\alpha_3 PB_2}{A} + \frac{6A^2B_2R_2}{p} + \frac{\alpha_1 12A^2B_4}{p} \]
\[ -\frac{2A^2B_2R_4}{p} - \alpha_3 PR_2, \]
\[ R_2 = \beta_1 \frac{8A^2B_2R_4}{p} - \beta_3 \frac{PR_4}{A} - \beta_1 \frac{12A^2B_2R_2}{p} + \beta_1 \frac{12A^2R_4}{p}, \]
\[ B_4 = -\alpha_1 \frac{58A^2B_2R_4}{p} + \frac{\alpha_1 PB_4}{A} + \frac{10A^2B_2^4}{p} \]
\[ -\frac{20A^2B_2R_2}{p} + \frac{\alpha_1 10A^2B_2R_2}{p} + \frac{10A^2B_2R_4}{p} \]
\[ + \frac{\alpha_1 10A^2B_2R_2}{p} + \frac{30A^2B_2}{p} + \frac{8A^2B_2M^2N_2}{p} \]
\[ -\frac{\alpha_1 10A^2B_2R_2}{p} + \frac{\alpha_1 16A^2M^2N_2}{p} - \frac{\alpha_1 PR_4}{A}. \]

The system (5)—(6) contains 10 equations with respect to 10 unknowns: \( A, P, M, B_2, R_2, N_2, B_4, R_4, N_4 \) and \( h \), all depending on \( t \).

Introduce the new time \( \tau \) by
\[
\frac{d}{(A^2B_2/P) dt} = \frac{d}{d \tau} = (\cdot)',
\]
and divide (6) by \( A^2B_2/P \). This conveniently transforms (6) into the form with linear terms:
\[ A' = -\alpha_1 2A - \alpha_3 \frac{P^2}{A^2B_2}, \]
\[ P' = -\beta_1 2PR_2 \frac{B_2}{B_2} - \beta_3 \frac{P^3}{A^3B_2}, \]
\[ M' = -2\chi MN_2 \frac{B_2}{B_2}, \]
\[ B_2' = -\alpha_1 10B_2 + \frac{\alpha_3 P^2}{A^3B_2} + \alpha_1 6R_2 + \alpha_1 12B_4 \frac{B_2}{B_2}, \]
\[ -\frac{2A^2B_2R_4}{p} - \alpha_3 PR_2, \]
\[ R_2' = -\beta_1 12R_2 + \frac{\beta_3 P^2R_2}{A^3B_2} + \beta_1 12R_4 \frac{B_2}{B_2}, \]
\[ -\frac{2A^2B_2R_4}{p} - \beta_3 PR_4 \frac{A}{A}, \]
\[ N_2' = -\chi 12N_2 + \frac{\chi 2P^2}{B_2} + \chi 6N_2R_2 + \chi 12N_4 \frac{B_2}{B_2}, \]

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Figure 1: Trajectories (different views) in the space of the energy variables.

\[
B_4' = -\alpha_1 58 B_4 + \alpha_1 \frac{P^2 B_4}{A^3 B_2} + \alpha_1 10 R_2^3 - \alpha_1 20 B_2 R_2 \\
+ \alpha_1 10 R_2^3 + \alpha_1 10 R_4 + \alpha_1 \frac{20 B_2 R_2}{B_2} + \alpha_1 \frac{30 B_6}{B_2} \\
+ \alpha_2 \frac{8M^2 N_2^2}{A} - \alpha_2 \frac{4M^2 N_2^2 R_2}{AB_2} - \alpha_2 \frac{16M^2 N_2 N_4}{AB_2} \\
- \alpha_1^2 R_1^2 \frac{A^2 B_2}{A^3 B_2}.
\]

(9)

Numerical solutions of the system (5), (8), (9) are displayed in Fig. 1–3. We used \(\alpha_1 = 0.09, \alpha_2 = 0.09, \alpha_3 = 1, \beta_1 = 0.07, \beta_2 = 0.13, \beta_3 = 1.92, \chi = 0.09\). The initial positions of the energy front, dissipation-rate front and velocity front coincided.

From the Figures we observe that some variables are fast and some are slow. The amplitudes \(A\) and \(P\) decay rapidly in comparison to \(B_2\) and \(R_2\). This decay is largely due to the terms with \(\alpha_4\) and \(\beta_1\), linked to the energy dissipation rate. The velocity amplitude \(M\), compared to \(N_4\), decays not as rapidly though. The variables \(B_4, R_1\) and \(N_4\) (and higher-order variables) decay rapidly compared to \(B_2, R_2\) and \(N_2\). Thus, the variables \(B_2, R_2\) and \(N_2\) are slow.

