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THE AXISYMMETRIC EQUATIONS FOR A BUOYANT JET IN A CROSSFLOW

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ABSTRACT

The entrainment into buoyant jets and plumes in a weak crossflow was investigated. The entrainment velocities were measured with a particle image velocimetry (PIV) technique. The experiments showed that in the irrotational region the entrainment velocities can be superimposed. This implied that, given the entrainment velocity for a plume in a still medium, the entrainment velocity in a moving medium can be calculated by adding the crossflow velocity to the entrainment velocity in a still medium. The implications of this lead to an additional term in the integral momentum equations and to a more detailed picture of the entrainment into the buoyant jet in a crossflow.

INTRODUCTION

At any radius the entrainment velocity into a buoyant jet in a stationary ambient is proportional to the maximum plume velocity. The equation of continuity implies that the entrainment velocity varies inversely with the radius. It is traditional to use the entrainment velocity at a radius of b (where $u_{eg}/U_{eg} = 1/e$) and this velocity is αU_{eg} (= U_α), where α is the traditional entrainment coefficient. If it is assumed that, when the plume is advected, its advected excess velocity distribution (u_{eg}) and the turbulence and the pressure distribution are unchanged by the advection, then the entrainment velocity relative to the crossflow velocity (U_∞) will be proportional to the maximum plume velocity (U_{eg}) measured relative to the crossflow velocity (Figure 1). Provided the entrainment velocity is greater that the crossflow velocity, then, in stationary coordinates on the axis of the plume and parallel to the crossflow, the entrainment velocity on the upstream side is U_α+ $U_\infty \sin\alpha_t$ and on the downstream side is U_α- $U_\infty \sin\alpha_t$, as shown in Figure 2.

Experiments examining the entrainment velocities into buoyant jets used a particle image velocimetry (PIV) technique, in which the ambient fluid was seeded with particles and a sheet of laser light illuminated a section through the flow. Pairs of images, separated by a small time interval, were analyzed using the PIV analysis of Stevens and Coates (1994). This used a pattern matching technique to find the displacement of sub-images, in the time interval between the images. The results were presented as a velocity vector map of the image. Figure 3 shows entrainment velocities of a horizontal buoyant jet with a densimetric Froude number of 10 in a stationary ambient. Figure 4a shows the same horizontal buoyant jet in an ambient flow of 2.13 mm/s. This is compared to Figure 4b, in which the ambient flow of 2.13 mm/s was added numerically to the entrainment velocities of the stationary ambient case. While there are some differences, particularly close to the port where the buoyant flow feels the wake from the port, the assumption, that the entrainment velocity in a flow is the normal entrainment velocity with the advected velocity superimposed, is within the experimental errors and it is worth exploring this advected buoyant jet assumption.

ENTRAINMENT WITH A VERY SMALL CROSSFLOW

Considering a stationary control volume perpendicular to the excess velocity, the assumed Gaussian velocity distribution and the self-similarity of the flow imply, as shown in Figure 5, a radial velocity distribution of

$$\frac{u_r}{\alpha U_{eg}} = \frac{1-\exp-\left(\frac{r}{b}\right)^2}{\frac{r}{b}} \qquad (1)$$

The equation of continuity is

$$d\int_0^R \left(u_{eg} + U_\infty \cos\alpha_r\right) 2\pi r\,dr$$
$$-\left(\int_0^{2\pi} \alpha U_{eg} b \frac{1-\exp-\left(\frac{R}{b}\right)^2}{\left(\frac{R}{b}\right)} \frac{R}{b} d\phi \right. \qquad (2)$$
$$\left. + U_\infty \sin\alpha_r \int_0^{2\pi} \cos\phi\, bR\, d\phi\right) ds = 0$$

Hence provided R/b is large such that all of the entrainment is included

$$d\int_0^R \left(u_{eg} + U_\infty \cos\alpha_r\right) 2\pi r\,dr - 2\pi b \alpha U_{eg} ds = 0 \qquad (3)$$

