14th Australasian Fluid Mechanics Conference
Adelaide University, Adelaide, Australia
10-14 December 2001

On Competition Between Modes at the Onset of Thermocapillary Convection Under a Uniform Vertical Magnetic Field

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Abstract

The stability of oscillatory thermocapillary (Marangoni) convection of electrically conducting fluid layers heated from below under the influence of a uniform vertical magnetic field is investigated numerically using linear stability theory. In particular we present an example of a situation in which there is competition between modes at the onset of convection when the layer is heated from below.

Introduction

The onset of thermocapillary-driven (Marangoni) convection in a layer of fluid which is heated (or cooled) from below is a fundamental model problem for several material processing technologies, such as semiconductor crystal growth from melt, in microgravity conditions. As Schwabe [8] describes, typically in microgravity, thermocapillary rather than buoyancy forces are the dominant mechanism driving the flow. Since the melts involved are often of electrically-conducting material, such as silicon, the technological need to postpone (or indeed eliminate entirely) the onset of undesirable convective motions has motivated considerable interest in studying the effect of externally-imposed magnetic fields on the onset of Marangoni convection. The motion of the electrically conducting melt under a magnetic field induces electric currents. Lorentz forces, resulting from the interaction between the electric currents and the magnetic field, affect the flow.

The effect of an externally-imposed uniform vertical magnetic field on the onset of Marangoni convection in a horizontal layer of electrically-conducting fluid was first addressed by Nield [7]. One of the most important dimensionless parameters is the Marangoni number (the ratio of surface tension gradients to viscous forces). Nield [7] studied the onset of steady Marangoni convection in the case of a flat free surface and showed that increasing the magnetic field strength has the stabilising effect of monotonically increasing the critical Marangoni number for the onset of convection. Wilson [9, 10, 11] derived explicit analytical expressions for the asymptotic behaviour of the critical Marangoni number and the corresponding critical wave number for the onset of steady convection in the limit of large magnetic field strength.

There has been much less work on the effect of a uniform magnetic field on the onset of oscillatory Marangoni convection. Wilson [9] investigated the case $P_1 = P_2 = 1$ (where P_1 and P_2 are the Prandtl number and the magnetic Prandtl number respectively) in detail numerically and concluded that, just as in the non-magnetic case, oscillatory convection is possible only when the layer has a deformable free surface and is cooled from below ($M < 0$), and that the effect of increasing the magnetic field strength is always a stabilising one. In addition, Wilson [9] conducted an unsuccessful numerical search for oscillatory convection for a range of parameter values when the layer is heated from below ($M > 0$), from which he concluded that no such convection was possible. However, in an important paper, Kaddame and Lebon [5] demonstrated that this conclusion was premature by showing that, even in the simplest case of a fluid layer with a flat free surface and perfectly electrically conducting boundaries, there are situations with $M > 0$ in which oscillatory convection not only occurs but is actually preferred to steady convection at the onset of instability. Motivated by their numerical results Kaddame and Lebon [5] suggested that oscillatory convection is possible only if $P_1 < P_2$. Unfortunately, Kaddame and Lebon's [5] solution for the onset of oscillatory convection is in error and has been reexamined and extended by Hashim and Wilson [3]. However, one of the most important results of Hashim and Wilson [3] is that Kaddame and Lebon's [5] conclusion that oscillatory Marangoni convection is possible if $P_1 < P_2$ is indeed correct.

An exciting recent development is the growth of interest in understanding how the nonlinear competition between different modes can lead to pattern formation near or at the onset of convection. For example, Johnson and Narayanan [4] studied experimentally thermocapillary- and buoyancy-driven convection in a layer of silicone oil with an upper air gap, which was placed between two plates, near a "codimension-two" point at which two steady modes coexist. Johnson and Narayanan [4] observed a dynamic switching between two different flow patterns near this point. VanHook *et al.* [12] investigated experimentally the formation of thermocapillary-dominated convection patterns in a thin layer of silicone oil bounded below by a heated rigid plane boundary and above by an air layer. Specifically, VanHook *et al.* [12] observed that both the long- and short-wavelength (hexagonal) modes can coexist for a range of liquid depths. VanHook *et al.* [12] observed that the presence of the hexagons suppresses the long-wavelength mode, while the presence of the long-wavelength mode may induce the formation of hexagons. Motivated in part by these results, Golovin *et al.* [1] recently studied theoretically the nonlinear evolution and secondary instabilities of Marangoni convection in a two-layer liquid-gas system with a deformable liquid-gas interface, bounded from below and from above by rigid plates. Golovin *et al.* [1] derived a system of amplitude equations describing the evolution of a short-wave mode and its interaction with the long-wave mode. The equations they obtained are valid when both instability modes coexist.

