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The effect of diffusive mass transfer on boundary-layer stability

I.A. Halatchev and J.P. Denier

Department of Applied Mathematics
Adelaide University, South Australia, 5005 AUSTRALIA

Abstract

We consider the linear stability of a boundary-layer flow over a permeable flat plate under conditions of intense interfacial mass transfer. The stability of the flow is governed by an Orr-Sommerfeld type equation coupled to a second-order differential equation for the concentration disturbance field through a flux boundary condition at the permeable surface. This is solved to determine the regions of parameter space in which the flow is linearly unstable. In particular, the critical Reynolds number for the flow is obtained.

Introduction

Porous media surround us everywhere, in reactors of all kinds, almost every possible chemical engineering process, in aeronautics, in waste disposal, in aquifers, in fossil fuel deposits, in the high intensity heat and interfacial mass transfer processes. We shall study diffusion driven flows in porous media in terms of a permeable, two-phase system in which fluid flows over a solid surface. Our attention will focus on the effect of diffusion on the hydrodynamic stability of the boundary layer that typically arises in high Reynolds number flows over porous surfaces.

According to thermodynamics of irreversible processes there will be a contribution to each flux in the system from each driving force in the system. However, the most important contribution to the mass flux is that resulting from the concentration gradient [2]. Here we will assume that the mass flux is a result only of a concentration gradient and neglect the other two potential mechanical driving forces - the pressure gradient and the external forces acting unequally on the various chemical species. This will allow us to focus on the effect of the diffusion-induced normal flux on the stability of the boundary-layer flow.

The problem of the boundary-layer flow under conditions of interfacial mass transfer, governed by the classical Prandtl equations, the laminar boundary layer convection-diffusion equation and the steady heat transfer equation was first treated by Hartnett & Eckert [8]. The boundary-layer equations were solved subject to conditions that took into account foreign fluid injection (with blowing velocity $V(x,0) \neq 0$) at a permeable surface. When $V(x,0) \sim x^{-1/2}$ it was shown that the flow develops in a self-similar fashion and that the velocity and temperature profiles are greatly influenced by "suction" and "blowing".

We consider a slightly different problem here, namely the Blasius boundary-layer flow over a semi-infinite, permeable, flat plate across which a concentration gradient exists. In this case the mass transfer is driven by molecular diffusion and the concentration profile within the boundary layer is dynamically linked to the momentum transport within the flow. For simplicity, we assume that there is no chemical reaction and no external forces acting on the flow; we also neglect emission and absorption of radiant energy into the boundary layer (see [2]). The kinetics of this model then describes, for example, the flow of species with different surface and free-stream concentrations, as occurs in a binary mixture of chemical species A and B flowing over a porous surface.

Formulation

Consider then the laminar flow of a viscous incompressible fluid over a flat, semi-infinite, permeable plate across which a concentration gradient exists. The concentration difference induces a mass flux at the permeable surface. The rate v_n^* of the induced flow can be defined in terms of the mass flux through the surface as

$$v_n^* = -\frac{MD}{\rho^*}\frac{\partial C^*}{\partial n}, \tag{1}$$

where M is the molecular mass, D is the diffusion coefficient, ρ^* the density of the fluid, C^* the concentration and $\partial/\partial n$ the derivative normal to the permeable surface; in the case of the flat-plate boundary layer $\partial/\partial n = \partial/\partial y^*$ (see Incropera & DeWitt [9]).

We define non-dimensional variables (an asterisk denotes a dimensional quantity)

$$(x^*, y^*) = L(x,y), \quad t^* = \frac{Lt}{U_\infty}, \quad (U^*, V^*) = U_\infty \mathbf{U},$$
$$C^* = C_\infty^* + (C_0^* - C_\infty^*)C \quad P^* = \rho^* U_\infty^2 P.$$

where x^* and y^* denote Cartesian coordinates aligned along and normal to the plate surface, U^* and V^* the corresponding velocity components, C^* the concentration, L a typical length (for example, the distance from the leading edge of the plate), U_∞ the free-stream speed, C_0^* the concentration at $y^* = 0$ and C_∞^* at $y^* = \infty$. The non-dimensional equations governing the flow are

$$\nabla \cdot \mathbf{U} = 0, \tag{2a}$$
$$\frac{\partial U}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{U} = -\nabla P + \frac{1}{Re}\nabla^2 \mathbf{U}, \tag{2b}$$
$$\frac{\partial C}{\partial t} + (\mathbf{U}\cdot\nabla)C = \frac{1}{ScRe}\nabla^2 C. \tag{2c}$$