It is notable that the term involving the entrainment does not involve U_∞. The maximum crossflow for which this equation applies can be determined by calculating the maximum radial velocity in a zero ambient velocity (Figure 5) and setting this equal to the crossflow velocity. The maximum crossflow velocity is then given by $U_\infty \sin\alpha_r / \alpha U_{eg}$ equal to 0.638. The normal two dimensional stream function, ψ, for this entrainment flow ($\perp$ to u_{eg}) is

$$\psi = -\alpha U_{eg} b \int_0^\phi \left(1-\exp-\left(\frac{r}{b}\right)^2 d\phi + U_\infty r \sin\alpha_r \sin\phi \right. \qquad (4)$$

It should be noted that there is three dimensional flow near the axis because, as the entraining fluid enters the buoyant flow, it gets carried up vertically by the motion of the buoyant flow. Due to this three dimensional aspect, the inflow decreases as the axis is approached and hence the value of ψ decreases along the two dimensional streamline. For this maximum crossflow velocity the entrainment into the buoyant flow is bounded by ψ equalling zero and hence

$$\left(1-\exp-\left(\frac{r}{b}\right)^2\right)\phi = \frac{U_\infty \sin\alpha_r}{\alpha U_{eg}} \frac{r\sin\phi}{b}$$
$$= 0.638 \frac{r\sin\phi}{b} \qquad (5)$$

When r/b tends to infinity and ϕ tends to π, this yields y/b equal to 4.92 and thus the complete entrainment comes from a width of approximately 10b. The plot of the lines enclosing the complete entrainment for the maximum crossflow velocity is shown in Figure 6 and it is worth noting that the entrainment function implies a cusp on the centreline. This may be the start of the obvious changing pattern always observed in a jet or buoyant flow in a crossflow. Also in Figure 6 is the case with a crossflow velocity of 0.5 of the maximum (0.319). In this case the line enclosing the complete entrainment is the same as a sink in a uniform flow and the cusp is no longer there.

THE CASE WHEN THE CROSSFLOW IS GREATER THAN THE MAXIMUM ENTRAINMENT VELOCITY

The position of zero radial velocity (i.e. the limit of the entrainment) is on the two dimensional streamline and this point can be determined by equating the entrainment velocity to the resolved part of the crossflow velocity along the radius. This yields

$$U_\infty \sin\alpha_r \cos\phi = \alpha U_{eg} \frac{1-\exp-\left(\frac{r}{b}\right)^2}{\left(\frac{r}{b}\right)} \qquad (6)$$

The maximum value of $(1 - \exp -(r/b)^2)/(r/b)$ is 0.638. Thus the minimum value of ϕ is ϕ_1 given by

$$\cos\phi_1 = 0.638 \frac{\alpha U_{eg}}{U_\infty \sin\alpha_r} \qquad (7)$$

A ϕ of zero occurs at r/b equal to 1.121 and, when $\alpha U_{eg}/(U_\infty \sin\alpha_r)$ is equal to 1.567. When the value of $\alpha U_{eg}/(U_\infty \sin\alpha_r)$ is less than 1.567, there is a minimum value of ϕ. For example, if $\alpha U_{eg}/(U_\infty \sin\alpha_r)$ has a value of, then ϕ_1 is 50°. This gives a minimum value or y_1/b of 0.86 (1.121 sin50°). From this point the trace of the zero two dimensional stream function ($\psi = 0$) is

$$\psi = 0 = -\alpha U_{eg} b \int_{\phi_1}^{\phi} \left(1-\exp-\left(\frac{r}{b}\right)^2\right) d\phi$$
$$+ U_\infty \sin\alpha_r \int_{y_1}^{y} dy \qquad (8)$$

When r/b tends to infinity, ϕ tends to π and this yields

$$\frac{y_\infty - y_1}{b} = \frac{\alpha U_{eg}}{U_\infty \sin\alpha_r}(\pi - \phi_1) \qquad (9)$$

For the case where $\alpha U_{eg}/(U_\infty \sin\alpha_r)$ is 1, the value of y_∞/b is 3.11. Thus the zero streamline goes from an x/b of infinity and y/b of 3.11 to x_1/b of 0.72 and y_1/b of 0.86. This is plotted in Figure 6 and shows the flux of volume at $x = \infty$ and suggests that this is

$$\frac{d}{ds}\int_0^{r=\infty}\int_0^{2\pi} \left(U_\infty \sin\alpha_r + u_{eg}\right) 2\pi r\,dr\,d\phi$$
$$= 2\left(\alpha U_\infty \sin\alpha_r (y_\infty - y_1) + U_\infty \sin\alpha_r y_1\right) \qquad (10)$$
$$= 2\left(\alpha U_{eg} b(\pi - \phi_1) + U_\infty \sin\alpha_r y_1\right)$$

