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Model for Anomalous Scaling of Turbulent Structure Functions

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Abstract

A model for inertial range intermittency and anomalous scaling of velocity structure functions is proposed. This model is similar to the Kolmogorov log-normal model except that velocity difference statistics are assumed instead of dissipation statistics. The Navier-Stokes equation is used to derive the basic law for the instantaneous velocity difference between two points. This gives incomplete information about the dependence on scale size and requires a statistical hypothesis in order to compute structure functions and other quantities. The specific assumptions made here relate the singular exponent to the velocity amplitude and give results which agree well with experiment.

Introduction

The purpose of this paper is to study the statistics of the velocity difference between two neighboring points in a turbulent flow. Structure functions of turbulence theory are defined by averages of the velocity difference between neighboring points:

$$B_p(r) = < (\mathbf{v}\cdot\hat{\mathbf{r}})^p > \qquad (1)$$

where $\mathbf{v}(\mathbf{r},\mathbf{x}) = \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$. We have restricted to longitudinal components in this definition. The classical result of Kolmogorov[1] is $B_p = C_p(\varepsilon r)^{p/3}$, where ε is the mean dissipation at a point. It is convenient to use a length scale L defined by $\varepsilon = U^3/L$ where U is the rms velocity fluctuation, then the Kolmogorov result is $B_p = C_p U^p (r/L)^{p/3}$. Experimentally it has been found that $B_p \sim U^p (r/L)^{\zeta_p}$ where $\zeta_p < p/3$. This has been interpreted as being caused by extreme intermittency in the instantaneous dissipation which has an effect on velocity difference statistics even at inertial range scales.

Asymptotic Analysis

Partial Lagrangian Coordinate System

The starting point for an analysis of structure functions is an equation for the difference in velocity between two points. Such an equation may be derived by letting one of the points, $\mathbf{x}$, be a Lagrangian fluid particle, moving with the fluid. The second point, which is not Lagrangian, is slaved to the motion of the first with fixed separation $\mathbf{r}$. It may be easily shown[3] that $\mathbf{v}$ satisfies the Navier-Stokes equation in the form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial \mathbf{r}} = -\frac{1}{\rho}\frac{\partial P}{\partial \mathbf{r}} + \nu\frac{\partial}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{r}}\mathbf{v} \ , \quad \frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{v} = 0 \ , \qquad (2)$$

in which only derivatives with respect to $\mathbf{r}$ occur, none with respect to $\mathbf{x}$. (In an accelerated coordinate system an inertial force appears on the right hand side of (2), which may be absorbed into the pressure.) Since $\mathbf{v} = 0$ at the origin of $\mathbf{r}$, these equations describe a generalized stagnation point flow. It is of some interest to note that Kolmogorov had such a partial Lagrangian system in mind. He considered a coordinate system fixed to a fluid particle and looked at the velocity of another point relative to this moving coordinate system. He did not actually write down (2), but no doubt had this equation in mind.

Outer Equation

Now express these equations in outer variables, using the rms velocity U as the characteristic velocity, and $L = U^3/\varepsilon$ as length. Define dimensionless outer variables $\mathbf{v}_o = \mathbf{v}/U, \mathbf{r}_o = \mathbf{r}/L, \tau = tU/L$. Scaling with these variables gives

$$\frac{\partial \mathbf{v}_o}{\partial \tau} + \mathbf{v}_o\cdot\nabla_o\mathbf{v}_o = -\nabla_o P_o + R_L^{-1}\nabla_o^2\mathbf{v}_o \ , \quad \nabla_o\cdot\mathbf{v}_o = 0 \qquad (3)$$

where $R_L = UL/\nu$ and $\nabla_o = \partial/\partial\mathbf{r}_o$. An outer expansion may be taken in the form

$$\mathbf{v} = U\mathbf{v}_o, \quad \mathbf{v}_o = \mathbf{v}_{o,1} + R_L^{-1}\mathbf{v}_{o,2} + \cdots \ . \qquad (4)$$

The first term $\mathbf{v}_{o,1}$ satisfies (3) without the viscous term.

