14th Australasian Fluid Mechanics Conference
Adelaide University, Adelaide, Australia
10-14 December 2001

The mean velocity profile for wall-bounded flows at low Reynolds number

T. B. Nickels

[1]Department of Engineering
Cambridge University, Cambridge, ENGLAND

Abstract

A useful form is proposed for the mean velocity profile in turbulent wall-bounded flows. This form is used to investigate the effects of pressure-gradients on low Reynolds number flows.

Introduction

Since the analysis which led to the logarithmic law followed by Coles' law-of-the-wall/law-of-the-wake there have been various variations on this basic formulation used by experimentalists and turbulence modellers to fit velocity profiles for turbulent flows. Despite this proliferation of functional forms there is no one form that correctly takes into account all of the proper asymptotic behaviour of these flows. Here a relatively simple, but asymptotically correct, form is proposed and used to analyse the effects of pressure-gradient on the inner boundary layer at low Reynolds numbers.

Background

One of the motivations for this paper and the analysis of data underlying it was the author's frustation in attempting to use existing "curve-fits" for the mean-velocity profiles for turbulent boundary layers. When attempting to examine the effects of changes in external conditions on boundary layers it is useful to reduce the description to a small number of parameters. Coles' law-of-the-wall/law-of-the-wake was a significant step forward in this direction. There have also been other formulations such as [12] and [8] which attempt to fit the whole of the boundary-layer with the inclusion of a wake function with varying success. Different formulations have advantages and disadvantages but all share some deficiencies.

Some of the deficiencies in existing curve-fits are;

1. Unrealistic behaviour for strongly accelerated flows with small wakes near the outer edge of the layer.
2. Difficulty in fitting standard log-law profiles to low Reynolds number profiles.
3. Incorrect asymptotic behaviour near the wall which becomes significant in large pressure-gradients
4. Difficulty with some wake profiles that do not have asymptotically zero gradient beyond the edge of the boundary layer.

The first arises from the fact that the wake is reduced in accelerating flows and may become close to zero. The actual profiles round-off to give close to zero velocity gradient near the edge of the layer whereas the in the standard log-law the gradient dies off rather slowly. This also leads to difficulty in identifying the edge of the layer since, without the wake the log-law continues indefinitely.

The second point relates to evaluating parameters such as the wake strength and the skin-friction (experimentally) by fitting the profiles. At low Reynolds number the increase due to the wake leads to an apparent change in the log-law intercept when trying to fit standard formulations since a log-law either doesn't exist, or is swamped by the wake component.

The third point refers to the observation that in low Reynolds number flows with very strong pressure-gradients the region close to the wall is affected by the pressure-gradient resulting in noticeable differences close to the wall.

The final point is, perhaps, minor. It relates to the fact that the mean-velocity for boundary layers reaches a constant at, or just beyond the edge of the layer. Coles' wake-function and others have zero gradient right at the edge but behave anomalously beyond (they are more appropriate for pipe or channel flows). This leads to small difficulties when trying to fit these functions to real data which must then be truncated at the edge.

The following functional form was developed to satisfy certain asymptotic conditions. The functional form for the mean-velocity profile of a boundary-layer and other wall-bounded flows must satisfy certain boundary conditions and achieve certain states asymptotically. In the inner region close to the wall it is well known that the Taylor-series expansion for a wall bounded flow is of the form

$$U^+ = y^+ + \tfrac{1}{2}p^+y^{+2} + a_1y^{+4} + \text{H.O.T.}\ y \to 0 \qquad (1)$$

where $U^+ = U/U_\tau$, $y^+ = yU_\tau/\nu$, $p^+ = (\nu/U_\tau^3)dp/dx$ and $U_\tau = \sqrt{\tau_o/\rho}$ is the wall-shear velocity. This expansion should apply for all wall-bounded flows close enough to the wall. At high Reynolds number the value of p^+ is frequently very small and so may be neglected, however it is strictly part of the expansion.

At the outer edge of the boundary layer $\partial U/\partial y \to 0$ as $y \to \infty$ or in the case of a pipe or channel flow $\partial U/\partial y = 0$ at $y = H$ where H is the distance from the wall to the centre of the flow (i.e. H is the channel half-width or H is the radius of the pipe).

Considering these two conditions first it may be noted that none of the traditionally proposed functional forms satisfy either. Near the wall the effect of pressure gradient is generally neglected, perhaps because it is small in many flows. In the outer part of the flow traditional forms assume a logarithmic law extending to infinity with a wake function superimposed. Hence while the wake function may have zero gradient (or asymptotically zero gradient) at the outer edge of the layer the gradient is at least $U_\tau/\kappa y$ which does approach zero - though very slowly.