Noting that the superposition of velocities implies that the $U_\infty \sin\alpha_r$ term in the double integral is zero and, defining the u_{eg} term in the double integral as q, leads to

$$\frac{q}{2\pi U_{eg} b} = \frac{\pi - \phi_1}{\pi} + \frac{U_\infty \sin\alpha_r}{\pi \alpha U_{eg}}\left(\frac{y_1}{b}\right)$$
$$= \frac{\pi - \phi_1}{\pi} + \frac{U_\infty \sin\alpha_r}{\pi \alpha U_{eg}} 1.1 \sin\phi_1 \qquad (11)$$

When $\cos\phi_1$ is 1, $\alpha U_{eg}/(U_\infty \sin\alpha_r)$ is greater that 1.567 and ϕ_1 is 0 and the dimensionless entrainment flux $q/(2U_\infty b)$ is the normal entrainment assumption. When $\alpha U_{eg}/(U_\infty \sin\alpha_r)$ is small, then $\cos\phi_1$ tends to 0 and ϕ_1 tends to $\pi/2$. Thus the dimensionless entrainment assumption is $q/(2U_\infty b)$ equals 1.1. There are two implications with this assumption. Firstly, the superposition of velocities in equation 10 allows the normal continuity to be used rather than the empirical spread function used by Wood (1993) and others. Secondly, the assumption, that in a crossflow the entrainment flux q equals the projected area defined by $2U_\infty b$ which was introduced by Frick in 1984 by assuming a wake behind the flow and has been used extensively (Frick 1984, Lee and Cheung 1990 etc.), is close to the entrainment when $(U_\infty \sin\alpha_r)/\alpha U_{eg}$ is large. Fric and Roshko (1994) showed that the flow around the transverse jet does not separate but closes in around the jet leaving little or no open wake, as implied by the vorticity equation. However, the assumption is close to being correct in spite of there being no discernable wake.

THE EFFECT OF THE ASYMMETRIC ENTRAINMENT ON THE MOMENTUM EQUATIONS

The continuity equation can be written as

$$d\int_0^R \left(u_{eg} + U_\infty \cos\alpha_r\right) 2\pi r\, dr = \left(\int_0^{2\pi} u_r R\, d\phi + U_\infty \sin\alpha_r \int_0^{2\pi} \cos\phi\, R\, d\phi\right) ds = 0 \tag{12}$$

and the horizontal momentum equation is

$$d\int_0^R \left(u_{eg} + U_\infty \cos\alpha_r\right)\left(U_\infty + u_{eg}\cos\alpha_r\right) 2\pi r\, dr = ds\int_0^{2\pi} \left(u_r + U_\infty \sin\alpha_r \cos\phi\right)\left(u_r \sin\alpha_r \cos\phi + U_\infty\right) R\, d\phi \tag{13}$$

The continuity equation multiplied by U_∞ is subtracted from the horizontal momentum equation

$$\frac{d}{ds}\int_0^R \left(u_{eg} + U_\infty \cos\alpha_r\right) u_{eg} \cos\alpha_r 2\pi r\, dr = U_\infty U_R R \sin^2\alpha_r \int_0^{2\pi} \cos^2\phi\, d\phi + U_R^2 R \sin\alpha_r \int_0^{2\pi} \cos\phi\, d\phi = \pi U_\infty U_R R \sin^2\alpha_r \tag{14}$$

When R is sufficiently large that the gaussian distribution of velocity and buoyancy tend to zero then

$$\frac{d}{ds}\left(I_q U_{eg} b^2\right) = 2\left(\alpha U_{eg} b(\pi - \phi_1) + 1.1 U_\infty b \sin\alpha_r \sin\phi_1\right) \tag{15}$$

$$\frac{d}{ds}\left(I_m U_{eg}^2 b^2 \cos\alpha_r + U_\infty I_q U_{eg} b^2 \cos^2\alpha_r\right) = \pi b U_\infty \alpha U_{eg} \sin^2\alpha_r \tag{16}$$

$$\frac{d}{ds}\left(I_m U_{eg}^2 b^2 \sin\alpha_r + U_\infty I_q U_{eg} b^2 \cos\alpha_r \sin\alpha_r\right) = \pi b U_\infty \alpha U_{eg} \sin\alpha_r \cos\alpha_r + I_\Delta \Delta b^2 \tag{17}$$

$$I_{q\Delta} U_{eg} \Delta b^2 + U_\infty \cos\alpha_r I_\Delta \Delta b^2 = q_{\Delta 0} \tag{18}$$

