Inertial oscillations and the thermal bar

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Abstract
Small bottom slope, inviscid solutions are found for a model of the thermal bar system. The model includes Coriolis effects and variable heat input. The solutions include inertial oscillations that have a significant effect on the circulation, especially for the case when the heating is instantaneously applied.

Introduction
At the end of winter, the temperature of the water in many temperate lakes is less than 4°C; the temperature at which water achieves its maximum density. As spring progresses and the water is warmed, the near shore shallow waters heat more rapidly than the deeper parts. As a consequence, the 4°C isotherm propagates out from the shore and to either side of it the horizontal pressure gradient has opposite signs. This leads to a double cell circulation pattern with downwelling in the vicinity of the 4°C isotherm. This isotherm is called the thermal bar and inhibits horizontal transport from the shallows to the deeper parts of the lake.

Previous analytical studies of the thermal bar can be divided into two categories. The first category concentrate on predicting the propagation of the thermal bar and are based on heat balance models [2, 11]. The second category of studies do not explicitly model the circulation associated with the thermal bar system [1, 7, 3, 4]. The thermal bar occurs in large lakes and can persist for several weeks thus Coriolis effects can be important in the dynamics of the thermal bar system. Most previous studies of the effects of rotation on the thermal bar have been either steady [7] or quasi-steady [9] in the sense that while the background temperature structure is unsteady, the model momentum equations do not include inertia terms. This means that the physical balance is largely a Coriolis-viscous balance. Scaling shows that it is unlikely that such a balance can occur in deep lakes and that the dynamics of the circulation associated with the thermal bar is principally a Coriolis-inertia balance. [5] considered the effects of inertia using both analytical and numerical methods. They found that the circulation included inertial oscillations that are significant enough to reverse the expected circulation, particularly ahead of the thermal bar. However, their model includes an unrealistic vertically uniform heat source (as do many other models for the thermal bar).

The present work builds on [5] by including a number of generalisations. These include a vertically non-uniform heat source and more general bathymetry. The model is also axisymmetric which is the arguably a more appropriate framework for modelling lakes.

Model Formulation
Figure 1 shows a schematic of the flow domain. The model equations are

\[
\frac{Du}{Dt} = \frac{v^2}{r} - f v = \frac{1}{\rho_0} \frac{\partial \rho}{\partial r} \quad (1)
\]

\[
\frac{Du}{Dt} = \frac{w}{r} + f u = 0, \quad (2)
\]

\[
\frac{Dw}{Dt} = -\frac{1}{\rho_0} \frac{\partial \rho}{\partial z} + g \beta \Delta T^2, \quad (3)
\]

\[
\frac{D\theta}{Dt} = Q(r, z, t), \quad (4)
\]

\[
\frac{1}{r} \frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0. \quad (5)
\]

The heat source term \(Q\) in (4) drives the system and is formulated below. The initial conditions are \(u = v = w = 0\) and \(T = T_0 < 4°C\). \(R\) and \(H\) are horizontal (radial) and vertical length scales derived below.

It is assumed here that a spatially uniform surface heat flux \(I_0(t/\tau)\) where \(I_0\) is a scale for the heat flux, \(I\) is a dimensionless function and \(\tau\) is a timescale is distributed vertically within the water column. If the water column is initially isothermal and less than 4°C then a surface heat flux is destabilising and thus vertical mixing will occur. It is assumed here that this vertical mixing occurs on a timescale shorter than that associated with the evolution of the system as a whole. This suggests that the heat flux at the surface should be uniformly distributed in the vertical. If, however, the water column is warmer than 4°C then heating at the surface is stabilising and (in the absence of other effects) mixing will not occur. If the main component of the heat flux is solar radiation then an appropriate model would be based on Beer’s law [10]. Under these assumptions, the heat source term in (4) is given by

\[
Q(r, z, t) = \begin{cases} 
\frac{I_0(t/\tau)}{\rho_0 C_p} \epsilon \sigma T^4 & T > 4°C, \\
\frac{I_0(t/\tau)}{\rho_0 C_p H \ln(\tau/R)} & T \leq 4°C 
\end{cases} \quad (6)
\]

where \(C_p\) is the specific heat of water and \(\eta\) is a representative attenuation coefficient for the solar radiation. A typical value for \(\eta \approx 0.3 \text{m}^{-1}\) [6].

