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Turbulent Boundary Layer Evolution Towards Equilibrium Conditions

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Abstract

The outline for a closure hypothesis to enable the computation of the streamwise evolution of two-dimensional turbulent boundary layers in arbitrary pressure gradients is presented. Utilising the Coles [1] logarithmic law of the wall and law of the wake formulation, in conjunction with the mean continuity and mean momentum equations, the important non-dimensional parameters which describe the state of a general non-equilibrium boundary layer are identified. These parameters form the basis of the closure hypothesis, which is achieved empirically from experiments. The range of application of the proposed scheme is examined by studying a zero-pressure-gradient flow leading to a sink flow.

Introduction

The closure scheme of Marusic *et al.* [5] is based upon classic similarity laws. These concepts provide a convenient means to derive an analytical expression for shear stress distribution. This expression, in turn, yields the important non-dimensional mean flow parameters. The relationship between these parameters is assumed to be universal and is obtained experimentally. As existing data is sparse, an interpolation and extrapolation scheme is devised to compute the development of flows not already observed. The result is a pair of coupled first order ODEs which allow the streamwise evolution of the layer to be computed when the initial velocity profile and the streamwise distribution of freestream velocity are known.

Closure Scheme

The mean velocity profile of a wall bounded turbulent boundary layer is classically described by the Coles [1] law of the wall and law of the wake formulation given by

$$\frac{U}{U_\tau} = \frac{1}{\kappa}\ln\left[\frac{zU_\tau}{\nu}\right] + A + \frac{\Pi}{\kappa}W_c\left[\eta,\Pi\right] \qquad (1)$$

where it is generally assumed that κ is equal to 0.41 and A is equal to 5.0. U is the mean streamwise velocity, U_τ is the friction velocity, z is the distance normal to the wall, ν is the fluid kinematic viscosity, Π is the Coles wake factor which generally varies with streamwise distance, W_c is the Coles wake function and $\eta = z/\delta_c$, where δ_c is the boundary layer thickness.

For analytical work, the wake function is expressed as a polynomial. To ensure that the gradient $\partial U/\partial z = 0$ at $\eta = 1$ a corner function is required. Jones [3] introduced the corner function $-\eta^3/3\kappa$ to achieve the required peel off from the law of the wall, thus

$$\frac{U}{U_\tau} = \frac{1}{\kappa}\ln\left[\frac{zU_\tau}{\nu}\right] + A - \frac{1}{3\kappa}\eta^3 + \frac{\Pi}{\kappa}2\eta^2(3-2\eta). \qquad (2)$$

From this velocity distribution, the continuity equation and the mean momentum equation, the expression for the shear stress distribution is obtained;

$$\frac{\tau}{\tau_0} = f_1\left[\eta,\Pi,S\right] + g_1\left[\eta,\Pi,S\right]\zeta + g_2\left[\eta,\Pi,S\right]\beta. \qquad (3)$$

The three functions f_1, g_1 and g_2 are known universal analytical functions. ζ is the wake strength gradient parameter given by $\zeta = S\delta_c d\Pi/dx$ and β is the Clauser pressure gradient parameter, $\beta = (\delta^*/\tau_0)(dp/dx)$, where $S = U_1/U_\tau$ is the skin friction parameter, U_1 is the freestream streamwise velocity, x is the streamwise coordinate, δ^* is the displacement thickness of the boundary layer, τ_0 is the wall shear stress and p is the average pressure. From equation (3) the appropriate non-dimensional parameters for calculating the streamwise evolution of the turbulent boundary layer are Π, S, β and ζ.

For general non-equilibrium flow, the closure problem becomes one of considering the following relation

$$\mathcal{F}\left[\Pi,S,\beta,\zeta\right] = 0, \qquad (4)$$

assuming that $\mathcal{F}$ is universal and only the four parameters are required in its formulation.

As experimental data is sparse, an interpolation and extrapolation scheme is developed to obtain a formulation for $\mathcal{F}$. Consider the $S-\beta$ plane for a fixed Π which contains an experimental data point for which S, β, ζ and Π are all known and from equation (3) τ/τ_0 as a function of η is also known. It is assumed the shear stress is fixed by a two-parameter family of the form

$$\frac{\tau}{\tau_0} = f\left[\eta,\Pi,\beta_a\right]. \qquad (5)$$

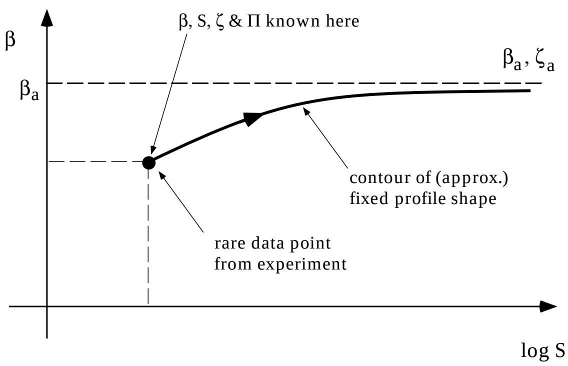

Figure 1: Fixed shear stress profile for a fixed Π.

