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Some Basic Aspects of the Triple Decomposition of the Relative Motion near a Point

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Abstract

The triple decomposition of the relative motion near a point (TDM) — based on the extraction of a so-called "effective" pure shearing motion — has been recently proposed by the present author. The relative motion near a point has been decomposed through the analysis of a "frozen" flow field at a given instant in time. This approach has been motivated by the fact that vorticity cannot distinguish between pure shearing motions and the actual swirling motion of a vortex. The TDM results in two additive vorticity parts and two additive strain-rate parts of distinct nature: the *shear* component associated with a pure shearing motion and the *residual* one. Some basic aspects of this novel decomposition technique are treated in the present paper. These aspects include: pure shearing motion and shearing elements in 3D, orbital compactness vs. arbitrary axial strain of a vortex, the interpretation of *residual* vorticity and the concept of *residual* circulation (i.e. a surface quadrature of the *residual* vorticity) for the description of vortex strength, and qualitative comparison of the TDM vorticity outcome with other known vorticity-decomposition techniques.

Introduction

The triple decomposition of the relative motion near a point (TDM) has been proposed by Kolář [10] in connection with vortex identification. The TDM is expressed through the corresponding triple decomposition of $\nabla \mathbf{u}$ constructed so that the strain-rate tensor $\mathbf{S}$ and vorticity tensor $\mathbf{\Omega}$ are cut down in magnitudes to "share" their portions through the third term $(\nabla \mathbf{u})_{\mathrm{SH}}$ associated with a pure shearing motion. In terms of the *residual* portions of $\mathbf{S}$ and $\mathbf{\Omega}$, it reads (cf. the conventional double decomposition $\nabla \mathbf{u} = \mathbf{S} + \mathbf{\Omega}$)

$$\nabla \mathbf{u} = \mathbf{S}_{\mathrm{RES}} + \mathbf{\Omega}_{\mathrm{RES}} + (\nabla \mathbf{u})_{\mathrm{SH}} \, . \tag{1}$$

The first term on the RHS of (1) represents an irrotational straining, the second one a rigid-body rotation. The third term of the triple decomposition denoted as $(\nabla \mathbf{u})_{\mathrm{SH}}$ representing a pure shearing motion is described by a "purely asymmetric tensor" fulfilling *in a suitable reference frame*

$$u_{i,j} = 0 \quad OR \quad u_{j,i} = 0 \quad \text{(for all } i, j\text{)}. \tag{2}$$

From the viewpoint of the double decomposition, $\nabla \mathbf{u} = \mathbf{S} + \mathbf{\Omega}$, the term $(\nabla \mathbf{u})_{\mathrm{SH}}$ itself is responsible for a specific portion of vorticity labelled "*shear* vorticity" and for a specific portion of strain rate labelled "*shear* strain rate" while the remaining portions of $\mathbf{S}$ and $\mathbf{\Omega}$ are labelled "*residual* strain rate" and "*residual* vorticity".

The TDM is closely associated with the so-called *basic reference frame* (BRF) where it is performed. The TDM results generated in the BRF are valid for all other frames rotated (not rotating!) with respect to the BRF under an orthogonal transformation. In the BRF, **(i)** an *effective* pure shearing motion is shown "in a clearly visible manner" described by the tensor form (2) under the definition condition that **(ii)** the effect of extraction of a "shear tensor" is maximized within the following — quite natural and straightforward — decomposition scheme applicable to an arbitrary reference frame

$$\nabla \mathbf{u} \equiv \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} = \begin{pmatrix} residual \\ tensor \end{pmatrix} + \begin{pmatrix} shear \\ tensor \end{pmatrix} \tag{3a}$$

where the residual tensor is given by

$$\begin{pmatrix} residual \\ tensor \end{pmatrix} = \begin{pmatrix} u_x & (\operatorname{sgn} u_y)\,\mathrm{MIN}(|u_y|,|v_x|) & \bullet \\ (\operatorname{sgn} v_x)\,\mathrm{MIN}(|u_y|,|v_x|) & v_y & \bullet \\ \bullet & \bullet & w_z \end{pmatrix} . \tag{3b}$$

In (3a, b) the following simplified notation is employed: u, v, w are velocity components, subscripts x, y, z stand for partial derivatives. The remaining two non-specified pairs of off-diagonal elements of the residual tensor in (3b) are constructed strictly analogously as the specified one, each pair being either symmetric or antisymmetric.