In this work we use classical linear stability theory to investigate the competition between modes at the onset of thermocapillary-driven convection in a horizontal layer of quiescent, electrically conducting fluid heated from below in the presence of a uniform vertical magnetic field. In particular, we found for the first time a situation in which there exists a competition between two steady and one oscillatory modes at the onset of convection in the case of a deformable free surface. The results in this work extend some of the numerical results of Hashim and Wilson [3] who studied the idealistic case of a flat free surface.

Problem Formulation

Subject to the Boussinesq approximation, the governing equations for an incompressible, electrically conducting, Newtonian fluid in the presence of a magnetic field, with buoyancy forces in the bulk of the fluid neglected are

$$\nabla\cdot\mathbf{U} = 0, \tag{1}$$
$$\nabla\cdot\mathbf{H} = 0, \tag{2}$$
$$\frac{D\mathbf{U}}{Dt} = -\frac{1}{\rho}\nabla\Pi + \nu\nabla^2\mathbf{U} + \frac{\mu}{4\pi\rho}(\mathbf{H}\cdot\nabla)\mathbf{H}, \tag{3}$$
$$\frac{D\mathbf{H}}{Dt} = (\mathbf{H}\cdot\nabla)\mathbf{U} + \eta\nabla^2\mathbf{H}, \tag{4}$$
$$\frac{DT}{Dt} = \kappa\nabla^2 T. \tag{5}$$

Here $D/Dt = \partial/\partial t + \mathbf{U}\cdot\nabla$, $\mathbf{U}$ is the fluid velocity, $\mathbf{H}$ is the magnetic field, T is the temperature and $\Pi = p + \mu|\mathbf{H}|^2/8\pi$ is the magnetic pressure, where p is the fluid pressure. The constant properties of the fluid are represented by the fluid density ρ, the kinematic viscosity ν, the magnetic permeability μ, the electrical conductivity σ, the electrical resistivity $\eta = 1/4\pi\mu\sigma$ and the thermal diffusivity κ.

We wish to examine the stability of a horizontal layer of quiescent fluid of thickness d subject to an externally-imposed uniform vertical magnetic field of strength H and a uniform vertical temperature gradient. To do this we choose rectangular axes with the x- and y-axes in the plane of the lower rigid boundary and the z-axis vertically upwards so that the lower boundary is given by $z = 0$ and in the undisturbed state the free surface is located at $z = d$. When motion occurs the free surface will be deformed and then we denote its position by $z = d + f(x,y,t)$. The layer of fluid is bounded below by a perfectly thermally conducting planar rigid boundary maintained at a constant temperature T_1 and above by a free surface initially at constant temperature T_2, which is in contact with a passive gas at constant pressure p_0 and constant temperature T_∞. The surface tension of the free surface τ is taken to be dependent on the temperature T according to the simple linear law $\tau = \tau_0 - \gamma(T - T_2)$, where τ_0 is the value of τ when $T = T_2$, and the constant γ is positive for most fluids. At the free surface we have the usual kinematic condition and conditions of continuity of the normal and tangential stresses, and the temperature obeys Newton's law of cooling, $-k\partial T/\partial\mathbf{n} = h(T - T_\infty)$, where k and h are the thermal conductivity of the fluid and the heat transfer coefficient between the free surface and the air, respectively, and $\mathbf{n}$ is the outward unit normal to the free surface. At the lower, rigid and plane, boundary we have the no-slip boundary condition. For simplicity, we follow Kaddame and Lebon [5] and assume that the media above and below the fluid are both perfect electrical conductors, i.e. no magnetic field can cross the boundary.

