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ACCURACY AND EFFICIENCY OF SOME HIGHER-ORDER SCHEMES ON NON-UNIFORM GRIDS FOR FLUID FLOW PROBLEMS

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ABSTRACT

A generalised formulation is applied to implement the quadratic upstream interpolation (QUICK) scheme, the second-order upwind (SOU) scheme and the second-order hybrid scheme (SHYBRID) on non-uniform grids. The accuracy and efficiency of these higher-order schemes on non-uniform grids are assessed. Three well-known bench-mark problems are revisited using non-uniform grids. These are: (1) heat transport in a recirculating flow; (2) 2D Burgers' equations; and (3) a 2D lid-driven cavity flow. The known exact solutions of these problems make it possible to thoroughly evaluate accuracies of various uniform and non-uniform grids. Higher accuracy is obtained for fewer grid points on non-uniform grids. The order of accuracy of the schemes examined is maintained for some test problems if the distribution of non-uniform grid points is properly chosen.

INTRODUCTION

Extension of higher-order convection schemes to non-uniform Cartesian grids is conceptually easy. However, the coefficients can be rather complex. Different forms of the schemes on non-uniform grids can be found in the literature. It is desirable to implement a number of higher-order schemes in a CFD code and to choose an "optimum" one for a particular flow problem. Li and Rudman (1995) presented a generalised formulation for four- point discretisation schemes on non-uniform grids. The central difference (CD) scheme, QUICK, SOU and SHYBRID fall within this formulation. There is a need to assess accuracy of higher-order schemes on non-uniform grids, but little work has appeared in the literature. On non-uniform grids, many schemes lose their order of accuracy, as may be shown by performing a truncation error analysis. Lack of detailed numerical experiments on this aspect gives rise to lack of confidence in using higher-order schemes on non-uniform grids. The higher-order schemes can also be implemented in a computational space with uniform grids which is transformed from a physical space. However, the influence of grid sizes on the accuracy is again introduced by the discretised metric coefficients. The objective of this paper is to assess accuracy and efficiency of higher-order schemes on non-uniform Cartesian grids.

GENERALISED FORM OF FOUR HIGHER-ORDER SCHEMES

The nomenclature here is the same as that in Li and Rudman (1995). The detailed implementation of the method can be found in Li and Baldacchino (1995). In Fig. 1, the grid-related sizes δ_{iw} ($i = 1, 2,$and 3) take different meanings when the velocity at the local face changes its direction.

The following geometrical parameters are defined:

$$\alpha_{1w} = \frac{\delta_{1w}}{\delta_{1w} + \delta_{2w}}; \qquad \beta_{1w} = \frac{\delta_{1w} + \delta_{2w}}{\delta_{3w} - \delta_{1w}}$$
$$\alpha_{2w} = \frac{\delta_{2w}}{\delta_{1w} + \delta_{2w}}; \qquad \beta_{2w} = \frac{\delta_{2w} + \delta_{3w}}{\delta_{3w} - \delta_{1w}} \qquad (1)$$

The generalised scheme can be summarised as

$$\begin{aligned} F_w\phi_w &= (\alpha_{2w}\phi_W + \alpha_{1w}\phi_P - q_w\tilde{\phi}_W)F_w^+ \\ &+ (\alpha_{1w}\phi_W + \alpha_{2w}\phi_P - q_w\tilde{\phi}_P)F_w^- \end{aligned} \qquad (2)$$

where

$$\begin{aligned} \tilde{\phi}_W &= \phi_P - \beta_{2w}\phi_W + \beta_{1w}\phi_{WW} \\ \tilde{\phi}_P &= \phi_W - \beta_{2w}\phi_P + \beta_{1w}\phi_E \end{aligned} \qquad (3)$$

The scheme parameters q_w take the appropriate forms for different schemes. For CD, $q_w = 0$; for

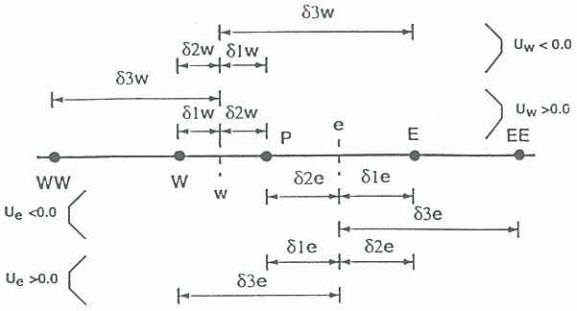

Figure 1: GRID SYSTEM WITH GRID RELATED PARAMETERS IN THE x-DIRECTION.