The natural time scale for this rotating problem is \(t \sim \tau = f^{-1}\), the inertial period. There is no specific length
scale associated with the general bathymetry. There is a length scale associated with the internal heating, namely the inverse of the attenuation coefficient. However, the dynamics of the thermal bar system is only weakly dependent on this quantity and a more relevant length scale can be found as follows. Balancing $T_t$ and $Q$ in (4) for $T < 4^\circ C$ yields an expression for $T$ which is independent of $z$. From this it can be shown that after a time $t$, the depth of the water column at the $4^\circ C$ isotherm is $H = \frac{\rho_0}{\rho} \Delta T_0$, where $\Delta T_0 = 4^\circ C - T_0$. Substituting $t = \tau$ into this scale gives a vertical length scale. Assuming that $H/R \sim R$, a scale for the bottom slope of the domain, gives a horizontal length scale.

Assuming a hydrostatic balance in (3) and balancing inertia against the horizontal pressure gradient in (1) yields a scale for the horizontal velocity $u \sim U = Ag3\Delta T_0^2/f$. Aka, from (5), $w \sim AU$. Using these scales and $\Delta T \sim \Delta T_0$ to nondimensionalise the governing equations yields

$$\frac{Du}{Dt} - \frac{Ro \nu^2}{r} - v = -p_r, \quad (7)$$
$$\frac{Dv}{Dt} + \frac{Ro \mu u}{r} + u = 0, \quad (8)$$
$$\frac{ Dw}{Dt} = -p_z / A^2 + (1 - T)^2 / A^2, \quad (9)$$
$$\frac{D T}{Dt} = Q(r, z, t), \quad (10)$$
$$\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{\partial w}{\partial z} = 0 \quad (11)$$

where all variables are now nondimensional, $\frac{\partial}{\partial r} = \frac{\partial}{\partial r} + Ro(u \frac{\partial}{\partial r} + w \frac{\partial}{\partial z})$, where $Ro = U/fR$ is the Rossby number. The dimensionless $Q$ is

$$Q(r, z, t) = \begin{cases} 1 \eta e^{n \eta} & T > 1, \\
\left( \frac{\nu}{U} \right)^n & T < 1 \end{cases} \quad (12)$$

where $\eta$ has been nondimensionalised by $H^{-1}$.

**Asymptotic solution**

Sadly, the system of equations (7)-(11) do not admit a general solution. However, expanding the dependent variables as series in $A^2$ and $Ro$ yields a system of equations that can be solved recursively. Only the zero order equations are solved here:

$$u_t - v = -p_r, \quad (13)$$
$$v_t + u = 0, \quad (14)$$
$$0 = -p_z - (1 - T)^2, \quad (15)$$
$$T_t = Q, \quad (16)$$
$$\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{\partial w}{\partial z} = 0. \quad (17)$$

Physically, the flow develops as follows. The unsteady term in the temperature equation balances the internal heating term. A hydrostatic pressure field is derived from this which then feeds into the horizontal momentum equations to drive the flow.

The zero order temperature is given by

$$T(r, z, t) = \begin{cases} 1 + \eta e^{n \eta} (J(t) - h(r)) & h(r) < J(t) \\
J(t) & h(r) > J(t) \end{cases} \quad (18)$$

where the function $J(t) = \int_0^t J(t') dt'$ has been introduced. The solutions of $h(r) = J(t)$ correspond to points in the $(t, r)$ plane where the temperature is exactly 1 and represent the position of the thermal bar at any particular time. Thus, the thermal bar position at any particular time is given by $r(t) = h^{-1}(J(t))$.