These must be solved subject to the initial conditions

$$\mathbf{U} = (1,0), \; C = 1 \quad \text{at} \quad x = 0$$

and boundary conditions

$$\mathbf{U} = \left(0, -\frac{\theta}{Sc\,Re}\frac{\partial C}{\partial y}\right), \; C = 1 \quad \text{on} \quad y = 0; \tag{3}$$
$$\mathbf{U} \to (1,0), \; C \to 0 \quad \text{as} \quad y \to \infty. \tag{4}$$

Here $Re = U_\infty L/\nu$ is the Reynolds number, $Sc = \nu/D$ is the Schmidt number and $\theta = M(C_0^* - C_\infty^*)/\rho^*$ is a parameter which characterises the intensity of the mass transfer across the permeable surface [3].

The boundary-layer flow

In the limit of large Reynolds number the flow develops a boundary layer of thickness $O(Re^{-1/2})$ attached to the leading edge of the plate. Introducing boundary-layer variables

$$y = Re^{-1/2}Y,\ U = U_B,\ V = Re^{-1/2}V_B,\ C = C_B, \quad (5)$$

where y is physical coordinate and Y the boundary-layer coordinate, the steady boundary-layer equations are

$$\frac{\partial U_B}{\partial x} + \frac{\partial V_B}{\partial Y} = 0, \quad (6a)$$

$$U_B\frac{\partial U_B}{\partial x} + V_B\frac{\partial U_B}{\partial Y} = \frac{\partial^2 U_B}{\partial Y^2}, \quad (6b)$$

$$\frac{\partial P_B}{\partial Y} = 0, \quad (6c)$$

$$U_B\frac{\partial C_B}{\partial x} + V_B\frac{\partial C_B}{\partial Y} = \frac{1}{Sc}\frac{\partial^2 C_B}{\partial Y^2}. \quad (6d)$$

For simplicity, we have assumed that the free-stream speed is uniform in which case $\partial P_B/\partial x = 0$. The boundary conditions appropriate to this system are, from (3),

$$U_B = 0,\ V_B = -\frac{\theta}{Sc}\frac{\partial C_B}{\partial Y},\ C_B = 1 \quad \text{on} \quad Y = 0; \quad (7)$$

$$U_B \to 1,\ C_B \to 0 \quad \text{as} \quad Y \to \infty. \quad (8)$$

Noting that $C_{BY}(0) < 0$, the mass transfer has the effect of proscribing a suction or blowing velocity at the surface, depending upon whether θ is negative or positive. Our concern is with how this diffusion-driven mass transfer affects the hydrodynamic stability of the flow.

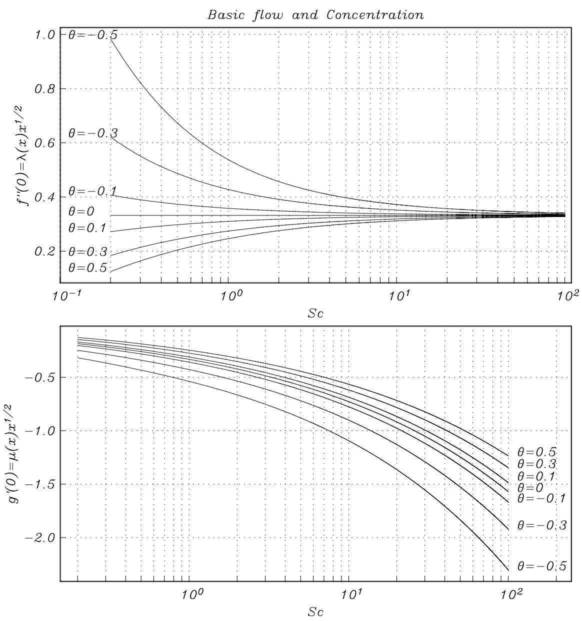

Figure 1: Graphs of $f''(0)$ and $g'(0)$ versus θ for different values of Sc.