By performing a curve-fit, the contour of a fixed shear stress profile shape can be traced out. As $S \to \infty$, β and ζ approach asymptotic values β_a and ζ_a, respectively, as shown in figure 1. The process of keeping the profile shape fixed is referred to as profile matching. The profile matching is achieved by using least-squares approximation.

By repeating the profile matching procedure for different values of Π, it is possible to map out lines of constant ζ_a in the $\Pi - \beta_a$ plane and thus obtain a known function

$$\psi[\Pi, \beta_a, \zeta_a] = 0. \tag{6}$$

This shear stress profile matching procedure produces isosurfaces of ζ which can be mapped out in $\Pi - \beta - S$ space. Hence $\mathcal{F}[\Pi, S, \beta, \zeta] = 0$ is known.

Evolution Equations

Evolution equations, forming a set of first order ordinary differential equations for S and Π, are obtained using the logarithmic law of the wall, law of the wake and the momentum integral equation.

The first evolution equation, derived from the momentum integral equation together with the law of the wall and law of the wake, is given by

$$\frac{dS}{dR_x} = \frac{\chi\left[R_x, R_L\right] R\left[S, \Pi, \zeta, \beta\right]}{SE\left[\Pi\right]\exp\left[\kappa S\right]}, \tag{7}$$

where

$$\begin{aligned} R &= \frac{S}{\kappa S^2 C_1 - \kappa S C_2 + C_2} \\ &+ \frac{\beta(2SC_1 - C_2)}{C_1(\kappa S^2 C_1 - \kappa S C_2 + C_2)} \\ &+ \frac{\zeta(\frac{dC_2}{d\Pi} - S\frac{dC_1}{d\Pi} - 2(C_2 - SC_1))}{\kappa S^2 C_1 - \kappa S C_2 + C_2}, \end{aligned}$$

$$C_1\left[\Pi\right] = \int_0^1 \frac{U_1 - U}{U_\tau} d\eta, \qquad C_2[\Pi] = \int_0^1 \left(\frac{U_1 - U}{U_\tau}\right)^2 d\eta$$

$$E[\Pi] = \exp\left[-\kappa\left(A + \frac{2\Pi}{\kappa} - \frac{1}{3\kappa}\right)\right]$$

where $R_x = xU_0/\nu$, U_0 is the freestream velocity at some initial point $R_x = 0$ where $x = 0$, $R_L = LU_0/\nu$ is the overall Reynolds number of the apparatus and $\chi = U_1/U_0$.

The evolution equation for Π is found from the definition of ζ in conjunction with the law of the wall and law of the wake, without reference to momentum balances, and is given by

$$\frac{d\Pi}{dR_x} = \frac{\zeta\chi\left[R_x, R_L\right]}{S^2 E\left[\Pi\right]\exp\left[\kappa S\right]}. \tag{8}$$

The definition of β, along with equation (2), leads to an auxiliary equation

$$S^2 E\left[\Pi\right]\exp\left[\kappa S\right]\frac{1}{\chi^2}\frac{d\chi}{dR_x} = -\frac{\beta}{C_1\left[\Pi\right]}. \tag{9}$$

Equations (7) and (8) form a set of coupled first order ODEs for S and Π and equation (9) is an auxiliary equation for β where χ is a known given function of R_x and R_L. Given equations (7), (8), (9) and the second auxiliary equation of $\mathcal{F}$, it is possible to compute the evolution of a boundary layer.

Sink Flow

The study of sink flow turbulent boundary layers is of particular interest as according to Townsend [7] and Rotta [6] they represent the only flow case which will evolve to precise equilibrium on a smooth wall. The freestream velocity distribution of a sink flow turbulent boundary layer corresponds to that of a potential sink. Figure 2 shows the sink flow with a virtual sink of strength Q located at a distance L from the origin.