The effect of extraction of the shear tensor is maximized where the absolute tensor value of the residual tensor is minimized by changing the reference frame under an orthogonal transformation. This extremal condition guarantees that a pure shearing motion — if considered separately — is recognized as a third elementary part of the triple decomposition. Then a pure shearing motion is labelled with the term "effective". For further details, quantitative TDM evaluation algorithm, discussion, and particularly for the qualitative description of the flow kinematics near a point adopted in the frame of the TDM, see [10].

Local Pure Shearing Motion near a Point in 3D

A pure shearing motion of arbitrary complexity in 3D (always exhibiting non-zero vorticity) is defined on the basis of (2) as the parallel relative motion of non-rotating undeformed shearing elements — material planes, lines, or points — depending on flow complexity in 3D. The term "parallel relative motion" has the following meaning. Firstly, local coordinate systems with initially parallel axes are assigned to all elements within the local (i.e. near a point) family of shearing elements. Secondly, throughout the process of the parallel relative motion these local systems must remain strictly parallel. Consequently, all shearing elements of the same family are mutually non-rotating. If the local family of shearing elements does not rotate as a whole, that is, the local coordinate system of an arbitrary element of the family does not rotate with respect to its initial state, a shearing motion is considered a pure shearing motion.

Three elementary motions of the TDM are depicted (for planar flows) in Fig. 1 showing explicitly a pure shearing motion.

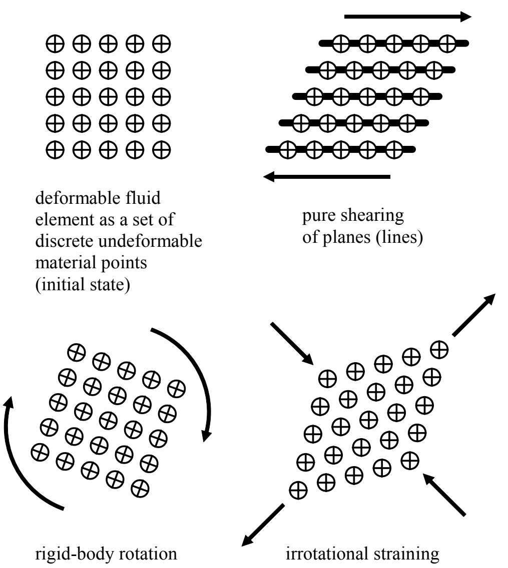

Figure 1. Qualitative model of three elementary motions of the TDM.

The geometry of a pure shearing motion — i.e. the type of shearing elements — is determined by the tensor structure of $(\nabla \mathbf{u})_{SH}$ as follows (the symbol $\bullet$ denotes below a non-zero tensor component):

(i) material *shearing planes*, e.g. for $(\nabla \mathbf{u})_{SH}$ of the form

$$\begin{pmatrix} 0 & \bullet & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \bullet & \bullet \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \bullet \\ 0 & 0 & \bullet \\ 0 & 0 & 0 \end{pmatrix}, \text{ or}$$

(ii) material *shearing lines*, e.g. for the structures

$$\begin{pmatrix} 0 & \bullet & 0 \\ 0 & 0 & \bullet \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \bullet & 0 \\ 0 & 0 & 0 \\ \bullet & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ \bullet & 0 & \bullet \\ \bullet & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \bullet & \bullet \\ 0 & 0 & \bullet \\ 0 & 0 & 0 \end{pmatrix}, \text{ or}$$

(iii) material *shearing points* for the structure

$$\begin{pmatrix} 0 & \bullet & 0 \\ 0 & 0 & \bullet \\ \bullet & 0 & 0 \end{pmatrix} \text{ and its transpose.}$$

The introduced pure shearing motion represents locally — near a point — the well-known shear flows as, for example, Couette and Poiseuille flows.

Orbital Compactness vs. Arbitrary Axial Strain of a Vortex

Wu et al. [15] performed an analytical diagnosis of four local vortex-identification criteria, demonstrated by the Burgers and Sullivan vortex, indicating that the Q-criterion [8] and λ_2-criterion [9] may cut a connected vortex into broken segments at locations with strong axial stretching. They emphasized the following requirements: a generally applicable vortex definition should be able to identify the vortex axis and allow for an arbitrary axial strain.