Notice that, except in the amplitude equations (8) the fast variables \(A\), \(P\) and the variable \(M\) appear in the right-hand sides only in ratios \(P^2/A^3\) and \(A/M^2\). We anticipate, and confirm later in the paper, that these ratios are slow. We define

\[
E = \frac{P^2}{A^3}, \quad S = \frac{A}{M^2}.
\]

Differentiating (10) and expressing the derivatives \(A', P'\) and

\[
M' \quad \text{from (8) we deduce the dynamic equations for } E \text{ and } S.
\]

Also, we add the dynamic equation for \(N_3\) so that all the 4th order variables, \(B_4, R_4\) and \(N_4\), now evolve according to their respective dynamic laws. We obtain

\[
S' = -\alpha_2 2S - \alpha_3 \frac{E S}{B_2} + \chi \frac{S N_2}{B_2},
\]

\[
E' = -\beta_4 4 \frac{R_2 E}{B_2} + \alpha_1 6 E + (3\alpha_3 - 2\beta_3) \frac{E^2}{B_2},
\]

\[
B_2' = -\alpha_1 10 B_2 + \alpha_3 E + \alpha_1 6 R_2 + \alpha_1 \frac{12R_4}{B_2}
\]

\[
- \alpha_4 \frac{4N_2^2}{SB_2} - \alpha_3 \frac{E R_2}{B_2},
\]

\[
R_2' = -\beta_1 12 R_2 + \beta_1 \frac{8R_2^3}{B_2} + \beta_3 \frac{E R_2}{B_2} + \beta_4 \frac{12R_4}{B_2}
\]

\[
- \beta_2 \frac{4N_2^2}{SB_2} - \beta_3 \frac{2ER_2}{B_2},
\]

\[
N_2' = -\chi 12 N_2 + \chi \frac{2N_2^2}{B_2} + \chi \frac{6N_2 R_2}{B_2} + \chi \frac{12N_4}{B_2},
\]

\[
B_4' = -\alpha_1 58 B_4 + \alpha_3 \frac{E R_4}{B_2} + \alpha_1 10 R_4^2 - \alpha_1 20 B_2 R_2
\]

\[
+ \alpha_1 10 R_2^3 + \alpha_1 10 R_4 + \alpha_1 \frac{20 B_2 R_2}{B_2} + \alpha_1 \frac{30 B_6}{B_2}
\]

\[
+ \alpha_2 \frac{8N_2^2}{S} - \alpha_2 \frac{4N_2 R_2}{SB_2} - \alpha_2 \frac{16N_2 N_4}{SB_2} - \alpha_1 \frac{E R_4}{B_2}
\]

\[
+ \alpha_4 \frac{4N_2^2}{SB_2} - \alpha_4 \frac{E R_4}{B_2},
\]

\[
E' = -\beta_4 4 \frac{R_2 E}{B_2} + \alpha_1 6 E + (3\alpha_3 - 2\beta_3) \frac{E^2}{B_2},
\]

\[
B_2' = -\alpha_1 10 B_2 + \alpha_3 E + \alpha_1 6 R_2 + \alpha_1 \frac{12R_4}{B_2}
\]

\[
- \alpha_4 \frac{4N_2^2}{SB_2} - \alpha_3 \frac{E R_2}{B_2},
\]

\[
R_2' = -\beta_1 12 R_2 + \beta_1 \frac{8R_2^3}{B_2} + \beta_3 \frac{E R_2}{B_2} + \beta_4 \frac{12R_4}{B_2}
\]

\[
- \beta_2 \frac{4N_2^2}{SB_2} - \beta_3 \frac{2ER_2}{B_2},
\]

\[
N_2' = -\chi 12 N_2 + \chi \frac{2N_2^2}{B_2} + \chi \frac{6N_2 R_2}{B_2} + \chi \frac{12N_4}{B_2},
\]

\[
B_4' = -\alpha_1 58 B_4 + \alpha_3 \frac{E R_4}{B_2} + \alpha_1 10 R_4^2 - \alpha_1 20 B_2 R_2
\]

\[
+ \alpha_1 10 R_2^3 + \alpha_1 10 R_4 + \alpha_1 \frac{20 B_2 R_2}{B_2} + \alpha_1 \frac{30 B_6}{B_2}
\]

\[
+ \alpha_2 \frac{8N_2^2}{S} - \alpha_2 \frac{4N_2 R_2}{SB_2} - \alpha_2 \frac{16N_2 N_4}{SB_2} - \alpha_1 \frac{E R_4}{B_2}
\]

\[
+ \alpha_4 \frac{4N_2^2}{SB_2} - \alpha_4 \frac{E R_4}{B_2},
\]
The dynamical system (11)–(12) is augmented to the closed form by the front equations

\[ 1 - B_2 h^2 - B_4 h^4 - B_6 h^6 = 0, \]
\[ 1 - R_2 h^2 - R_4 h^4 - R_6 h^6 = 0, \]
\[ 1 - N_2 h^2 - N_4 h^4 = 0. \]

Fig. 4 demonstrates that \( E, S \) and \( B_2 \) behave linearly against each other and therefore can be identified as slow variables.