The solution (18) can be used to calculate a hydrostatic pressure. By introducing $q = u + \nu$ the velocities can be found as a solution of a 1st order linear ODE. For brevity, the details are omitted.

**Discussion**

**Introductory remarks**

The solution above includes general heating $J(t)$ and arbitrary bathymetry $h(r)$. Most previous modelling of the thermal bar has considered $l(t) = 1$, that is, the heating is constant and instantaneously applied at $t = 0$. However, the seasonal heating that occurs in lakes is likely to be gradual [8] so a more appropriate form for the heating would be $l(t) = t$. One form for the bathymetry is considered here: $h(r) = r_0 - r, r < r_0$. This represents a conical basin with the shore located at $r = r_0$.

**Temperature structure**

The temperature structure can be divided into two parts. In the deeper regions, the water column is cooler than unity, the temperature at which the density maximum occurs. The temperature there is vertically uniform. In the shallower regions, the water column is warmer than unity and there is a significant stratification associated with the vertically non-uniform heat source term. The thermal bar defines the boundary between these regions.

Figure 2(a) shows a snapshot of the isotherms for $l = 1$ at $t = 15$ for $\eta = 0.1$. At this time, the thermal bar is at $r = 15$ and is denoted by the solid contour. For $r \leq 15$, the isotherms are vertical. For $r > 15$, significant stratification is evident. There is still a significant horizontal temperature gradient in the shallows however it is weaker than that associated with a vertically uniform heat source used in [5]. The remaining two panels of figure 2 are discussed below.
Circulation structure

Since it is assumed in the asymptotic solutions above that the fluid is inviscid, the velocities have a simple vertical structure. For regions where \( h(r) > J(t) \), the velocities have a simple linear vertical profile. Where \( h(r) < J(t) \) the structure is a little more complicated as there is an exponential component to the velocity profile. The structure of the circulation can be largely characterised by the velocities at the surface \( z = 0 \).

Figures 3 and 4 show contours in the \((t, r)\)-plane of the offshore and longshore surface velocities for the two cases considered here with \( \eta = 0.1 \). Note that the vertical coordinate in figures 3 and 4 is \( r_0 - r \) so the bottom left hand corner of those plots corresponds to the shore at \( t = 0 \). The dashed line in these figures indicates the position of the thermal bar for each case. The position of the thermal bar also indicates the position where the pressure gradient that drives the circulation changes sign. On the shore side of the thermal bar the pressure gradient is favoring negative offshore surface velocities (that is away from the shore). On the other side of the thermal bar the pressure gradient is favoring positive offshore surface velocities (that is towards the shore). This leads to the classic picture of downwelling in the vicinity of the thermal bar.

Figure 3(a) shows contours of the offshore surface velocity for \( l = 1 \). This case has the same heating (ahead of the thermal bar) and similar geometry (i.e. linear) to the model considered in [5]. The position of the thermal bar is as predicted in [2]. For small times \( (t < 1) \), the dynamics are essentially the same as for the non-rotating case. There is a shoreward surface flow ahead of the thermal bar and a relatively weak longshore flow. The \( u_{l=0} = 0 \) contour indicates the position where either downwelling or upwelling is occurring. For \( l > 1 \), Coriolis effects become important. In this case, this leads to significant inertial oscillations in the flow. These oscillations are strong enough to lead to reversals in the circulation both ahead of and behind the thermal bar. For \( l > 1 \), there are regions of both upwelling as well as downwelling with (for example at \( t = 15 \)) many circulation cells. These reversals lead to the thermal bar spending most of its time with a surface velocity advecting it further offshore. In other words, the circulation has the effect of increasing the propagation speed of the thermal bar. This is consistent with [5] but in contrast to the non-rotating result [3, 4]. However, in a departure from the results of [5], there are reversals of the circulation on the shore side of the thermal bar as well. The inertial oscillations on the shore side of the thermal bar persist indefinitely and lead to repeated reversals of the circulation there. This is due to the difference in the thermal forcing in that region. The results of [5] also have inertial oscillations on the shore side of the thermal bar but they sit upon an accelerating mean flow swamping their signal. The flow here is "ringing" in response to the initiation of the heating. Another difference between the present results and those of [5] is that the offshore velocities are bounded, including as \( h \to 0 \).