SOU, $q_w = \alpha_{1w}$; and for QUICK,

$$q_w = \frac{\delta_{2w}}{\delta_{2w} + \delta_{3w}} \alpha_{1w} \tag{4}$$

and for the second-order hybrid scheme (SHYBRID), if the local Peclet number $Pe_w = 0$, $q_w = 0$; if $Pe_w \neq 0$, then

$$q_w = \max[0.0, \alpha_{1w} - \frac{1}{|Pe_w|}] \tag{5}$$

RESULTS OF THE TEST CASES

Heat Transport in a Recirculating Flow

The problem of Beier et al. (1983) with an exact solution (Fig. 2) presents two important flow features in practical problems, namely recirculation and a temperature boundary layer. The problem involves a recirculating flow in a heated cavity with the upper surface of the cavity being adiabatic, and the left and bottom surfaces being defined by a varying temperature profile.

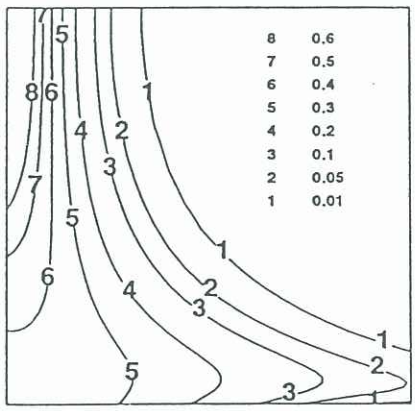

Figure 2: EXACT SOLUTION OF TEMPERATURE FIELD FOR THE FIRST TEST PROBLEM.

Three grid configurations each with four different grid sizes are used: a uniform grid, a non- uniform grid consisting of two uniform sections (non-uniform grid 1), and a stretched grid (non-uniform grid 2); see Fig. 3 for the two non-uniform grids. The results are plotted in Fig. 4 for each grid configuration.

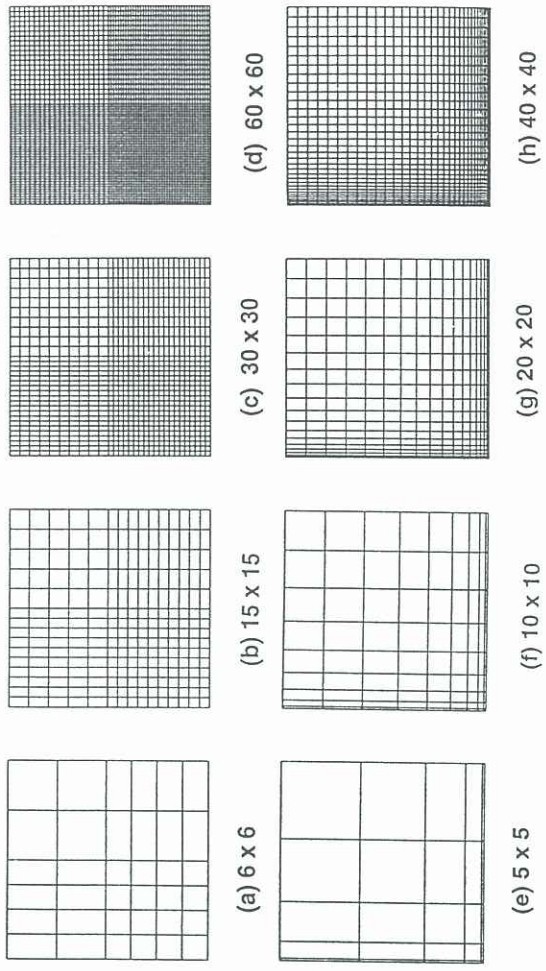

Figure 3: FOUR GRID SIZES OF THE TWO NON-UNIFORM GRIDS FOR THE FIRST TEST PROBLEM.

The higher-order schemes are consistently superior to first-order upwind (FOU). It is notable that higher-order schemes maintain higher than first order accuracy on all non-uniform grids studied here. SOU displays significantly less accuracy than SHYBRID and QUICK, although it is notably superior to its first-order counterpart. It is also notable that SHYBRID does yield a better result to QUICK on the finest non-uniform grid 1. By comparing CPU times (not shown here) required for the first two grids, it can be seen that the non-uniform grid solutions are more efficient.

Two-dimensional Burgers' Equations

By using the Cole-Hopf transformation, Fletcher (1983) constructed an interesting exact solution of the two-dimensional Burgers' equations. We use an exact solution with a severe internal gradient (Fig. 5). Five

grid configurations are used, each with four grid sizes (Fig. 6). The relative accuracy and CPU time spent on each scheme for all the grids are summarised (not shown), and the RMS errors of the u-component for each scheme are also plotted in Fig. 7.