The numerical experiments show that the linear terms dominate on early stages of the dynamics forcing \( B_4, R_4 \) and \( N_4 \) to decay. Accordingly, the linear terms quickly drop to a level comparable to the rest of the terms.

**Simple example of an attractor**

This behaviour resembles the dynamics near attractors called centre manifolds. A centre manifold attracts trajectories of a dynamical system where some, slow, variables have zero linear decay rates, while the other, fast, variables have negative linear decay rates [2]. We illustrate this on a simple example from [7]:

\[
\begin{align*}
\dot{x} &= -px - xy, \\
\dot{y} &= -y + x^2 - 2y^2.
\end{align*}
\]

If \( p = 0 \), then it can be shown that the attractor is precisely

\[ y = x^2. \]

Driven by the linear term (\( -y \)) the variable \( y \) quickly drops and trajectories fall onto the attracting manifold (15) on which the nonlinear terms \((x^3 - 2x^2)\) are comparable to the linear term (\( -y \)). Observe that the variable \( y \) depends on \( t \) via the slow variable \( x \).

If \( p \) is positive but relatively small, the attracting manifold can be found as a perturbation of (15). The case of small \( p > 0 \) is similar to our situation in (12).

Remarkably, in the unperturbed case \( p = 0 \) the attractor (15), in its leading order, can be obtained by simply replacing the time derivative \( \dot{y} \) by zero: \( 0 = -y + x^2 - 2y^2 \) giving \( y = x^2 + o(x^2) \) \( \rightarrow \) \( x^2 \) when \( x \rightarrow 0 \). The motion on the attractor is obtained from the first equation (14). In the leading order \( \dot{x} = -x^3 \).

If \( p > 0 \), then, strictly speaking, the derivative \( \dot{y} \) must be taken into account. However, if \( p \) is small enough, that is the spectral gap between the linear decay rate \( 1 \) of \( y \) and \( p \) of \( x \) is considerable, then \( y = x \) can acceptably approximate the attractor.

**Attractor for the turbulent jet**

We take the similar approach in our turbulence problem.

Replace in (11)–(13) the time derivatives of \( B_4, R_4 \) and \( N_4 \) by zeroes. This gives 6 algebraic equations to determine the 6 variables: \( B_4, R_4, N_4, B_6, R_6 \) and \( h \) in terms of the slow variables \( E, \ldots \) equations to determine the 6 variables: \( B_4, R_4, N_4, B_6, R_6 \) and \( h \) in terms of the slow variables \( E, \ldots \)
and a trajectory obtained as described above where the values of the slow variables are taken from the solution of the full system. The latter trajectory therefore constitutes the projection of the actual trajectory onto an attractor.

The comparison is shown in Fig. 5–7. For the energy and dissipation-rate variables the actual solution curve approaches its projection very closely. For the velocity variables the actual curve and its projection are also close although to a lesser extent. Overall, the attraction is quite evident.

The higher-order coefficients of series (2), $B_i$, $R_i$ and $N_i$ for $i = 6, 8, \ldots$, can be expressed through the slow variables $E$, $S$, $B_2$, $R_2$ and $N_2$ in the similar way as $B_4$, $R_4$ and $N_4$ above. As new equations are added into the slow variables are taken from the solution of the full system (11)–(13) more terms are to be added in the front equations (13). As a result, any number of the coefficients of the series (2) can be determined as implicit functions of the slow variables; such functions would constitute the sought attractor.

Conclusions

We analysed the K–ε model of the expanding turbulent jet shaped as a plane layer. The profiles of energy, dissipation rate and velocity across the jet are sought in the form of power series. The series coefficients satisfy a nonlinear dynamical system with a few slow variables. Using these variables, we found an approximate form of attractor in the form of a system of algebraic equations connecting higher-order variables and slow variables. The convergence of the trajectories to the attractor is demonstrated.
Figure 7: Actual behaviour of $N_4$ (solid line) and its projection.

References