For \( l = t \), there is a dramatic change in the character of the circulation. The influence of the inertial oscillations on the flow is very much reduced with no repeated reversals of the circulation ahead of the thermal bar. The downwelling region (indicated by the \( u_{l=0} = 0 \) contour in figure 3(b)) is still ahead of the thermal bar for most of the time. It is initially behind the thermal bar when Coriolis effects are small. The effects of the inertial oscillations are still evident in the waviness of this contour. Also, there is a degree of waviness in the contours after the thermal bar has passed but not enough to reverse the circulation except for a small region close to \( r = r_0 \). Since the \( u_{l=0} = 0 \) contour is largely ahead of the thermal bar, non-linear advection would lead to the surface signature of the thermal bar moving out from the shore more quickly than predicted by [2].

The reason for the difference between the \( l(t) = 1 \) and \( l(t) = t \) cases can be more clearly seen by examining the initial value problem for \( u \). For small times (that is before the passage of the thermal bar so that the heating is vertically uniform), \( u \) can be written as \( u = f(t, z)g(t) \) where (from (13) and (14)) \( g(t) \) satisfies the forced wave equation \( g'' + g = F(t) \) where \( F(t) = \frac{1}{h}J(t)(h - J(t)) \). The solutions for the two forms for the forcing are

\[
g(t) = \begin{cases} 
1 \cos t & \text{for } l(t) = 1, \\
(t^3 + (h + 6)\sin t - t) & \text{for } l(t) = t.
\end{cases}
\]

Both of these solutions include inertial oscillations however only the \( l(t) = 1 \) case allows for multiple zeros of \( g(t) \) for fixed \( h \). In fact, solving \( g(t) = 0 \) gives the \( u_{l=0} = 0 \) contours (for \( h(r) > J(t) \)) in figure 3. For \( l(t) = t \), the circulation includes inertial oscillations but they sit upon an accelerating mean flow (of order \( t^2 \)) that soon swamps their signal.

Figure 4 shows a set of contours in the \((t, r)\)-plane of the
The circulation is much reduced with no reversals on either side of the thermal bar (figure 3(b)). The longshore oscillations drive an accelerating longshore circulation that ultimately swamps any signal from the inertial oscillations.

As for the offshore circulation, there are significant differences in the longshore structure for \( l = 1 \) when compared with \( l = 1 \). The effect of the inertial oscillations on the circulation is very much reduced with no reversals on either side of the thermal bar (figure 3(b)). The longshore circulation is close to a geostrophic balance with reversals occurring in all cases close to the reversal of the pressure gradient associated with the passing of the thermal bar.

It was mentioned above that the velocities do not have much internal structure and thus they can be largely characterised by the surface flow. The structure that does occur is on the shore side of the thermal bar and this is shown in figures 2(b) and 2(c). Figure 2(b) shows the offshore streamfunction at \( t = 15 \) for \( l = 1 \). This time shows the multiple cell circulation structure associated with the reversals discussed above. On the shore side of the thermal bar, not all of the dividing streamlines are vertical reflecting the exponential with depth structure of the radial pressure gradient there. The longshore circulation (figure 2(c)) has a linear profile left of the thermal bar. To the right, the exponential with depth structure of the pressure gradient leads to stronger flow near the surface.

**Conclusions**

The main conclusion from the asymptotic solutions is that inertial oscillations are less important in the dynamics if the heating is gradual. For both the gradual and instantaneous heating cases, the thermal bar spends most of its time with a surface velocity that is advecting the thermal bar further from the shore.

The influence of the vertically non-uniform heating is restricted to the shore side of the thermal bar where it is applied. The circulation behind the thermal bar where there is some stratification is weaker than for a vertically uniform model since less energy is available to drive a flow.

**References**