In what follows we will employ a similarity solution to the boundary-layer equations as our "basic flow". Introducing the similarity variable $\eta = Y/x^{1/2}$ we find that the boundary-layer equations admit similarity solutions of the form

$$U_B = f'(\eta),\ V_B = \frac{1}{2\sqrt{x}}(\eta f' - f),\ C_B = g(\eta), \quad (9)$$

where the functions f and g are solutions of

$$f''' + \frac{1}{2}ff'' = 0, \quad g'' + \frac{Sc}{2}g'f = 0 \quad (10)$$

subject to the boundary conditions

$$f(0) = \frac{2\theta}{Sc}g'(0),\ f'(0) = 0,\ g(0) = 1; \quad (11)$$

$$f'(\infty) = 1,\ g(\infty) = 0. \quad (12)$$

A detailed description of the quantitative effect of the mass transfer on the boundary-layer flow can be found in Boyadjiev & Vulchanov [5] who demonstrated that the secondary flow, with flow rate $f(0)$, does not change the qualitative character of the boundary-layer flow but simply serves to modify the shape of the velocity profile $f'(\eta)$ through the change in the value of the skin friction; see also the plots of reduced skin friction $f''(0)$ and wall concentration gradient $g'(0)$ in Fig. 1.

Linearized instability of the boundary layer flow

In order to consider the linear stability of the flow we superimpose an infinitesimally small two-dimensional disturbance on the basic boundary-layer flow. In this case the total flow field is written as

$$(U, V, P, C) = (U_B, Re^{-1/2}V_B, P_B, C_B) + \epsilon(u, v, p, c) + \ldots.$$

where ϵ is the infinitesimally small disturbance amplitude. In order to obtain the equations governing the disturbance amplitude we write

$$(u, v, c) = (F'(y), -i\alpha F(y), i\alpha G(y))e^{i\alpha(x-ct)},$$

and define

$$F^{(n)}(y) = \delta_*^n \varphi^{(n)}(\eta), \quad G^{(n)}(y) = \delta_*^n \sigma^{(n)}(\eta)$$

where $\delta_* = 1.720Re/R_\delta$ and $R_\delta = 1.720(xRe)^{1/2}$ is the Reynolds number based on the local boundary layer thickness. Here $2\pi/\alpha$ is the streamwise wavelength of the disturbance and $c = c_r + ic_i$ is the complex wavespeed (to be determined). The disturbance equations reduce to

$$\left(f' - c\right)D^2\varphi - f'''\varphi =$$
$$-\frac{1.720i}{AR_\delta}\left\{D^4\varphi - \frac{1}{2}\left[\left(\eta f' - f\right)D^2 - \left(\eta f''' + f''\right)\right]\varphi'\right\},$$
$$\left(f' - c\right)\sigma + ig'\varphi = -\frac{1.720i}{AR_\delta}\left\{\frac{1}{Sc}D^2\sigma - \frac{1}{2}\left(\eta f' - f\right)\sigma'\right\},$$

where

$$D^2 = \frac{\partial^2}{\partial \eta^2} - A^2$$

and $A = \alpha/\delta^*$. This system must be solved subject to the boundary conditions

$$\varphi(0) = \frac{1.720\theta}{ScR_\delta}\sigma'(0),\ \varphi'(0) = 0,\ \sigma(0) = 0;$$
$$\varphi(\infty) = 0,\ \varphi'(\infty) = 0,\ \sigma(\infty) = 0.$$

The preceding equations, together with their boundary conditions, constitutes an eigenvalue problem for c_r as a function of A and R_δ. The relationship between R_δ and x can be interpreted as follows; in determining a critical Reynolds number R_δ (beyond which the flow is unstable) we will, in effect, be determining a critical position x_{crit} at which the boundary layer becomes linearly unstable to wave-like disturbances.

The disturbance equations are coupled, not through the field equations but through the inhomogeneous boundary condition on the vertical disturbance velocity $\varphi(0)$. Previous work on this problem by Boyadjiev and co-workers [3, 4, 6] assumed that the parameter θ was small and thus the boundary condition relating $\varphi(0)$ to the disturbance concentration gradient could be approximated by $\varphi = 0$. This assumption has the appealing effect of decoupling the momentum and concentration fields for the disturbance thus resulting in a classical Orr-Sommerfeld eigenvalue problem for the complex wave-speed c. This approximation, although capturing the effect of mass transfer on the boundary-layer flow, cannot correctly account for the forcing of the disturbance momentum transport due to the diffusion through the permeable surface. We, however, retain the full coupling of the disturbance fields here.