In their instantaneous-streamline analysis Chakraborty et al. [3] enhanced the swirling-strength criterion of Zhou et al. [17], based on the Δ-criterion [5], by including a local approximation of the non-local property proposed by Cucitore et al. [6], requiring that the swirling material points inside a vortex have bounded separation remaining small. They introduced an idea of orbital compactness of a vortex in terms of the so-called spiralling compactness of the motion projected on the vortex plane given by the complex-conjugate eigenpairs of $\nabla \mathbf{u}$. They further presented an example of the rapid radial spreading out of instantaneous streamline which does not appear to qualify as a vortex (see Fig. 1b of [3]).

The allowance for an arbitrary axial strain of Wu et al. [15] has become a subject of recent debate (Chakraborty et al. [4]; Wu et al. [16]) as this requirement, basically, does not conform to the orbital compactness proposed in [3]. Note that for an incompressible flow the axial strain is directly related to the spiralling compactness [3, 4]. According to [3, 4], the spiralling compactness requires for vortex-identification purpose an appropriate threshold dictated by the length and time scales of the given problem. Following [16], however, adding a threshold value to the local axial strain or to the orbital compactness is subjective and cannot be rationalized.

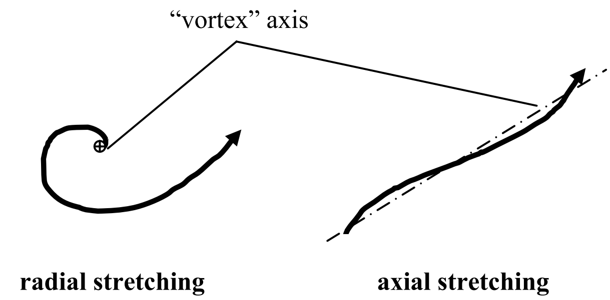

Figure 2. Vortex stretching

The TDM provides an alternative insight into the local relative motion near a point. The controversial aspect of the orbital compactness vs. arbitrary axial strain of a vortex is examined below for the TDM outcome. It is assumed, similarly as in [3], that a reasonable estimate of the non-local viscous-flow vortical features can be inferred from the local (pointwise) characteristics.

Let us consider a uniaxial isochoric stretching or contraction coupled with a specific amount of rotation round the principal stretching axis. This axisymmetric flow situation, clearly seen in the system of principal axes in terms of the conventional double decomposition, is recognized in the corresponding BRF in terms of the triple decomposition as a specific 3D pure shearing motion of the following tensor form

$$\nabla \mathbf{u} = \begin{pmatrix} -\frac{a}{2} & 0 & -\frac{\sqrt{3}}{2}a \\ 0 & a & 0 \\ \frac{\sqrt{3}}{2}a & 0 & -\frac{a}{2} \end{pmatrix}^{\text{PRINCIPAL AXES}} \leftrightarrow \nabla \mathbf{u} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & a \\ a & 0 & 0 \end{pmatrix}^{\text{BRF}} . \tag{4}$$

Here $a > 0$ stands for the uniaxial stretching (coupled with a radial contraction), and $a < 0$ stands for the radial stretching (coupled with a uniaxial contraction), schematically sketched in Fig. 2.

The similar flow situation as in (4) with a greater amount of uniaxial stretching results (in terms of the TDM) in the same shearing motion as in (4) and, moreover, in the non-zero *residual* irrotational straining motion (responsible for a non-zero *residual* strain rate). The *residual* vorticity is zero in this case, consequently, not qualifying the examined point as a vortex.

The similar flow situation as in (4) with a greater amount of rotation results in the same pure shearing motion as in (4) and, moreover, in the non-zero *residual* rigid-body rotation (responsible for a non-zero *residual* vorticity) qualifying the local flow near a point as a vortex. An obvious analogy holds for the case of the radial stretching with a uniaxial contraction.

One can conclude that in the frame of the TDM, while stretching (uniaxially or radially) the local vortical motion near a point, there is an *inherent* objective physical bound for the amount of stretching to identify the examined point as part of a vortex. In the present context, this bound is just the local 3D pure shearing motion described by the form (4). Consequently, no subjective threshold value to the local axial strain or to the spiralling compactness is necessary to fulfil qualitatively the requirement of orbital compactness proposed by Chakraborty et al. [3].