In order to address these deficiencies a new functional form for the mean velocity profile is proposed that retains the overall general structure embodied in the traditional functional forms but has the correct asymptotic behaviour.

The proposed functional form

The proposed form has a three part structure in which the boundary layer profile is the sum of three separate boundary layers i.e. a viscous sublayer, an overlap layer and a wake layer. Each individually has asymptotically zero gradient at the outer edge and hence the sum satisfies the outer boundary condition.

Sublayer region

The functional form chosen for the sublayer region is as follows.

$$U^+ = \tfrac{1}{a_o}(1 - (1 + 2a_o y^+ + \tfrac{1}{2}(3a_o^2 - a_o p^+)y^{+2} - \tfrac{3}{2}a_o^2 p^+ y^{+3})e^{-3a_o y^+}) \quad (2)$$

This function asymptotes to a constant value at large y^+ and has a Taylor series expansion of

$$U^+ = y^+ + \tfrac{1}{2}p^+ y^{+2} - \tfrac{9}{4}a_o^2(\tfrac{1}{2}a_o + p^+)y^{+4} + \text{H.O.T.} \quad (3)$$

One interesting feature of this type of function is that it is possible to modify the function to achieve the desired Taylor series expansion (which is what has been done here). It also asymptotes to a constant value at large y^+ and so further from the wall the effect is simply to shift the profile up. This has some similarity with the approach of [8]. The coefficient $1/a_o$ may be considered to be be a measure of the thickness of the sublayer or alternatively, following a suggestion of [1] a critical Reynolds number of the sublayer. Various measurements suggest that this thickness varies with p^+ and also with other parameters such as roughness. The value is directly related to the intercept of the log-law in the traditional formulation. It should be noted that in the Taylor series expansion the coefficient of the fourth order term is directly proportional to the first term in the expansion for the Reynolds shear-stress (i.e. $\overline{-uv}^+ = 4a_1 y^{+3}$)— if we neglect the streamwise gradients of wall shear-stress and turbulence intensities. This is exactly true in the case of pipe or channel flows. Hence the expansion suggests that the increase of the Reynolds shear-stress at the wall depends on p^+. This point will be addressed later.

Overlap region

The function chosen for the overlap region is such that its expansion does not affect the correct terms in the near wall expansion and the log-law "rounds off" near the edge of the layer sufficiently quickly.

$$U^+ = \frac{1}{6\kappa}\ln\left(\frac{1 + (0.66a_o y^+)^6}{1+\eta^6}\right) \quad (4)$$

where $\eta = y/\delta$ and δ is a measure of the overall boundary layer thickness (the value of δ is similar to the 99%–thickness, though not based on this). The first term in the expansion of this function is of order y^{+6} and hence does not affect the first terms in the near wall region. This function also asymptotes to a constant for large y as required by the outer boundary condition. In the intermediate range when y^+ is large ($(0.66a_o y^+)^6 >> 1$)and η is small ($\eta << 1$) the form of this function combined with the sublayer function is approximately given by

$$U^+ = \frac{1}{\kappa}\ln(y^+) + \frac{1}{\kappa}\ln(0.66a_o) + 1/a_o \quad (5)$$

and hence it reduces to the traditional log-law. The log-law intercept is thus $\frac{1}{\kappa}\ln(0.66a_o) + 1/a_o$. This is very important in accelerated boundary layers where the wake may become negligible and the standard log-law continues to increase. The constant of 0.66 has simply been chosen to give a slightly better fit to the data in the inner region.

Wake region

The form for the wake has been chosen such that it approaches a constant value outside the layer and so that it has a form that is very similar to that of a two-dimensional wake in free-shear flow. It is given by

$$U^+ = a_2(1 - e^{-2(\eta^2+\eta^6)}), \quad (6)$$

where a_2 is the wake factor that varies with pressure gradient in a similar manner to Coles' wake factor Π. In fact the value is roughly equal to $2\Pi/\kappa$ although the rounding-off of the log-law means the correspondence is not exact. This form of wake function is not a great advance over that proposed by Coles and others except that it quickly asymptotes to the free-stream velocity at the edge of the boundary layer. This is the correct behaviour for boundary layers, whereas Coles function is more appropriate to pipe and channel flows. It is interesting to note that the form is quite similar to that of turbulent free wake profiles and the η^6 term may be considered to be a correction due to intermittency near the edge of the layer. As noted, this is not the most appropriate form for pipe and channel flows, although it does fit the data very well except for a very small error right at the edge. In these flows a form such as Coles' wake function may be more appropriate since the boundary condition is different. It is also possible to modify the above function using different co-ordinates such that y reduces as the centre of the flow is exceeded (for example $y = |R - r|$ for a pipe-flow) but this will not be pursued here.