The solution procedure for the eigenvalue problem is a modification of that developed by Keller [10] and full details can be found in [7].

Results and discussion

Representative curves of neutral stability in (R_δ, A) and (R_δ, c_r) plane are presented in Figures 2 - 3. Consider firstly the results for a Schmidt number of 0.7 presented in Figure 2. Following the usual convention the curves

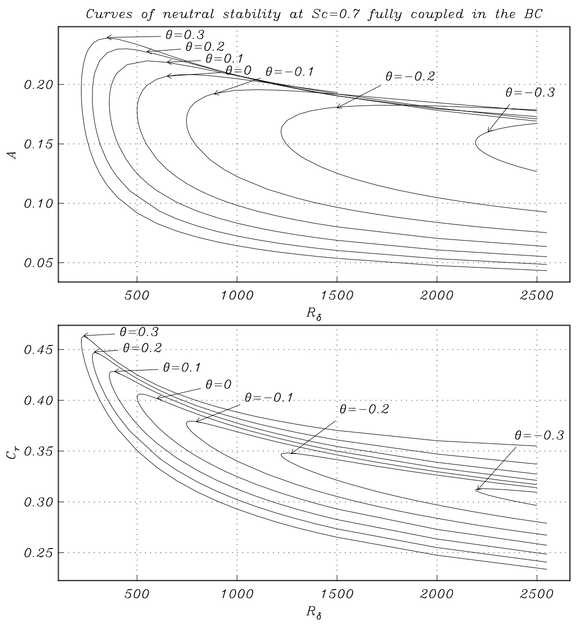

Figure 2: Curves of neutral stability at $Sc = 0.7$ for different values of θ.

of neutral stability delineate the boundary in the parameter space between stable and unstable disturbances; the flow is unstable for values of the parameter that lay inside the neutral curve. From Figure 2 we see that the effect of positive mass transfer (i.e. blowing) is to reduce the critical Reynolds number and consequently destabilise the boundary layer; the point of neutral stability therefore moves towards the leading edge of the plate which hints

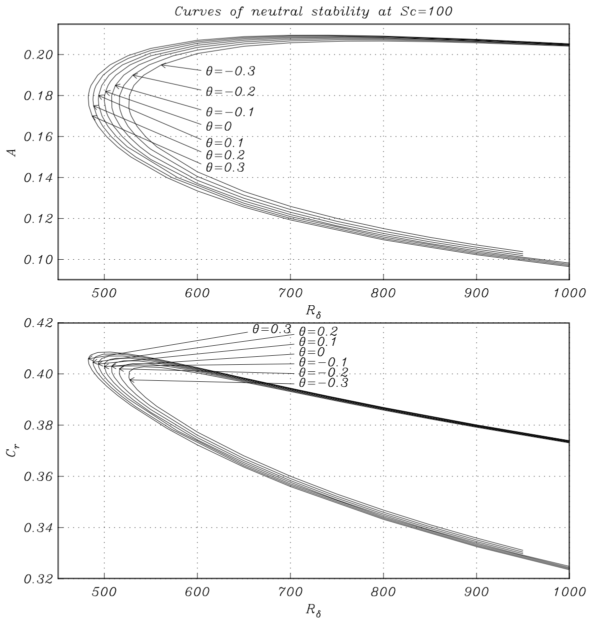

Figure 3: Curves of neutral stability at $Sc = 100$ for different values of θ.

at an earlier transition to turbulence within the flow. This conclusion is in agreement with that made in the work of Boyadjiev *et al.* [3, 4, 6] but corrects the errors which arise due to the erroneous decoupling of the disturbance momentum and concentration fields. We note that, for the present case ($Sc = 0.7$), the critical Reynolds number, for a value of $\theta = -0.3$, is 2.19264×10^3 as compared to the value of 2.23226×10^3 predicted by the analysis of [3] whereas for $\theta = 0.3$ the respective values are 2.210018×10^2 (current results) and 2.18419×10^2 (results of [3]). Further from Figure 2 we observe that blowing serves to increase the critical wavenumber and wave-speed. These general conclusions also hold for higher values of the Schmidt number as is seen from Fig. 3. The changes in the critical Reynolds number, wavenumber and wave-speed as a function of θ are summarised in Tables 1 - 2.