Interestingly enough, a uniform dilatation does not affect the TDM results [10] and can be removed prior to a further analysis of $\nabla \mathbf{u}$ without loss of generality and applicability to compressible flows.

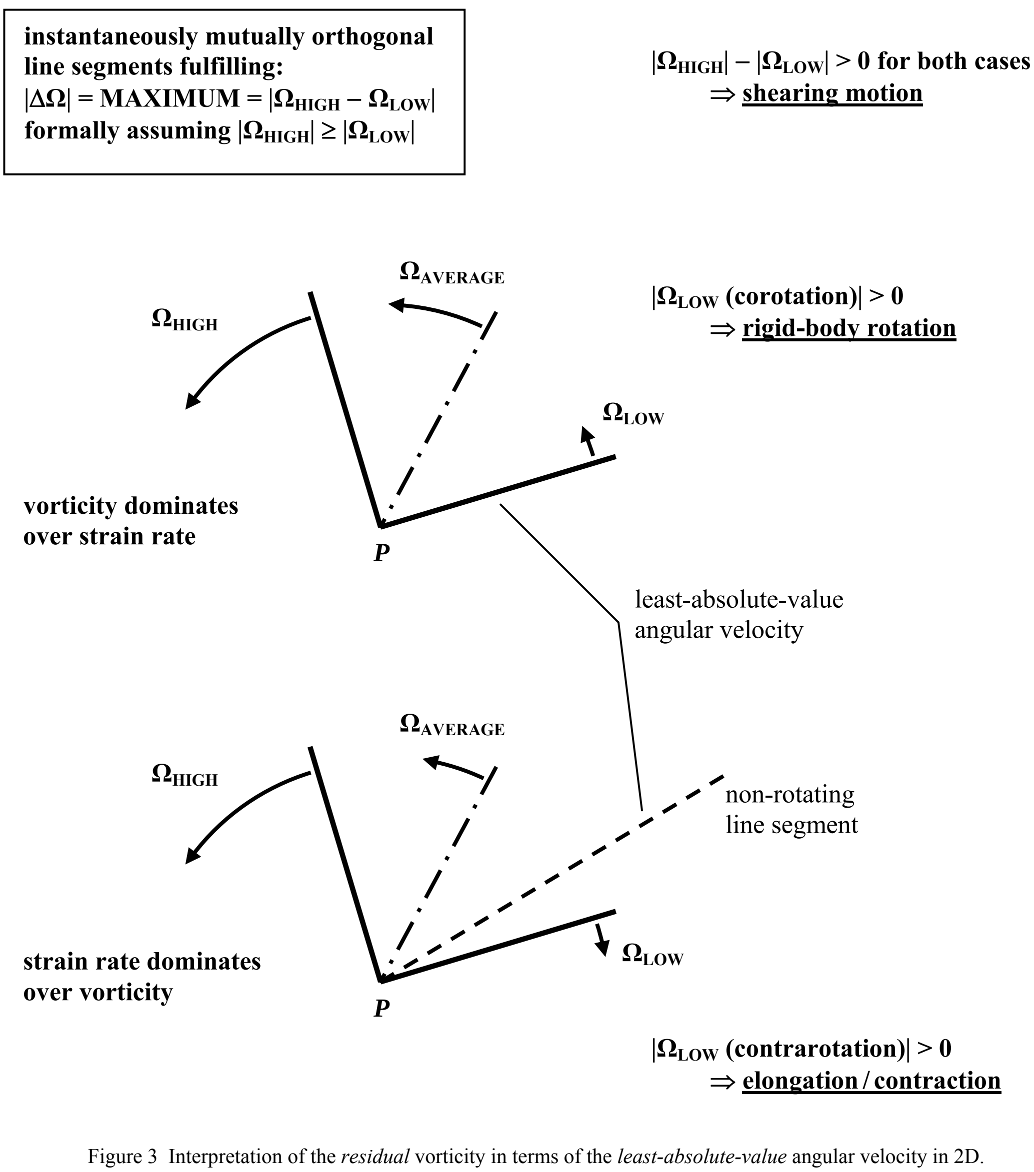

Figure 3 Interpretation of the *residual* vorticity in terms of the *least-absolute-value* angular velocity in 2D.