Hence we have a three-layer structure which is the same as the traditional framework. The buffer region is contained in the overlap between the two inner layers. When all is combined we have a boundary layer profile with only 3 parameters, the constant a_o, which is essentially directly related to the log-law intercept, κ the universal log-law constant and a_2 which is directly related to Coles' wake parameter. The only additional parameter that is involved is p^+ which is an external parameter and that must be involved due to the expansion of the mean flow near the wall. The other point to note is that the combined function naturally splits into $U^+ = f(y^+) + g(\eta)$ where f and g contain logarithmic parts, which is again neatly in line with the traditional approach.

The final functional form is the sum of these three parts and hence is given by

$$\begin{aligned} U^+ = {} & \tfrac{1}{a_o}(1 - (1 + 2a_o y^+ + \tfrac{1}{2}(3a_o^2 - a_o p^+)y^{+2} \\ & - \tfrac{3}{2}a_o^2 p^+ y^{+3})e^{-3a_o y^+}) \\ & + \frac{1}{6\kappa}\ln\left(\frac{1 + (0.66a_o y^+)^6}{1+\eta^6}\right) \\ & + a_2(1 - e^{-2(\eta^2+\eta^6)}) \end{aligned} \quad (7)$$

Whilst this may appear complicated the only parameters are a_o, which is dependent on p^+ (where p^+ is small a_o tends to a universal constant), κ, which is a universal constant, and a_2, which is a wake parameter. It may be noted that this has the correct asymptotic behaviour, both near the wall and near the edge of the layer — this is an improvement over many forms in common use. It also includes the dependence of the sublayer thickness on p^+ which is a necessary requirement in order to correctly reproduce data.

A least squares curve-fitting procedure was used to find the values of the unknown coefficients and examine their universality. Comparison of this function to a large amount of data suggests that a universal value of κ provides an excellent fit to the velocity profile. At low Reynolds number and with strong pressure-gradients the constant a_o varies with p^+. Hence $a_o = f(p^+)$ where f denotes some unknown function. This is not surprising and the shifting of the log-law intercept has been noted before. In many cases of moderate-to-high Reynolds number boundary layers the effect of p^+ may be neglected and so the functional form is slightly simplified.

The values of the coefficients

A very large amount of data from different flows has been used to evaluate the values of the coefficients and their universality or variation. DNS data was used for many of the cases. This avoids the uncertainties in the value of the wall shear-stress and probe wall corrections that are generally (though not always) associated with experimental data. DNS data also has sufficient accuracy and resolution near the wall to examine the behaviour of the inner flow. It should be pointed out that DNS data does have its own uncertainties related to treatment of the inlet and outlet conditions and potential spatial resolution problems. This has been noted in some cases when comparing different simulations for the same flow with matched parameters: the velocity profiles are not always the same. One reason for this may be that the final profile depends on the method used to initiate the flow.

Despite this caveat DNS data provides a good test bed to evaluate the values and universality of the constants.

Examination of a wide range of data has shown that a universal constant value of $\kappa = 0.395$ provides a very good fit to all the data.

Since, as has already been noted, a_o varies with p^+ then we first examine zero pressure-gradient boundary layer data. In particular the simulations of [11]. From this we find the value of 0.0857 for $\kappa = 0.395$. This leads to an effective log-law intercept of 4.397. If we consider $1/a_o$ to be some measure of the sublayer thickness this value corresponds to $y^+ = 11.7$ which is close to the position of maximum production in the layer.

These two coefficients define the functional form completely except for the wake parameter which appears to vary with Reynolds number for zero-pressure gradient flow and is very sensitive to pressure-gradients. As a matter of interest the wake parameter for Spalart's highest Reynolds number in this formulation is 1.597 (also as noted by Spalart it does not vary much for the three Reynolds numbers examined).