Inner Equation

Now rescale variables so that the viscous term is retained as $R_L \to \infty$, by defining inner variables

$$\mathbf{r}_i = R_L^{-\alpha}\mathbf{r}_o, \ \ \mathbf{v}_i = R_L^{-\beta}\mathbf{v}_o, \ \ P_i = R_L^{-2\beta}P_o \ . \qquad (5)$$

Then (3) may be rewritten as

$$R_L^{-\beta+\alpha}\frac{\partial \mathbf{v}_i}{\partial \tau} + \mathbf{v}_i\cdot\nabla_i\mathbf{v}_i = -\nabla_i P_i + R_L^{-1-\beta-\alpha}\nabla_i^2\mathbf{v}_i \ . \qquad (6)$$

The conditions that the viscous terms be retained as $R_L \to \infty$ gives the condition $\beta + \alpha = -1$. Clearly another condition would be required to complete this, but it is left undetermined. Equation(6) takes the form (using $\beta = -1-\alpha$)

$$R_L^{2\alpha+1}\frac{\partial \mathbf{v}_i}{\partial \tau} + \mathbf{v}_i\cdot\nabla_i\mathbf{v}_i = -\nabla_i P_i + \nabla_i^2\mathbf{v}_i \qquad (7)$$

which suggests an inner expansion

$$\mathbf{v} = UR_L^{-1-\alpha}\mathbf{v}_i, \quad \mathbf{v}_i = \mathbf{v}_{i,1} + R_L^{1+2\alpha}\mathbf{v}_{i,2} + \cdots \ . \qquad (8)$$

Matching

These two asymptotic expansions describe the same function in different variables. If there is an overlap region where both are valid the method of matched asymptotic expansions can be used. Van Dyke's matching principle states that one should take the inner expansion of the outer expansion and set that equal to the outer expansion of the inner expansion. That prescription was carried out in detail in [3], yielding the functional relation

$$\mathbf{v}_{o,1}(\mathbf{r}_o) = R_L^{-1-\alpha}\mathbf{v}_{i,1}(R_L^{-\alpha}\mathbf{r}_o) \ . \qquad (9)$$

This can be satisfied in the limit $R_L \to \infty$ only if

$$\mathbf{v}_{o,1} = \mathbf{V}r_o^q \ \text{ and } \ \mathbf{v}_{i,1} = \mathbf{V}r_i^q \ . \qquad (10)$$

Here $q = -(1+\alpha)/\alpha$ is undetermined (as is α). That is, substituting these into (9), with $r_i = R_L^{-1/(1+q)}r_o$ satisfies it exactly and any power other than q would give zero or infinity as $R_L \to \infty$. In these equations $\mathbf{V} = \mathbf{V}(\mathbf{x}, t, \hat{\mathbf{r}})$ is a dimensionless random variable; $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$. The significant result is

$$\mathbf{v} = U\mathbf{V}\left(\frac{r}{L}\right)^q \tag{11}$$

in which both $\mathbf{V}$ and q are random variables, i.e. they depend on the realization of the turbulent flow. Longitudinal structure function ensemble averages at a fixed point $\mathbf{x}$ are then given by

$$B_p = \left\langle (\mathbf{v}\cdot\hat{\mathbf{r}})^p \right\rangle = U^p \left\langle (\mathbf{V}\cdot\hat{\mathbf{r}})^p \left(\frac{r}{L}\right)^{qp} \right\rangle \ . \tag{12}$$

Any further conclusions depend on assumed statistics of $\mathbf{V}$ and q. This is addressed in the following section.

Anomalous Scaling

Statistical Assumptions

For longitudinal velocity components we take the velocity function in the form

$$v = V_0 U C(q) \left(\frac{r}{L}\right)^q \tag{13}$$

where

q is a random variable with pdf $f(q)$,

C is a random variable which depends only on q,

V_0 is an independent dimensionless random variable,

U is the rms velocity giving dimension to the expression.