Residual Vorticity and *Residual* Circulation

Let us recall the well-known interpretation of vorticity in terms of a local angular velocity. Vorticity, perpendicular to a flow plane through a given point, is (twice) the *mean* angular velocity of any two instantaneously mutually orthogonal line segments, within the flow plane, going through the given point and, consequently, it is (twice) the *mean* angular velocity of all line segments, within the flow plane, going through the given point (Cauchy [2], according to Truesdell and Toupin [13]; Truesdell [12]; Green [7]). Consistently with this interpretation, the *residual* vorticity in 2D can be viewed in terms of (twice) the *least-absolute-value* angular velocity of all line segments, within the flow plane, going through the given point [10], see Fig. 3. Moreover, this figure relates the angular velocities of the key orthogonal line segments (i.e. those with the maximum difference of angular velocities) to the three elementary types of the relative motion near a point.

The *residual* vorticity is zero for all the planar cases where strain rate dominates over vorticity as one portion of line segments rotates in one direction while the other portion of line segments in the opposite sense and two separating non-rotating line segments exist, namely saddle separatrices of the corresponding flow patterns. Further, for a simple shear one non-rotating line segment exists. The *shear* vorticity represents just the difference between (twice) the *mean* angular velocity and (twice) the *least-absolute-value* angular velocity of all line segments.

The integral strength of a vortex is usually calculated as the circulation along the vortex boundary or equivalently, due to Green's theorem, as the surface integral of vorticity over the vortex region. However, it is well known that vorticity is misrepresenting the local intensity of the actual swirling motion of a vortex. For example, one obtains a net circulation for the region of a simple shear due to a net vorticity. In spite of this fact, such vortex measures as the frequently employed initial circulation or downstream circulation still frequently represent basic characteristics in free shear flows as, for example, in bluff-body wakes.

The *residual* circulation is given by a surface quadrature of the *residual* vorticity. Note that for an arbitrarily chosen threshold level the region of the *residual* vorticity forms a subdomain of the vorticity region. The concept of the *residual* circulation can be easily applied to 2D or quasi-2D problems (for the application to jets in crossflow see [11]). The application of the *residual* circulation to arbitrary planar cross-sections of 3D vortical flow fields is also possible, though less direct (that is, if not using a quasi-2D approach for a given cross-section).

The application of the *residual* circulation may reveal interesting flow features. Let us compare a downstream behaviour of the jet in crossflow. Note that the secondary-flow vortical structures form a contrarotating vortex pair, even for twin jets in crossflow with a limited nozzle separation [11].

For all three arrangements of twin jets in crossflow (an oblique one is for 45º) as well as for the single jet in crossflow, the results (averaged for the oblique case) in Fig. 4 indicate — unlike vorticity behaviour — almost universal constant downstream behaviour of the *residual* circulation of the secondary-flow vortices within the measured downstream range (no smoothing). The portion of circulation associated with the shearing effect decays much faster than that based on the *residual* vorticity which remains almost constant. The turbulent vorticity transport across the contrarotating-vortex-pair centreline and the corresponding circulation decay deal predominantly with the *shear* vorticity rather than the *residual* vorticity, at least within the measured downstream range. The behaviour of the *residual* vorticity is a plausible reason for the well-known fact that the contrarotating vortex pair of a jet in crossflow persists far downstream.

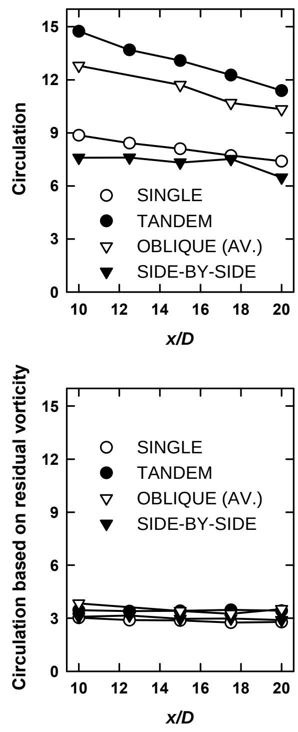

Figure 4. Comparison of circulation with *residual* circulation for jets in crossflow.