We shall investigate the linear stability of a basic state in which the fluid is at rest, $\mathbf{U} \equiv \mathbf{0}$, the free surface is flat, $f \equiv 0$, the temperature gradient across the layer is uniform, $T(z) = T_1 - \beta z$, where $\beta = (T_1 - T_2)/d$, the magnetic field is uniform, $\mathbf{H} = (0,0,H)$, and the pressure is constant. To simplify the analysis we non-dimensionalise the variables using the scales d, d^2/ν, ν/d, $\beta d\nu/\kappa$, $\nu H/\eta$ for length, time, velocity, temperature and magnetic field respectively. As a result the following dimensionless group arise: the Marangoni number $M = \gamma\beta d^2/\rho\nu\kappa$, the Chandrasekhar number $Q = \mu H^2 d^2/4\pi\rho\nu\eta$, the capillary number $C_\mathrm{r} = \rho\nu\kappa/\tau_0 d$, the bond number $B_\mathrm{o} = \rho g d^2/\tau_0$, where g is acceleration due to gravity, the Biot number $B_\mathrm{i} = hd/k$, the Prandtl number $P_1 = \nu/\kappa$ and the magnetic Prandtl number $P_2 = \nu/\eta$. Note that this choice of scaling was chosen for consistency with the work of Kaddame and Lebon [5], but differs from that of Wilson [9, 10, 11] who used the notation $P_\mathrm{r} = P_1$ for the Prandtl number and $P_\mathrm{m} = P_1/P_2$ for an alternative magnetic Prandtl number. However, the latter formulation can easily be recovered by multiplying the present time, velocity, temperature and magnetic field variables by $1/P_1$, P_1, P_1 and P_2 respectively.

Linearised Problem

We analyse the linear stability of the basic state in the usual manner by seeking perturbed solutions for any quantity $\Phi(x,y,z,t)$ in terms of normal modes in the form

$$\Phi(x,y,z,t) = \Phi_0(x,y,z) + \phi(z)\mathrm{e}^{\mathrm{i}(\alpha x+\beta y)+st}, \tag{6}$$

where Φ_0 is the value of Φ in the basic state and $a = (\alpha^2 + \beta^2)^{1/2}$ is the total horizontal wave number of the disturbance. The unknown temporal exponent s will, in general, be complex.

Substituting into equations (1)–(5) and neglecting terms of the second and higher orders in the perturbations we obtain the corresponding linearised equations involving only the z-dependent parts of the perturbations to the temperature and the z-components of the velocity and the magnetic field, denoted by T, w and h_z respectively, namely

$$(D^2 - a^2 - sP_1)T + w = 0, \tag{7}$$
$$(D^2 - a^2 - sP_2)h_z + Dw = 0, \tag{8}$$
$$(D^2 - a^2)\left[(D^2 - a^2 - s)w + QDh_z\right] = 0, \tag{9}$$

where the operator $D = \mathrm{d}/\mathrm{d}z$ denotes differentiation with respect to z. The corresponding linearised boundary equations are

$$sf - w = 0, \tag{10}$$
$$P_1C_\mathrm{r}\Big[(D^2 - 3a^2 - Q - s)Dw + sP_2Qh_z\Big] - a^2(a^2 + B_\mathrm{o})f = 0, \tag{11}$$
$$P_1(D^2 + a^2)w + a^2M(P_1T - f) = 0, \tag{12}$$
$$h_z = 0, \tag{13}$$
$$P_1DT + B_\mathrm{i}(P_1T - f) = 0, \tag{14}$$

at $z = 1$ and

$$w = 0, \tag{15}$$
$$Dw = 0, \tag{16}$$
$$h_z = 0, \tag{17}$$
$$T = 0, \tag{18}$$

at $z = 0$.

Solution of the Linearised Problem

The complete solution of the linear stability problem is determined once we have solved equations (7)–(9) subject to the boundary conditions (10)–(18).

In the general case $s \neq 0$ we proceed in the same manner as Wilson [9] and seek solutions in the forms

$$w(z) = AC\mathrm{e}^{\xi z}, \tag{19}$$
$$h_z(z) = BC\mathrm{e}^{\xi z}, \tag{20}$$
$$T(z) = C\mathrm{e}^{\xi z}, \tag{21}$$

where the complex quantities A, B and C and the exponent ξ are to be determined. Substituting these forms into the equations

(7)–(9) and eliminating A, B and C we obtain an eight-order algebraic equation for ξ, namely

$$\begin{aligned}(\xi^2-a^2)(\xi^2-a^2-sP_1)\times \\ \left[(\xi^2-a^2-s)(\xi^2-a^2-sP_2)-Q\xi^2\right] &= 0, \qquad (22)\end{aligned}$$

with eight distinct roots $\xi_1,\ldots,\xi_8$. Denoting the values of A, B and C corresponding to ξ_i for $i=1,\ldots,8$ by A_i, B_i and C_i we can use equations (7) and (8) to determine A_i and B_i to be