Where $\cos\phi_1$ was given by eq. 7 and

$$\frac{dz}{ds} = \frac{U_{eg}\sin\alpha_r}{U_\infty \cos\alpha_r + U_{eg}} \tag{19}$$

$$\frac{dx}{ds} = \frac{U_\infty + U_{eg}\cos\alpha_r}{U_\infty \cos\alpha_r + U_{eg}} \tag{20}$$

The assumption that the buoyant flow is advected implies that, moving with the crossflow velocity, the entrainment is the same as the stationary case. It should be noted that, for a stationary ambient whether the flow is momentum or buoyancy driven, the rate of spread is, within the limits of experimental measurement, constant (0.11). However, the entrainment constant α depends on the type of flow. With a vertical buoyant jet in a stationary ambient, the value of α (Jirka 1979, Wood 1993) can be written

$$\alpha = \frac{I_q}{2\pi}\left(0.11 + \frac{b}{2 I_m U_{eg}^2 b^2}\frac{d(I_m U_{eg}^2 b^2)}{ds}\right) = \frac{I_q}{2\pi}\left(0.11 + \frac{I_q I_m^{1/2}}{2 I_{q\Delta}} b^2 \frac{q_{\Delta 0}}{(I_m U_{eg}^2 b^2 \sin\alpha_r)^{3/2}}\right) \tag{21}$$

For a pure jet, it can be shown that α equals $0.11\, I_q/(2\pi)$ and for a pure plume α equals $(5\, I_q/6\pi)\, db/ds$. The second term in the α equation deals with the buoyancy forces and thus it was reasonable to include $\sin\alpha_r$ in the buoyancy term.

There are sufficient equations to solve for b, U_{eg}, α_r, x and z as a function of s. The equations have been checked for the limiting cases with a crossflow tending to zero of a pure plume and a pure jet.