Comparison of the TDM Vorticity Outcome with Other Vorticity-Decomposition Techniques

Though several methods of vorticity decomposition were published during last fifty years, see Astarita [1], Wedgewood [14], and the references therein, the only one partially suitable for qualitative comparison purposes is that of Wedgewood [14]. He derived a vorticity decomposition into two parts, the so-called *deformational* vorticity and the *rigid* vorticity. His analysis employs the cross product of a particle's velocity and acceleration, $\mathbf{u}\times \mathrm{D}\mathbf{u}/\mathrm{D}t$, and leads to the evolution equation for the *deformational* vorticity. For the local flow field near a point, Wedgewood [14] adopted the assumption that the quantity $\mathbf{u}\times \mathrm{D}\mathbf{u}/\mathrm{D}t$ must vanish — on average — along three orthogonal axes (according to [14], the same results can be obtained by volume averaging). He derived the evolution equation for the *deformational* vorticity tensor $\boldsymbol{\omega}_\mathrm{D}$ while decomposing the vorticity tensor $2\boldsymbol{\Omega}=\boldsymbol{\omega}_\mathrm{D}+\boldsymbol{\omega}_\mathrm{R}$ where $\boldsymbol{\omega}_\mathrm{R}$ is the *rigid* vorticity tensor (notation $\boldsymbol{\omega}_\mathrm{D}$ and $\boldsymbol{\omega}_\mathrm{R}$ is retained following [14]). As the solution of this equation depends on both space and time derivatives of the velocity-gradient tensor $\nabla\mathbf{u}$, it should be emphasized that the application of the Wedgewood criterial quantity $\mathbf{u}\times \mathrm{D}\mathbf{u}/\mathrm{D}t$ results in a necessity of knowing the temporal changes (time derivatives) of experimentally and/or numerically determined velocity-gradient fields. The prognostic 'Wedgewood equation' employs the temporal Jaumann derivative and requires $\mathbf{S}$ and $\boldsymbol{\omega}_\mathrm{D}$ to be differentiable in both space and time.

Wedgewood [14] formulated a flow classification and general objective (i.e. observer-independent) constitutive equations for the description of complex rheological fluids based on invariants of $\mathbf{S}$ and $\boldsymbol{\omega}_{\mathrm{D}}$, and on the so-called *rigid*-rotational derivative which is objective and quite similar to the objective Jaumann derivative, i.e. $\frac{\mathcal{D}}{\mathcal{D}t}\mathbf{T} = \frac{\mathrm{D}}{\mathrm{D}t}\mathbf{T} + (\boldsymbol{\Omega}\mathbf{T} - \mathbf{T}\boldsymbol{\Omega})$ where $\mathbf{T}$ denotes an arbitrary second-order tensor. The *rigid*-rotational derivative is formally obtainable by substituting the vorticity tensor $\boldsymbol{\Omega}$ in the Jaumann derivative by the tensor $\boldsymbol{\omega}_{\mathrm{R}}$.

Note that the TDM represents a decomposition of the relative motion near a point through the analysis of a “frozen” flow field at a given instant in time. Moreover, unlike the Wedgewood procedure, the velocity-gradient tensor $\nabla\mathbf{u}$ is decomposed as a whole rather than the vorticity tensor itself.

Comparison of the results of Wedgewood [14] with the TDM vorticity outcome for the simplest flows, basic homogeneous flows, leads to the following observation. The solution of the prognostic ‘Wedgewood equation’ for the deformational vorticity tensor is for all the three basic homogeneous flows considered in [14], i.e. for rigid-body rotation, elongational shear-free flow, and simple shear, consistent with the results of the TDM procedure. However, it should be emphasized that the already stated consistency is, at the present state of knowledge, strictly limited to the above mentioned homogeneous flows. That is, we have no justification to extend this similarity obtained for the simplest limiting flow cases to more complex flows. Let us recall that in the TDM procedure quite different arguments — the extraction of an “effective” pure shearing motion determined for a “frozen” flow field at a given instant in time — are employed instead of Wedgewood’s sophisticated time-dependent analysis based on the criterial quantity $\mathbf{u}\times\mathrm{D}\mathbf{u}/\mathrm{D}t$.

Conclusions

This paper deals with the recently proposed triple decomposition of the relative motion near a point. This new kinematic decomposition technique has in 2D and 3D a number of interesting and useful properties, some of them investigated in the present study. The following aspects are examined: pure shearing motion and shearing elements in 3D, orbital compactness vs. arbitrary axial strain of a vortex, the interpretation of *residual* vorticity and the concept of *residual* circulation for the description of vortex strength, and comparison of the TDM vorticity outcome with other vorticity-decomposition techniques.