The nature of the solution renders particularly appropriate for use with non-uniform grids. It is clear that higher accuracy is maintained for fewer grid points on non-uniform grids, which also results in less CPU time. By plotting the error distribution of the u -solution on uniform and non-uniform grids (not shown), it is found as expected that grid refinement using non-uniform grids reduces the error in the refined region.

In Fig. 7, a downward shift in the RMS profile as the grid in the larger-gradient region is refined, could indicate a gain in solution accuracy using fewer grid points. For FOU, this is true at a large number of grid points, but not true at a small number of grid points. But for the higher-order schemes QUICK and SHYBRID, this is true at a small number of grid points and not true at a large number of grid points. For SOU the results are different and complex. With a very large grid-aspect ratio $\frac{x_{i+1}-x_i}{x_i-x_{i-1}}$ of 0.0625 in non-uniform grid 4, the solution accuracy does not improve compared to non-uniform grid 3. For very large numbers of grids tested, even a grid-aspect ratio of 0.25 in non-uniform grid 2 does not give any better results for QUICK and SHYBRID. The importance of these results can be seen from the fact that in most engineering calculations, coarse grids are generally used. The results discussed above imply that non-uniform grid, together with higher-order schemes such as QUICK and SHYBRID, can be a useful approach. The detailed analysis of the influence of grid-aspect ratios can be expected to provide guidance for the behavior of non-uniform grids applied to real fluid flows.

Lid-driven Cavity Like Flow

This flow with an artificially designed body force, whose exact solution was constructed by Shih et al. (1989), is qualitatively similar to the classical lid-driven cavity flow. Shih et al . (1989) obtained a numerical solution only for $\nu = 0.1$, since a central difference scheme was used. Solutions are obtained here for $\nu = 0.001$. Two grid configurations are used, namely a uniform grid and a non-uniform grid consisting of two uniform sections in the vertical direction. The solutions are summarised in Table 1.

Even though the use of higher-order schemes improves the solution accuracy, the higher-order accuracy of SOU, SHYBRID and QUICK is not maintained. The solution accuracy on non-uniform grids with higher-order schemes is better than that with FOU. It appears that QUICK and SOU predict a better pressure gradient than SHYBRID. The results indicate the complexity of numerical prediction in flow problems. The reason for loss of order of accuracy of higher-order schemes is unknown to the authors. It may be hypothesised that this is due to the use of FOU at near-boundary points. Application of FOU at near-boundary points in the previous two problems studied did not result in a great loss of order of accuracy. The reason that it happens in the cavity flow problem here may be due to the elliptic behaviour of pressure.

Uniform	FOU	QUICK	SOU	SHYBRID
20 × 20	4.745E-2	4.433E-2	5.758E-2	4.678E-2
40 × 40	2.808E-2	2.462E-2	2.130E-2	2.737E-2
80 × 80	2.224E-2	2.100E-2	2.035E-2	2.173E-2
Non-U	FOU	QUICK	SOU	SHYBRID
20 × 16	5.330E-2	5.253E-2	6.018E-2	5.286E-2
40 × 30	3.089E-2	2.948E-2	2.606E-2	3.224E-2
80 × 60	2.374E-2	2.441E-2	2.380E-2	2.515E-2

Table 1: RMS ERRORS OF $\frac{\partial p}{\partial y}$ FOR THE LID-DRIVEN LIKE FLOWS.

CONCLUSION

Overall, higher accuracy is obtained for fewer grid points on non-uniform grids. It is confirmed that the order of accuracy of the schemes examined here can be maintained if the non-uniform grid points are properly chosen to be in small-variation regions of the dependent variables, and the grid-aspect ratio is small. The application of a first-order upwind scheme at near-wall points does not result in a great loss of order of accuracy for all the convection-diffusion problems, but seems to do so for the cavity flow problem, which requires further investigation.

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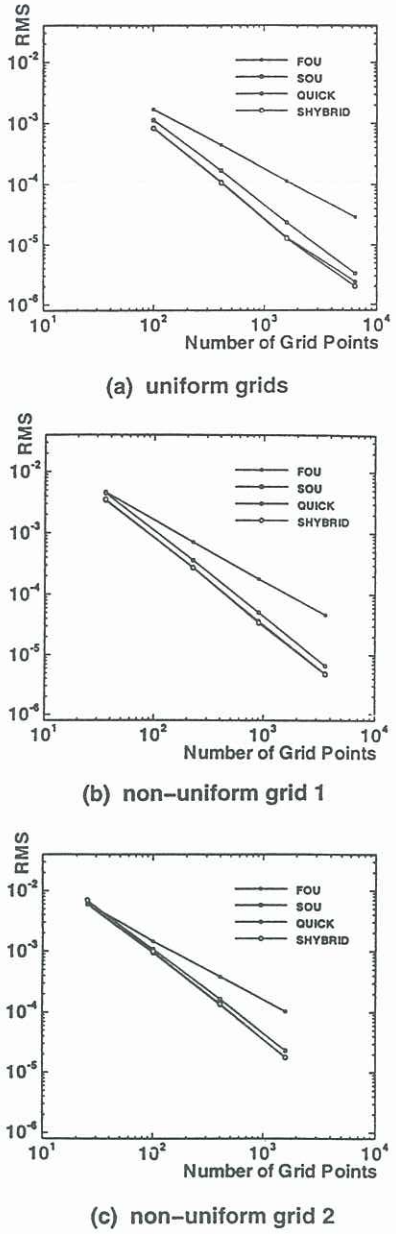