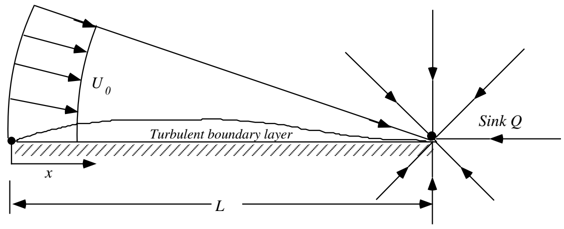

Figure 2: Sink flow.

For sink flow, the acceleration parameter $K = (\nu/U_1^2)(dU_1/dx)$ is a constant. When K is known, the auxiliary equation for β simplifies to an algebraic equation of the form

$$-\beta = KC_1 S^2 E[\Pi]\exp[\kappa S]. \tag{10}$$

The following functional form of $\psi[\Pi, \beta_a, \zeta_a] = 0$ proposed by Marusic *et al.* [5] is based upon the experimental data of Jones [3] for equilibrium sink flow,

$$\zeta_a = (0.85 - 6.9\Pi + 8\Pi^2)\Delta\beta_a \tag{11}$$

where

$$\Delta\beta_a = \beta_a - \beta_{ae}$$

and

$$\beta_{ae} = -0.5 + 1.38\Pi + 0.13\Pi^2$$

where β_{ae} is the value of β_a for $\zeta_a = 0$.

It must be noted that for sink flow the evolution equations become autonomous and can therefore be displayed on a $S - \Pi$ phase plane where solution trajectories cross only at critical points. The coupled evolution equations (7) and (8) become autonomous by an appropriate change in the variable R_x (i.e. $T_x = -\ln(1 - R_x K)/K$), therefore R_x does not appear explicitly. The autonomous evolution equations are therefore of the form

$$\frac{dS}{dT_x} = \psi_1[\Pi, S, K], \qquad \frac{d\Pi}{dT_x} = \psi_2[\Pi, S, K]. \tag{12}$$

Experimental Method

Experimental data was obtained from boundary layer measurements taken in an open return blower wind tunnel. The overall working section length was 5.4m. An overpressure existed at the beginning of the working section, allowing the initial pressure gradient to be controlled by bleeding air from the central stream via a series of four adjustable louvres. A sink flow pressure gradient produced by an inclined ceiling followed on from the louvred section. For the experiments, the louvres were closed, resulting in a zero-pressure-gradient leading to the sink flow, as shown in figure 3. The reference location $x = 0$ was at the commencement of the sink flow,

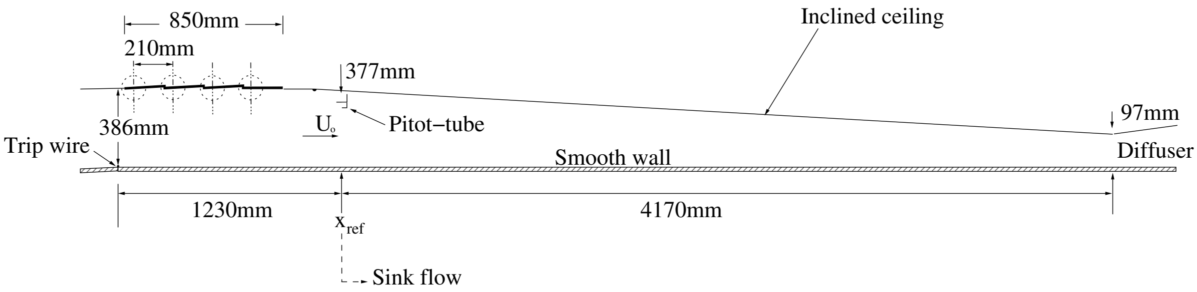

Figure 3: Schematic of the working section with the louvres closed.

1.23m downstream from the beginning of the working section, where a Pitot-static tube was fixed in the ceiling to measure U_0.

The required smooth wall turbulent boundary layer developed along a varnished and polished wood floor in the louvred section and a smooth acrylic floor in the favourable pressure gradient section. Care was taken to ensure that the join between the two floors was smooth.

Mean velocity profiles were measured at two acceleration levels, $K = 5.39 \times 10^{-7}$ and 2.70×10^{-7}, using a Pitot-static tube connected to a MKS Baratron manometer, type 310CD-00010. The measurements were taken at 13 streamwise stations from $x = -800$mm (zero-pressure-gradient flow) to $x = 3600$mm. The Pitot tube results were corrected for shear by applying the MacMillan [4] correction, i.e. an addition of $0.15d$ to locate the effective centre of the Pitot tube, where d is the outer diameter of the Pitot-tube ($d = 1.0$mm).