$$A_i = -(\xi_i^2-a^2-sP_1), \qquad (23)$$

$$B_i = \frac{\xi_i(\xi_i^2-a^2-sP_1)}{\xi_i^2-a^2-sP_2}, \qquad (24)$$

for $i=1,\ldots,8$. We can use equation (11) to eliminate the free surface deflection

$$f = \frac{P_1C_\mathrm{r}\left[(D^2-3a^2-Q-s)Dw+sP_2Qh_z\right]}{a^2(a^2+B_\mathrm{o})},$$

evaluated on $z=1$, leaving the eight boundary conditions (10), (12)–(18) to determine the eight unknowns $C_1,\ldots,C_8$ (up to an arbitrary multiplier), and the general solution to the stability problem is therefore

$$w(z) = \sum_{i=1}^{8} A_iC_i\mathrm{e}^{\xi_iz},$$

$$h_z(z) = \sum_{i=1}^{8} B_iC_i\mathrm{e}^{\xi_iz},$$

$$T(z) = \sum_{i=1}^{8} C_i\mathrm{e}^{\xi_iz}.$$

The dispersion relation between M, a, s, C_r, Q, B_o and B_i is determined by substituting these solutions into the boundary conditions and evaluating the resulting 8×8 complex determinant of the coefficients of the unknowns, which can be written in the form $M=-D_1/D_2$, where the two 8×8 complex determinants D_1 and D_2 are independent of M.

In the special case $s=0$ (corresponding to the onset of steady convection) the magnetic field h_z can be eliminated entirely from the problem. As a consequence the magnetic boundary conditions do not have to be imposed and so the same procedure as that outlined above in the general case $s\neq0$ yields an equation for M in terms of two 6×6 real determinants which are independent of M, P_1 and P_2.

The marginal stability curves in the (a,M) plane on which $\mathrm{Re}(s)=0$ separate regions of unstable modes with $\mathrm{Re}(s)>0$ from those of stable modes with $\mathrm{Re}(s)<0$. In all the cases investigated in the present work $M>0$ and the region above the marginal stability curve corresponds to unstable modes and the region below the curve corresponds to stable modes. Hence the critical Marangoni number for the onset of convection, denoted by M_c is simply the global minimum of M on the marginal curves. The corresponding critical values of $\omega=\mathrm{Im}(s)$ and a are denoted by ω_c and a_c respectively. If $\omega_\mathrm{c}=0$ then the onset of convection is steady and if $\omega_\mathrm{c}\neq0$ then it is oscillatory. The marginal stability curves are calculated by setting $\mathrm{Re}(s)=0$ and solving the complex equation $D_1+MD_2=0$ for the values of ω and M on the marginal curve. This procedure was implemented numerically using NAG routine F03ADF.

Numerical Results

In this section we shall present the numerical results which show the existence of competition between modes at the onset of convection in the presence of free surface deformation and a magnetic field.

In figure 1 we reproduce Hashim and Wilson's [3] numerically-calculated marginal stability curves which give an example of a situation in which two different modes coexist at the onset of convection in the case of a flat upper surface $C_\mathrm{r}=0$, $Q=250$, $P_1=1$ and $B_\mathrm{i}=0$. In the case investigated the two modes occur simultaneously when $P_{2\mathrm{c}}\simeq12.92$.

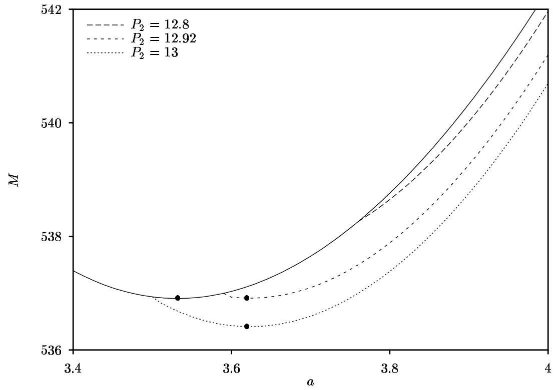

Figure 1: Numerically-calculated marginal stability curves near the critical value $P_{2\mathrm{c}}$ for a range of values of P_2 in the case $C_\mathrm{r}=0$, $Q=250$, $P_1=1$ and $B_\mathrm{i}=0$.