Figure 4: RMS ERROR PLOTS FOR HEAT TRANSPORT IN RECIRCULATING FLOW PROBLEM.

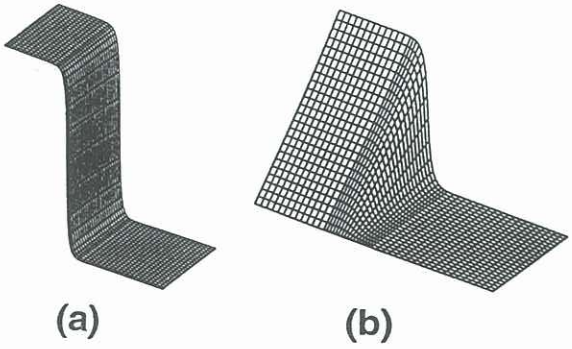

Figure 5: AN EXACT SOLUTION OF 2D BURGERS' EQUATIONS.

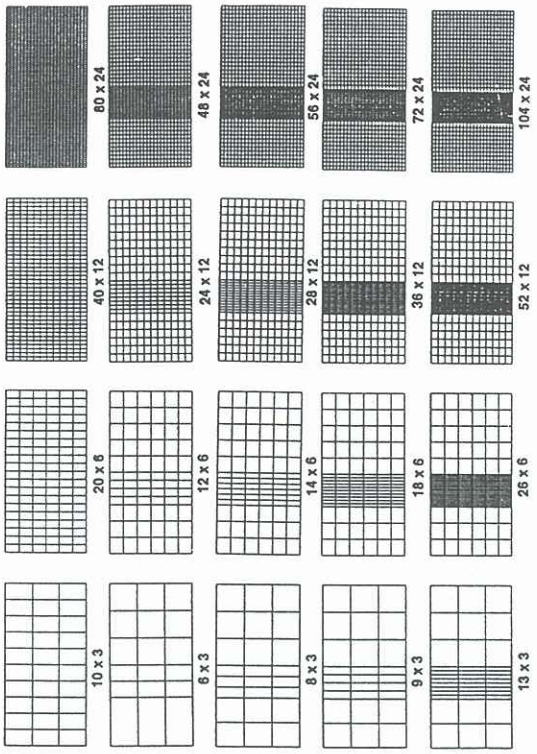

Figure 6: FOUR GRID SIZES USED FOR BURGERS' EQUATIONS. THE FOUR NON-UNIFORM GRIDS ARE REFERRED AS GRID 1, 2, 3 AND 4 IN THE TEXT.

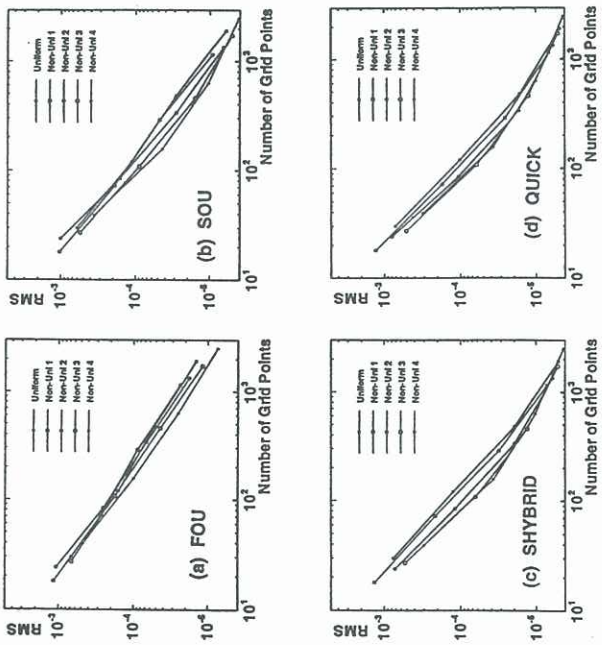

Figure 7: RMS PLOTS FOR FOUR DIFFERENT SCHEMES IN COMPUTING u OF THE 2D BURGERS' EQUATIONS.