The function $C(q)$ will be assumed to be large for smaller q for reasons which will become apparent below. C can be thought of as a velocity amplitude function which depends on the exponent q, or perhaps it is better to think of q as dependent on the amplitude C. If v is interpreted as the azimuthal velocity in an axially symetric vortex, the vorticity would be given by

$$\omega = V_0 \frac{U}{L} C(q)(q+1)\left(\frac{r}{L}\right)^{q-1} \ , \tag{14}$$

a singular vortex with amplitude which is greater when the singularity is greater (smaller q). Therefore $C(q)$ specifies the distribution of vortex singularities.

Longitudinal structure functions are then given by

$$B_p = < v^p > = U^p < V_0^p > \int_0^\infty \left(\frac{r}{L}\right)^{qp} C^p f(q) dq \ . \tag{15}$$

The idea here is that $f(q)$ has a peak near $q = 1/3$. If $C(q)$ is larger on the left of the peak and smaller on the right then the product $C^p f$ shifts the peak to smaller values as p increases, thus selecting a smaller q in $(r/L)^{qp}$. If the selected q were $1/3$, $(r/L)^{p/3}$ results, the K41 result. A smaller value than $1/3$ gives $(r/L)^{\zeta_p}$ with $\zeta_p < 1/3$, an anomalous result.

In order to illustrate this specific assumptions for $f(q)$ and $C(q)$ have to be made: the pdf $f(q)$ is assumed to be log-normal,

$$f(q) = \frac{1}{\sqrt{(2\pi)}\sigma q} \exp\left(-\frac{(\ln(q/q_0))^2}{2\sigma^2}\right) \ , \tag{16}$$

and C is assumed to be an inverse power of q

$$C(q) = \left(\frac{q}{q_0}\right)^{-\beta} \ . \tag{17}$$

The parameter values have been taken to be $q_0 = .373$ and $\sigma^2 = .0144$ for reasons to be discussed presently and β is taken to be 2. In Figure 1, $C^p f$ is plotted versus q for various values of p. The curves have been normalized so that each has an integral of unity. This quantity is like an effective pdf for q which depends on the order of the structure function.

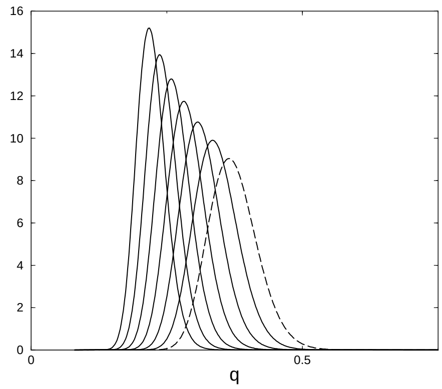

Figure 1: Effective pdf, $C(q)^p f(q)$. The dashed line is the actual log-normal pdf. The curves which peak at smaller q are for $p = 3, 6, 9, \cdots$.

The structure function exponents have been computed from Eq.(15) with these values. For each value of p the integration was carried out for $.02 < r/L < .2$. The resulting function of r/L is close to, but not exactly equal, a power law and was curve-fit to a power law formula to determine the best exponent. For $p = 3$ the result is suposed to be linear, $\zeta_3 \equiv 1$, to agree with the exact Kolmogorov "4/5" law, $B_3 = -.8\varepsilon r \equiv -.8U^3(r/L)$. This was achieved by adjusting the parameter q_0. The parameter σ was adjusted to give $\zeta_6 = 1.80$, which is close to the observed experimental value, and is the usual value assumed. The computed values of ζ_p are plotted in Figure 2 for p as large as 20. The She-Leveque curve agrees with experiments out to about $p = 18$ and can be regarded here as a surrogate for the experiments. The K62 log-normal result levels off at about $p = 16$ and decreases after that. The present result agrees well with the present level of experimenta but finally levels off around $p = 30$.