Results and Discussion

The mean velocity profiles were fitted to the law of the wall and law of the wake formulation given by equation (2). Figure 4 shows the good fit obtained when $\kappa = 0.41$ and $A = 5.0$.

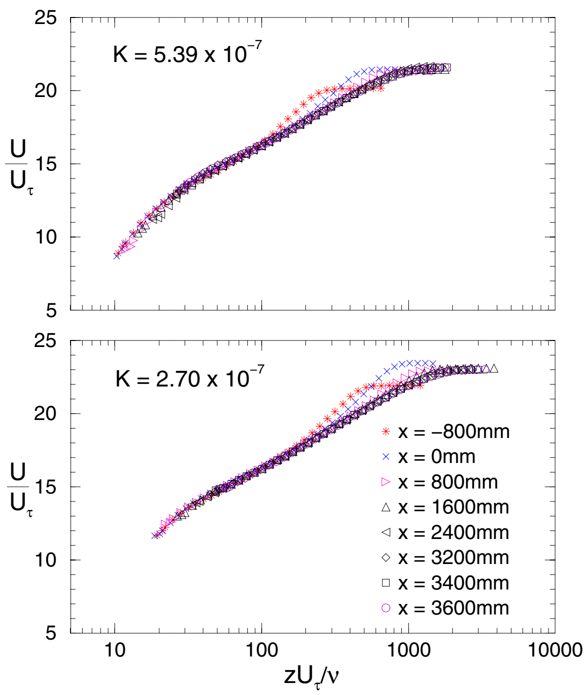

Figure 4: Mean flow results.

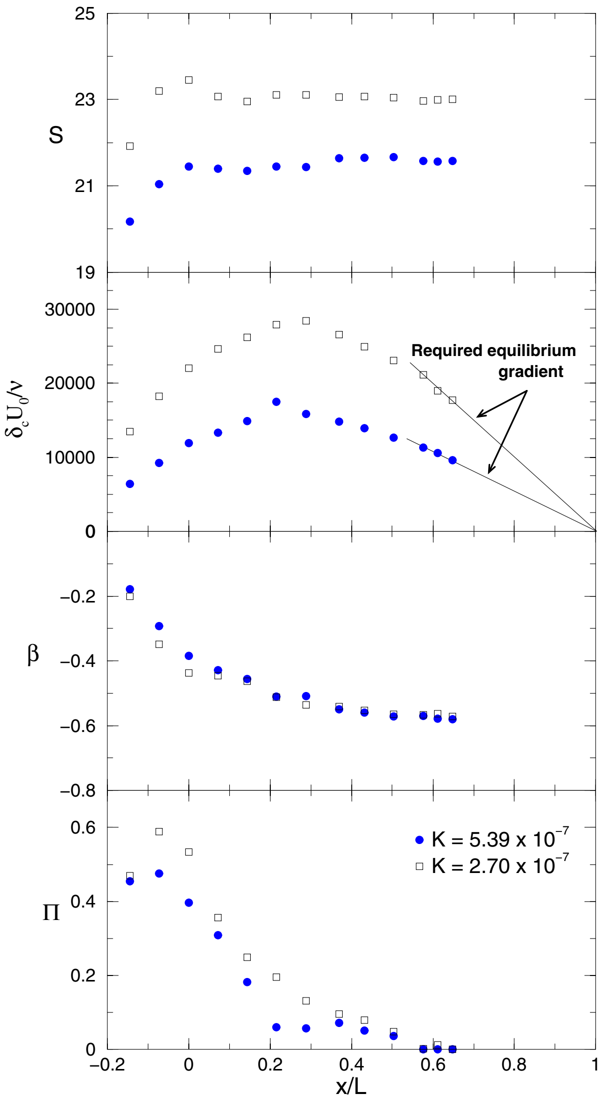

Figure 5: Evolution of the mean flow parameters.

For both flow cases, the profiles became self-similar at approximately $x/L = 0.57$. (Note that L is the location of the virtual sink and was equal to 5.55m based upon the C_p distribution.) The evolution of the mean flow parameters is shown in figure 5 and it can clearly be seen that equilibrium was reached at approximately $x/L = 0.57$. For precise equilibrium, as defined by Rotta [6], the following conditions must be satisfied

$$S = \text{constant}, \qquad \frac{d\delta_c}{dx} = \text{constant}, \qquad \beta = \text{constant}.$$

Also, Coles [2] proposed that a turbulent boundary layer will evolve to an equilibrium solution of $\Pi = 0$, which corresponds to 'pure wall' flow. For the sink flow experiments conducted by Jones [3], only the $K = 5.39 \times 10^{-7}$ case achieved pure wall flow. In the present study, the last three profiles for both K values satisfied the stated equilibrium conditions beyond $x/L = 0.57$. Figure 5 shows that S, $d\delta_c/dx$ and β are all constant and $\Pi = 0$.