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FIGURES

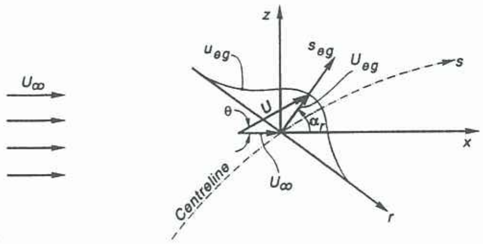

FIGURE 1: ASSUMED GAUSSIAN DISTRIBUTION OF EXCESS VELOCITY IN AN ADVECTED BUOYANT JET

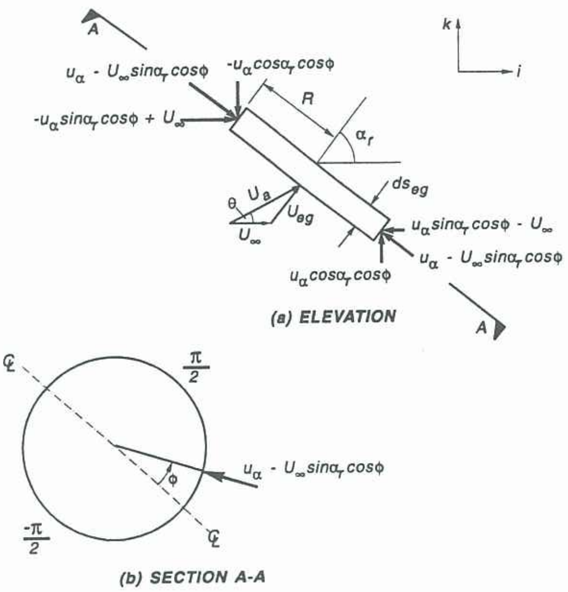

FIGURE 2: THE CONTROL VOLUME USED IN THE ANALYSIS OF AN ADVECTED BUOYANT JET

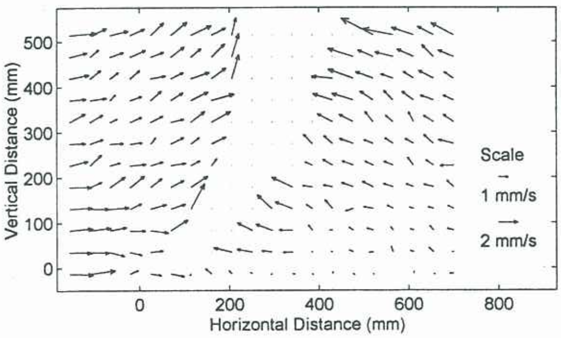

FIGURE 3: THE ENTRAINMENT VELOCITIES OF A HORIZONTAL BUOYANT JET (FR=10) IN A STATIONARY AMBIENT

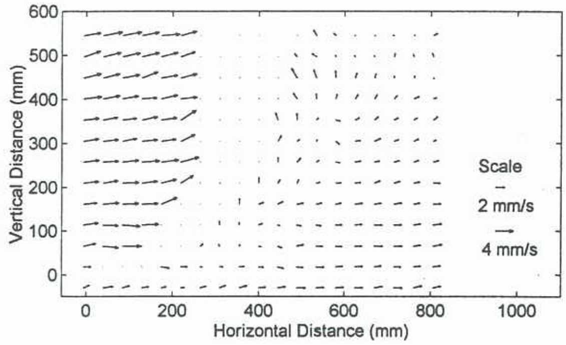

FIGURE 4A: ENTRAINMENT VELOCITIES OF A HORIZONTAL BUOYANT JET (FR=10) IN A CROSSFLOW OF 2.13 mm/s

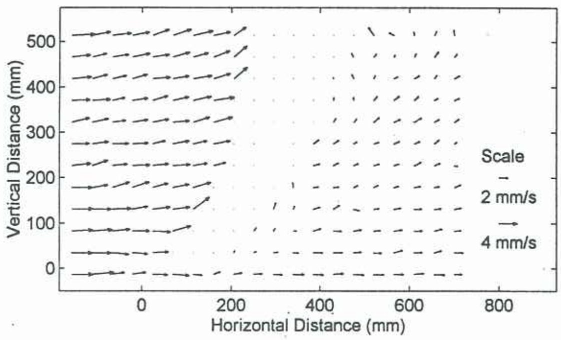

FIGURE 4B: ENTRAINMENT VELOCITIES OF A HORIZONTAL BUOYANT JET (FR=10) WITH A NUMERICALLY ADDED CROSSFLOW OF 2.13 mm/s

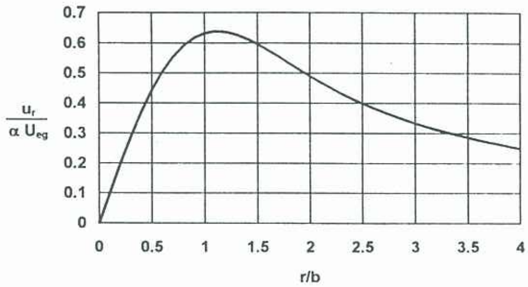

FIGURE 5: THE ENTRAINMENT VELOCITY DISTRIBUTION

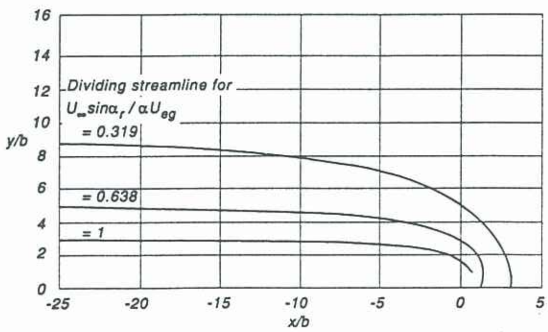

FIGURE 6: THE ENTRAINMENT BOUNDS FOR $U_\infty \sin\alpha_r/\alpha U_{eg}$ of 0.319, 0.638 and 1. 0.319 AND 0.638 ARE THE NORMAL JET ENTRAINMENT ASSUMPTION AND 1 IS THE ADVECTED BUOYANT JET ENTRAINMENT ASSUMPTION