The variation of a_o with pressure-gradient

If the pressure-gradient parameter p^+ is small (say, less than 0.002) then a_o is constant and the only free parameter is the wake strength a_2. The parameter p^+ decreases with increasing Reynolds number and hence at moderate Reynolds number it is acceptable to neglect the effect - this leaves us with a universal law of the wall $U^+ = f(y^+)$ and a_o may then be considered to be a universal constant (at least for flows on smooth walls that are nominally two-dimensional or axisymmetric). At low Reynolds numbers this is not necessarily the case and the effect of p^+ cannot be neglected. Cases of interest where this may occur are in highly accelerated flows approaching relaminarisation and in flows approaching separation (as $U_\tau \to 0,\ p^+ \to \infty$). This is a good argument for attempting to get the asymptotic behaviour of the profiles correct.

Figure 1 shows the variation of a_o with p^+ for a range of flows. The data shown are from [10], [11], [9] and [6] for boundary layers, [7] and [4] for channel flows and [5] and [3] for pipe flows. Whilst there is a some scatter a definite trend is apparent. A line of best fit is also shown for later use. It should be noted that it is not clear that a_o is a function only of p^+ and analysis of data suggests that there may be some memory of the boundary layer in flows where p^+ is changing quickly. This issue will not be addressed here but the values shown are for flows in which p^+ varies only slowly. The scatter may well be due to memory effects.

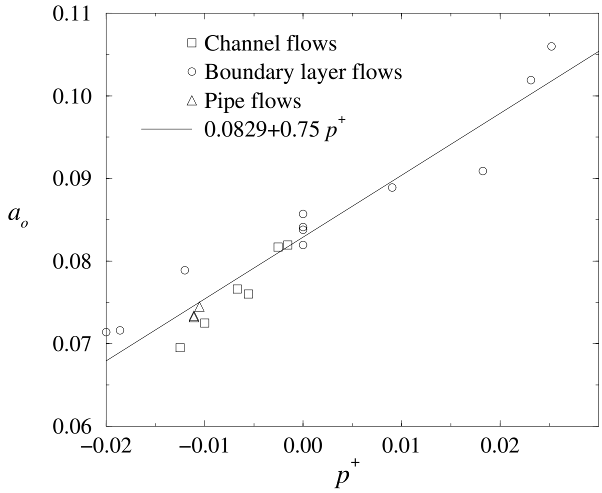

Figure 1: Variation of the coefficient a_o versus p^+

An example of the fit to the data of [6] is shown in figure 2.

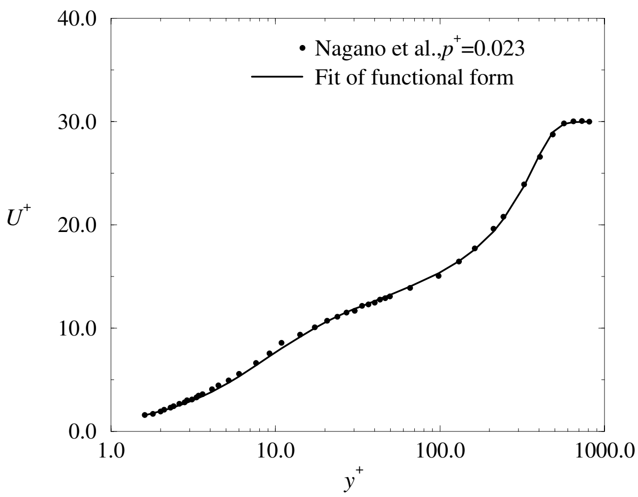

Figure 2: Fit to strong adverse pressure-gradient data

The first order behaviour of the Reynolds shear stress

As pointed out earlier the above form, if correct, has implications for the leading order term in the expansion of the Reynolds shear-stress. As already stated the coefficient of the fourth order term in the expansion of the mean flow, a_1, is directly proportional to the leading term in the expansion of the Reynolds shear-stress. If we take the value of $a_o = 0.0829 + 0.75p^+$ as given above then substitute into the coefficient as given in (3) and truncate for small p^+ we find that the Reynolds shear-stress, to leading order should behave approximately as $\overline{-uv}^+ = (0.0026 + 0.132p^+)y^{+3}$. It is well known that the leading order term in the Reynolds shear-stress is affected by pressure-gradient and increases in adverse pressure-gradients and decreases in favourable pressure-gradients. Figure 3. shows data from the same authors as above for the leading order term, $4a_1$ plotted against p^+. Also shown is a line of best fit and the truncated function above. The results suggest that this functional form correctly captures the leading order behaviour of the Reynolds shear-stress as it varies with pressure-gradient.