The momentum integral equation, given by

$$\frac{d\theta}{dx} + \frac{(H+2)\theta}{U_1}\frac{dU_1}{dx} = \frac{C_f'}{2}, \tag{13}$$

can be utilised to calculate the local coefficient of skin friction at the equilibrium stations. For equilibrium sink flow R_θ is constant and therefore equation (13) becomes

$$K R_\theta (H+1) = C_f'/2. \tag{14}$$

Table 1 shows the results for the last three stations from the momentum balance using equation (14). Good agreement exists between the momentum values for U_τ and the Clauser chart values. The Preston tube values were lower than the momentum values. This same trend was found in the results of Jones [3].

$K = 5.39 \times 10^{-7}$					
x(mm)	Momentum U_τ m/s	Clauser chart U_τ	$\%\epsilon$	Preston tube U_τ	$\%\epsilon$
3200	0.585	0.571	2.43	0.563	3.80
3400	0.634	0.617	2.63	0.608	4.058
3600	0.689	0.673	2.39	0.663	3.842

$K = 2.70 \times 10^{-7}$					
x(mm)	Momentum U_τ m/s	Clauser chart U_τ	$\%\epsilon$	Preston tube U_τ	$\%\epsilon$
3200	1.070	1.059	1.01	1.045	2.32
3400	1.148	1.141	0.61	1.131	1.48
3600	1.306	1.288	1.37	1.269	2.82

Table 1: Comparison of U_τ from momentum balance, Clauser chart and Preston tube.

It has been stated that the evolution equations for sink flow are autonomous and the solution trajectories can be displayed on a $S - \Pi$ phase plane. Using the formulation for ψ given by equation (11), the phase plane for $K = 2.70 \times 10^{-7}$ was mapped out with solution trajectories only crossing at critical points. Figure 6 shows two stable critical points and a saddle exist, however, the true equilibrium solution is at $\Pi = 0$. The second stable node and saddle are a result of applying the ψ formulation beyond its range of validity. Equation (11) is based upon the experimental data of Jones [3] and thus applies to the associated $\Pi - \zeta_a - \beta_a$ functional space, where the range for Π is from 0.0 to 0.3.

Marusic *et al.* [5] showed that good agreement exists between the experimental data of Jones [3] and the predicted evolution of the mean flow parameters using equation (11). Shown in figure 6 are the sink flow results from the current study. Within the restricted space for which the formulation (11) is valid, the experimental data agrees well with the theoretical evolution trajectories. To compute the evolution for the current $K = 2.70 \times 10^{-7}$ flow case, the initial conditions chosen were those at $x/L = 0.144$. The conditions at earlier stations sent

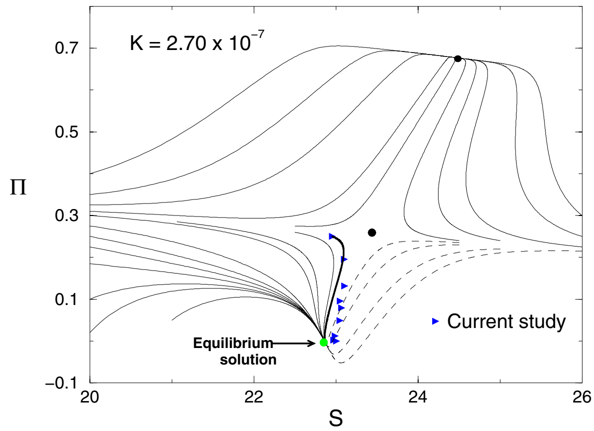

Figure 6: Phase plane for $K = 2.70 \times 10^{-7}$. —— Predicted solution trajectory using initial conditions from experiment at $x/L = 0.144$.

the solution trajectory shooting up to the second critical point. This sensitivity to the initial values of S and Π was due to the proximity of the saddle node. In other words, being close to the border of Jones' [3] restricted functional space where $\Pi = 0.3$.

Conclusion

The present work describes a framework for formulating closure for a turbulent boundary layer evolving in an arbitrary pressure gradient. The mathematical machinery is working and in the Jones [3] restricted $\Pi - \zeta_a - \beta_a$ functional space equation (11) is a valid estimate. However, a more robust and generally more applicable functional form of ζ_a is needed. To achieve this, more experimental data is required.

Acknowledgements

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