$Sc = 0.7$					
θ	$R_\delta \times 10^3$	A	c_r	A_{min}	$c_{r\,min}$
-0.3	2.1926	0.151	0.3116	0.1672	0.3130
-0.2	1.2196	0.160	0.3462	0.1825	0.3481
-0.1	0.7462	0.170	0.3772	0.1956	0.3795
0.0	0.5000	0.177	0.4028	0.2083	0.4061
0.1	0.36102	0.184	0.4242	0.2198	0.4283
0.2	0.27633	0.188	0.4414	0.2300	0.4470
0.3	0.22100	0.192	0.4563	0.2390	0.4626

Table 1: Values of critical Reynolds number R_δ, corresponding wave velocity c_r, wave number A and A_{min} and $c_{r\,min}$ at $Sc = 0.7$.

These results suggest that, in the range of Reynolds numbers and mass-transfer rates considered, at high Schmidt numbers Sc, the coupling effect has a relatively minor role in the instability process (Fig. 3). This is a simple consequence of the factor Sc^{-1} appearing in the boundary condition which forces the vertical momentum transport. However, at low to "moderate" values of the Schmidt number, the effect of coupling is considerable and must be taken into account if a reasonable and accurate estimate of the critical parameter values is to be obtained.

$Sc = 100$					
θ	$R_\delta \times 10^3$	A	c_r	A_{min}	$c_{r\,min}$
-0.3	0.5264	0.174	0.3985	0.2065	0.4027
-0.2	0.5161	0.175	0.4001	0.2072	0.4040
-0.1	0.5078	0.180	0.4029	0.2078	0.4051
0.0	0.5000	0.177	0.4028	0.2083	0.4061
0.1	0.4935	0.178	0.4038	0.2087	0.4070
0.2	0.4878	0.178	0.4045	0.2091	0.4078
0.3	0.4827	0.179	0.4055	0.2094	0.4085

Table 2: Values of the critical Reynolds number R_δ, corresponding wave velocity c_r, wave number A and A_{min} and $c_{r\,min}$ at $Sc = 100$.

Conclusions

At fixed values of the Schmidt number Sc, a change in the direction of the mass transfer (i.e. from blowing $\theta > 0$ to suction $\theta < 0$) has a stabilizing influence on the flow in that the critical value of the Reynolds number is increased. The stabilizing effect of "suction" is more significant than the destabilizing effect of "blowing" at the same absolute values of the mass-transfer parameter (Figure 2).

For higher values of the Schmidt number Sc the influence of the mass transfer on the hydrodynamic stability of the boundary-layer flow becomes less significant. This result is perhaps not surprising given the fact that the Schmidt number $Sc = \nu/D$ plays a role analogous to the Prandtl number in heat transfer, i.e. it is a measure of the relative importance of the mass transfer and momentum transfer in the flow. For large values of the Schmidt number the relative thickness of the concentration and momentum boundary layers is small and thus in this case diffusion is important only within the thinner concentration boundary layer at the wall.

Our results also demonstrate a significant difference when compared to those of earlier work [3], which employed an *ad hoc* assumption regarding the "smallness" of the mass-transfer parameter θ in order to simply the resulting eigenvalue problem for the complex wavespeed $c = c_r + ic_i$. In the case of low Schmidt numbers Sc there is a significant variation in the critical Reynolds numbers R_δ, although there is no significant change in the critical wavenumber A or critical wave-speed c. Interestingly, the crude approximation employed in [3] can, in some sense, be considered valid in the limit of large Schmidt number. This behaviour can be traced to the simple fact that the reciprocal of the Schmidt number appears in the boundary condition for both the boundary layer and the disturbance equations. Thus in the limit of large Schmidt number the momentum and concentration fields, for both the boundary layer and the disturbance, decouple. In this case diffusion has no significant effect upon the boundary-layer flow.

From a theoretical standpoint the present work has allowed us to account for the effect of coupling between the momentum and concentration fields in a self-consistent manner. This in turn has allowed for the determination of the correct values of the critical parameters (Reynolds number, wavenumber and wave-speed) as well as the corresponding eigenfunctions. These results can now be used to extend this analysis into the nonlinear regime and to consider the question of finite amplitude disturbances to the such flows. This is the subject of ongoing work.

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