In particular, it has been found that the vortex-identification requirement of orbital compactness is satisfied. This requirement asks, basically, for the existence of a finite bound for the (uniaxial or radial) stretching of a local flow near a point to be qualified as part of a vortex. The examined TDM vortex-identification outcome contains this bound as an *inherent* feature. Further, the concept of the *residual* circulation given by a surface quadrature of the *residual* vorticity enables to describe a vortex strength effectively and may improve the insight in various flow problems with large-scale vortical structures. The *residual* vorticity provides in 2D a straightforward interpretation in terms of the *least-absolute-value* angular velocity of line segments. Note that the widely used conventional circulation is based on vorticity which is inevitably more or less misrepresenting the local intensity of the swirling motion and, consequently, misrepresenting the vortex geometry and vortex strength.

Finally, apart from the promising properties of the TDM presented above including those mentioned in [10], there is no doubt that the 3D aspects and 3D-data-based evaluation of the proposed method need further and closer examination.

Acknowledgments

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References

[1] Astarita, G., Objective and generally applicable criteria for flow classification. J. Non-Newtonian Fluid Mech. **6**, 1979, 69–76.

[2] Cauchy, A.-L., Mémoire sur les dilatations, les condensations et les rotations produits par un changement de forme dans un système de points matériels. Ex. d'An. Phys. Math. **2**, 1841, 302–330 (Oeuvres (2) **12**, 1841, 343–377).

[3] Chakraborty, P., Balachandar, S., Adrian, R.J., On the relationships between local vortex identification schemes. J. Fluid Mech. **535**, 2005, 189–214.

[4] Chakraborty, P., Balachandar, S., Adrian, R.J., Comment on “Axial stretching and vortex definition” [Phys. Fluids 17, 038108 (2005)]. Phys. Fluids **18**, 2006, 029101-1–029101-2.

[5] Chong, M.S., Perry, A.E., Cantwell, B.J., A general classification of three-dimensional flow fields. Phys. Fluids A **2**, 1990, 765–777.

[6] Cucitore, R., Quadrio, M., Baron, A., On the effectiveness and limitations of local criteria for the identification of a vortex. Eur. J. Mech. B/Fluids **18**, 1999, 261–282.

[7] Green, S.I., Introduction to vorticity. In: Green, S.I. (Ed.), Fluid Vortices. Kluwer, 1995.

[8] Hunt, J.C.R., Wray, A.A., Moin, P., Eddies, stream, and convergence zones in turbulent flows. Center for Turbulence Research Report CTR-S88, 1988, 193–208.

[9] Jeong, J., Hussain, F., On the identification of a vortex. J. Fluid Mech. **285**, 1995, 69–94.

[10] Kolář, V., Vortex identification: New requirements and limitations. Int. J. Heat Fluid Flow **28**, 2007, 638–652.

[11] Kolář, V., Savory, E., Dominant flow features of twin jets and plumes in crossflow. J. Wind Eng. Ind. Aerodyn. **95**, 2007, 1199–1215.

[12] Truesdell, C., The Kinematics of Vorticity. Indiana University Press, 1954.

[13] Truesdell, C., Toupin, R.A., The Classical Field Theories. In: Flügge, S. (Ed.), Encyclopedia of Physics, Vol. III/1, Principles of Classical Mechanics and Field Theory. Springer-Verlag, 1960.

[14] Wedgewood, L.E., An objective rotation tensor applied to non-Newtonian fluid mechanics. Rheol. Acta **38**, 1999, 91–99.

[15] Wu, J.-Z., Xiong, A.-K., Yang, Y.-T., Axial stretching and vortex definition. Phys. Fluids **17**, 2005, 038108-1–038108-4.

[16] Wu, J.-Z., Xiong, A.-K., and Yang, Y.-T., Response to “Comment on ‘Axial stretching and vortex definition’” [Phys. Fluids 18, 029101 (2006)], Phys. Fluids **18**, 2006, 029102-1–029102-2.

[17] Zhou, J., Adrian, R.J., Balachandar, S., Kendall, T.M., Mechanisms for generating coherent packets of hairpin vortices in channel flow. J. Fluid Mech. **387**, 1999, 353–396.