It was noticed in doing these computations that the results were not exactly power laws. The ζ_p exponents depended slightly on the range of r/L used for the curve fitting. This can be looked at analytically by letting $\beta\sigma^2$ be constant as σ^2 tends to zero. That is as σ^2 gets smaller β becomes larger. This is set up by first writing

$$C(q)^p f(q) = \frac{1}{\sqrt{(2\pi)}\sigma q} \exp\left(-\beta p \ln(q/q_0) - \frac{(\ln(q/q_0))^2}{2\sigma^2}\right)$$

$$= \frac{1}{\sqrt{(2\pi)}\sigma q} \exp\left(\frac{(\beta\sigma^2 p)^2}{2\sigma^2} - \frac{(\ln(q/q_0) + \beta\sigma^2 p)^2}{2\sigma^2}\right)$$

The second line follows by completing the square in the first exponent. Then by a simple change of variables in the integral (15) may be written

$$B_p = U^p < V_0^p > \exp\left(\frac{(\beta\sigma^2 p)^2}{2\sigma^2}\right) \quad \text{times}$$

$$\int_{-\infty}^{\infty} \left(\frac{r}{L}\right)^{\zeta_p \exp(\sigma\xi)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\xi^2}{2}\right) d\xi$$

where, jumping ahead for notational purposes,

$$\zeta_p = p q_0 \exp(-\beta\sigma^2 p) \ . \tag{18}$$

Now expanding in powers of σ, the result is

$$B_p = U^p < V_0^p > \exp\left(\frac{(\beta\sigma^2 p)^2}{2\sigma^2}\right) \quad \text{times}$$

$$\left(\frac{r}{L}\right)^{\zeta_p}\left(1+\left(\zeta_p \ln r/L + \zeta_p^2(\ln r/L)^2\right)\sigma^2/2\cdots\right) \tag{19}$$

Therefore exact power laws result in the limit as $\sigma^2 \to 0$ but there are logarithmic corrections for small but finite σ^2.

In the limit ζ_p is given by (18). This formula has two adjustable parameters. We want $\zeta_3 = 1$, which gives $q_0 = \exp(3\beta\sigma^2)/3$ and we want $\zeta_6 = 2-\mu$, which gives $2\exp(-3\beta\sigma^2) = 2-\mu$. These therefore result in $q_0 = (1/3)(1-\mu/2)^{-1}$ and $\beta\sigma^2 = -(1/3)\ln(1-\mu/2)$. When these are substituted into (18) the result is

$$\zeta_p = (p/3)(1-\mu/2)^{(p/3)-1} \ . \tag{20}$$

This is plotted in Figure 2, where it seen to lie slightly below the computed values.

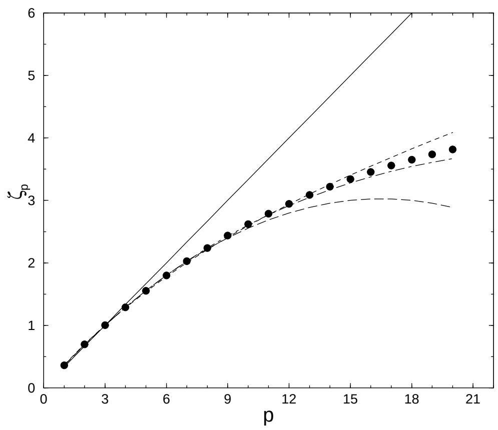

Figure 2: Anomalous exponents ζ_p versus p; Black dots, present computation; Solid straight line is K41, $\zeta_p = p/3$; Long dashed curve, K62 log-normal model $\zeta_p = p/3 + \mu(3p - p^2)/18$, $\mu = .2$; dashed curve is She-Leveque[4], $\zeta_p = p/9 + 2 - 2(2/3)^{p/3}$; Dot-dashed curve is Eq.(20), $\zeta_p = (p/3)(1 - \mu/2)^{(p/3)-1}$, $\mu = .2$.