The linear stability of steady thermocapillary-driven convection in the case of deformable free surface with a magnetic field was studied by Wilson [9] when $P_1=P_2=1$. Figure 2 shows typical numerically-calculated marginal stability curves for a range of values of P_2 in the case $C_\mathrm{r}=10^{-4}$, $Q=250$, $P_1=1$, $B_\mathrm{o}=0.1$ and $B_\mathrm{i}=0$. We note that in the case studied convection first sets in as oscillatory instabilities. Whereas figure 3 shows that convection first sets in as steady instabilities in the case $C_\mathrm{r}=10^{-3}$, $Q=250$, $P_1=1$, $B_\mathrm{o}=0.1$ and $B_\mathrm{i}=0$. The results presented in

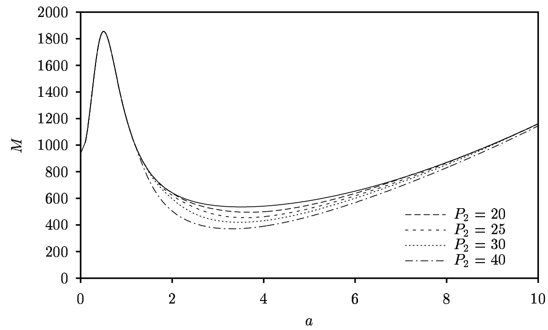

Figure 2: Numerically-calculated marginal stability curves for a range of values of P_2 in the case $C_\mathrm{r}=10^{-4}$, $Q=250$, $P_1=1$, $B_\mathrm{o}=0.1$ and $B_\mathrm{i}=0$.

figures 2 and 3 suggest an alternative possibility which, to the best knowledge of the authors, has not been identified before when a magnetic field is present, namely competition between three different modes. We found two steady modes and an oscillatory mode all occur simultaneously at the onset of convection in the case $C_\mathrm{r}=1.75\times10^{-4}$, $Q=250$, $P_1=1$, $B_\mathrm{o}=0.1$ and $B_\mathrm{i}=0$. We note that in the case investigated the three modes coexist when $P_{2c}\simeq13.38$, and the critical wavenumber on the oscillatory branch is larger than the critical wavenumber for the steady mode. The critical Marangoni number for the long-wave steady mode is in complete agreement with the analytical expression obtained by Wilson [9] in the limit $a\to0$. Similar

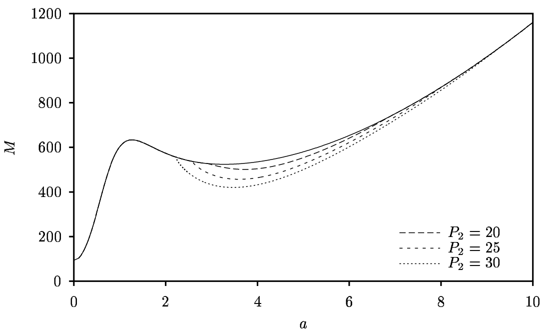

Figure 3: Numerically-calculated marginal stability curves for a range of values of P_2 in the case $C_r = 10^{-3}$, $Q = 250$, $P_1 = 1$, $B_o = 0.1$ and $B_i = 0$.

competition between three different modes was recently found by Hashim [2] for the buoyancy- and thermocapillary-driven convection without a magnetic field.

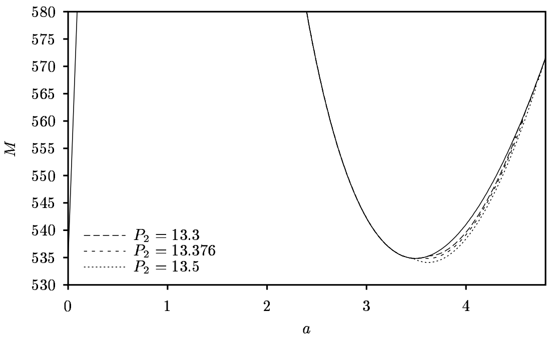

Figure 4: Numerically-calculated marginal stability curves for a range of values of P_2 in the case $C_r = 1.75 \times 10^{-4}$, $Q = 250$, $P_1 = 1$, $B_o = 0.1$ and $B_i = 0$.

Conclusions

In this work we used classical linear stability theory to investigate the competition between modes at the onset of thermocapillary convection in a horizontal planar layer of fluid heated from below in the case of a deformable free surface and in the presence of a magnetic field. The linear analysis presented in this work revealed for the first time a situation in which two steady modes and an oscillatory mode compete at the onset of convection. The answer as to which of these modes dominates the flow certainly warrants a further weakly nonlinear analysis. It would be very interesting to attempt an experimental verification of the novel feature revealed by the analysis in this paper.

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