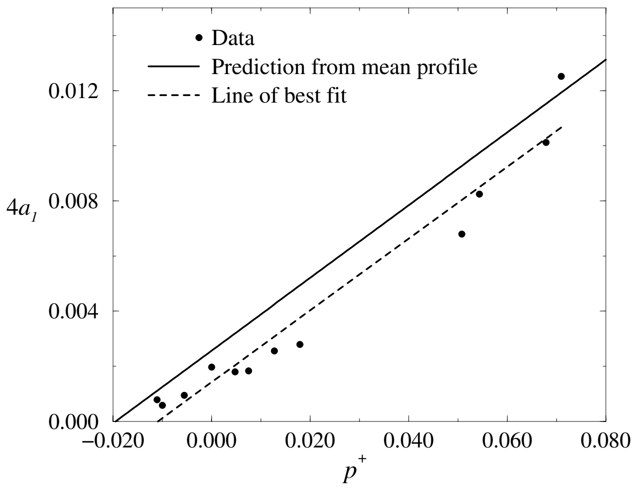

Figure 3: Variation of the coefficient $4a_1$ versus p^+

Conclusions

A useful functional form for the mean velocity profile has been developed that provides both an excellent fit to data and has the correct asymptotic behaviour. It also has a nice conceptual structure that is consistent with the physical and intuitive understanding of boundary layer structure. It is consistent with the changes that occur in leading order behaviour of the Reynolds shear stress observed in measurements. It is hoped that this will provide a useful tool for analysing data and may also be of some use in numerical schemes.

Acknowledgements

This type of work depends on a close examination of a wide range of data. The author has had access to this data through the generosity of various researchers in making their data freely available. This generosity has been further organised through the establishment of a range of databases that have been established throughout the world and can be readily accessed via the internet. In particular the author has made extensive use of the following databases. The ERCOFTAC database, the Data-Base of Turbulence and Heat Transfer (supported by the Ministry of Education Science and Culture in Japan), the AGARD database of test-cases for large eddy simulation. In addition Dr Martin Skote provided me with a link to his thesis data for which I am very grateful.

References

[1] Clauser, F. H., The turbulent boundary layer, *Advances in Mechanics*, **4**, 1956, 1–51.

[2] Coles, D. E., The law of the wake in the turbulent boundary layer, *J. Fluid Mech.*, **1**, 1956, 191–226.

[3] Eggels, J., Unger, F., Weiss, M. H., Westerweel, J., Adrian, R. J., Freidrich, R. and Nieuwstadt, F. T. M., Fully developed pipe-flow: A comparison between direct numerical simulation and experiment., *J. Fluid Mech.*, **268**, 1994, 175–207.

[4] Kuroda, A., Kasagi, K. and Hirata, M., Investigation of dynamic effects of the mean shear rate on the wall turbulence via direct numerical simulation, *Proc. 27th Natl. Heat Transfer Symp. of Japan*, **1**, 1990, 46–48.

[5] Loulou, P., Moser, R., Mansour, N. and Cantwell, B., Direct simulation of incompressible pipe flow using a b-spline spectral method., *NASA Tech Rep TM 110436*.

[6] Nagano, Y., Tagawa, M. and Tsuji, T., Effects of adverse pressure gradients on mean flows and turbulence statistics in a boundary layer, *Proc. 8th Symp. on Turb. Shear flows*, 7–20.

[7] Ohtsuka, A., abd T. Tsuji, O. I. and Nagano, Y., The mechanism of retransition in wall turbulence, *Proc. 9th Symp on Computational Fluid Dynamics*, 499–500.

[8] Reichardt, H., Vollstandige darstellung der turbulenten geschwindigkeitsverteilung in glatten leitungen, *ZAMM*, **31**, 1951, 208–219.

[9] Skote, M., Henkes, R. A. W. M. and Henningson, D. S., Direct simulation of self-similar turbulent boundary layers in adverse pressure gradients., *Flow, Turbulence and Combustion*, **60**, 1999, 47–83.

[10] Spalart, P. R., Numerical study of sink flow boundary layers, *J. Fluid Mech.*, **172**, 1986, 307–328.

[11] Spalart, P. R., Direct numerical study of a turbulent boundary layer up to $Re_\theta = 1410.$, *J. Fluid Mech.*, **187**, 1988, 61–98.

[12] Van Driest, E. R., On turbulent flow near a wall, *J. Aerosp. Sci*, **23**, 1956, 1007–1012.