Relationship to K62 Theory

The Kolmogorov refined similarity theory (K62) depends on the hypothesis that the instantaneous dissipation averaged over a sphere centered at **x** should replace the mean dissipation in the K41 theory. The average dissipation over a sphere of radius l is defined by

$$\varepsilon_l = \frac{3}{4\pi l^3}\int_{r<l} 2\nu D_{ij}(\mathbf{x}+\mathbf{r})D_{ij}(\mathbf{x}+\mathbf{r})d\mathbf{r} \tag{21}$$

where D_{ij} is the rate of strain tensor. It was shown in [3] that by using the relative velocity **v** and the energy equation in the partial Lagrangian coordinate system that (21) may be written, as $R_L \to \infty$,

$$\varepsilon_l = -\frac{3}{4\pi l^3}\int_{r=l} n_i v_i\left(P + \frac{1}{2}v_i v_i\right)dS \ . \tag{22}$$

This relates the average dissipation in the sphere to the flux of energy into the sphere through the outer surface. Now, using (13), and the equivalent expression for the other velocity components, we get

$$\varepsilon_r = \frac{U^3}{L}A_0 C(q)^3\left(\frac{r}{L}\right)^{3q-1} \ , \tag{23}$$

where A_0 is an independent random variable, similar in nature to V_0, but not simply V_0^3 because other velocity components occur in the surface integral. Equation(23) implies $\varepsilon_r = (A_0/V_0^3)v^3/r$ or $v \sim (\varepsilon_r r)^{1/3}$, which is the Kolmogorov refined hypothesis. It follows from (23) that

$$< \varepsilon_r^n > \sim \left(\frac{r}{L}\right)^{\tau_n} \quad \text{with} \quad \tau_n = \zeta_{3n} - n \ , \tag{24}$$

a result which is due to Kolmogorov[2]. Kolmogorov made a statistical assumption for ε_r (log-normal), determined τ_n, and used (24) to determine ζ_p. In the present work the statistical assumptions are on v and τ_n is determined from (24).

Conclusion

A self consistent derivation of turbulent structure function exponents has been made by the method of matched asymptotic expansions using the Navier-Stokes equation, to determine the velocity power law relation (11). This contains two undetermined parameters, an amplitude and a singularity exponent. Because the equations under cosideration are instantaneous, not averaged, these could be random variables depending on the realization of the turbulent flow. Assumptions about the statistical relationship between the amplitude C and the singularity exponent q was made at this stage. The choice made gave results which agree with experiment, but this choice is not unique and perhaps a better motivated idea could throw some light on the nature of the cascade process and give similar results.

One result is a simple expression for the structure function exponent ζ_p,

$$\zeta_p = \frac{p}{3}\left(1-\frac{\mu}{2}\right)^{\frac{p}{3}-1} \tag{25}$$

which satisfies $\zeta_3 = 1$ and $\zeta_6 = 2-\mu$ and agrees with experiments over a wide range.

These results suggest a number of further researches. It should be possible to include viscous effects along the lines of the work done in [3], which should show a slow approach to the limit as Reynolds number tends to infinity.

Given the velocity difference function (13) and the assumed statistics it should be possible to study the velocity pdf. This will require an an additional assumption about the independent random variable V_0.

References

[1] Kolmogorov, A.N., The local structure of turbulence in incompressible viscous fluid for very large Reynolds number, *Dokl. Akad. Nauk. SSSR* **30**, 1941, 299-303. Reprinted in *Proc. Roy. Soc. Lond. A* **434**, 1991, 9-13.

[2] Kolmogorov, A.N., A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. *J. Fluid Mech.* **13**,1965, 82-85.

[3] Lundgren, T.S., Kolmogorov turbulence by matched asymptotic expansion, *Phys. Fluids* **15**, 2003, 1074-1081.

[4] She, Z-S and Leveque, E., Universal scaling laws in fully developed turbulence, *Phys. Rev. Lett.* **72**, 1994, 336-339.