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A SIMPLE ALGORITHM FOR GENERATING THE CONTINUITY AND NAVIER-STOKES RELATIONS IN THE SERIES-EXPANSION SOLUTION OF THE NAVIER-STOKES EQUATIONS

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1 Introduction

A technique for generating local solutions of the Navier-Stokes equations using Taylor-series expansions to arbitrary orders is described in Perry (1984) and Perry & Chong (1986a) (details given in Perry & Chong, 1986b). The technique provides a simple method of synthesizing and generating steady and time-dependent flow patterns which are asymptotically exact at the origin of the expansion. The method is useful for investigating the properties of the Navier-Stokes equations and the topology of complex flow patterns. For example, Danielson and Ottino (1990) applied the Taylor-series-expansion method to the study of chaotic particle trajectories (Lagrangian turbulence). The usefulness of the Taylor series expansion method relies on the generation of the relationship between the coefficients of the expansion from the Navier-Stokes equations. Perry (1984) used tensor analysis to generate the necessary equations. The algorithm described, although elegant and rigorous, is difficult to follow and complicated to use and to convert into a computer code for generating the neccessary relations. For example, one of the rules from Perry (1984) is: *In each tensor combination $\{J\}\{K\}$, q always leads the indices of one tensor and i always leads the other. There are always two q's and they never occur together in the one tensor. Whenever there is an i in a tensor it must always be accompanied by a q. The free indices i, α, β, δ, must be cycled* — —. A simple algorithm is described in this paper for generating the relevant continuity relations and Navier-Stokes relations[1]. The relations are 'almost analytical' and can be used to generate all the necessary relations for the series-expansion solution of the Navier-Stokes equations.

2 Theory

2.1 Basic Equations

The Navier-Stokes equations for incompressible, constant density flow can be expressed as

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{1}$$

where $P = p/\rho$ is the kinematic pressure, p is the pressure, ρ is the fluid density, ν is the kinematic viscosity, u_i is the velocity tensor and x_i is the space coordinate tensor. The continuity equation is

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{2}$$

The velocity field can be expanded as

$$u_i = \sum_{n=0}^{N} \mathcal{R}.[a_i, b_i, c_i]_i.x_1^{a_i}.x_2^{b_i}.x_3^{c_i} \tag{3}$$

where $[a_i, b_i, c_i]$ represents a coefficient in the Taylor-series expansion. a_i, b_i and c_i are the powers of of x_1, x_2 and x_3 respectively and are used to index the coefficients of the expansion. N is the highest order of the expansion. For each n, $a_i + b_i + c_i = n$, such that a_i, b_i and c_i are in every possible permutation and combination. It can been shown that the $\mathcal{R}$'s are factors which allow for the different permutation of the indices (Perry & Chong, 1982 [2]) and are given by

$$\mathcal{R} = \frac{(a_i + b_i + c_i)!}{a_i!b_i!c_i!} \tag{4}$$

The velocity expansions are shown in tabular form in Table 1.

[1]Note the use of *relations* and *equations*. Relationships between the coefficients of the expansion such that the continuity *equations* are satisfied will be referred to as continuity *relations*. A similar convention is adopted for the Navier-Stokes *relations* as distinct from the Navier-Stokes *equations*.

[2]The analysis can be carried out without this constant. However, this has been included so that the relations generated are consistent with the relations given in Perry & Chong(1986).

	$u_1 =$	$u_2 =$	$u_3 =$
1	$[0,0,0]_1$	$[0,0,0]_2$	$[0,0,0]_3$
2	$+[1,0,0]_1$	$+[1,0,0]_2$	$+[1,0,0]_3$
3	$+[0,1,0]_1$	$+[0,1,0]_2$	$+[0,1,0]_3$
4	$+[0,0,1]_1$	$+[0,0,1]_2$	$+[0,0,1]_3$
5	$+[2,0,0]_1$	$+[2,0,0]_2$	$+[2,0,0]_3$
6	$+[0,2,0]_1$	$+[0,2,0]_2$	$+[0,2,0]_3$
7	$+[0,0,2]_1$	$+[0,0,2]_2$	$+[0,0,2]_3$
8	$+2[1,1,0]_1$	$+2[1,1,0]_2$	$+2[1,1,0]_3$
9	$+2[1,0,1]_1$	$+2[1,0,1]_2$	$+2[1,0,1]_3$
10	$+2[0,1,1]_1$	$+2[0,1,1]_2$	$+2[0,1,1]_3$
11	$+[3,0,0]_1$	$+[3,0,0]_2$	$+[3,0,0]_3$
12	$+[0,3,0]_1$	$+[0,3,0]_2$	$+[0,3,0]_3$
13	$+[0,0,3]_1$	$+[0,0,3]_2$	$+[0,0,3]_3$
14	$+3[2,1,0]_1$	$+3[2,1,0]_2$	$+3[2,1,0]_3$
15	$+3[2,0,1]_1$	$+3[2,0,1]_2$	$+3[2,0,1]_3$
16	$+3[1,2,0]_1$	$+3[1,2,0]_2$	$+3[1,2,0]_3$
17	$+3[1,0,2]_1$	$+3[1,0,2]_2$	$+3[1,0,2]_3$
18	$+3[0,2,1]_1$	$+3[0,2,1]_2$	$+3[0,2,1]_3$
19	$+3[0,1,2]_1$	$+3[0,1,2]_2$	$+3[0,1,2]_3$
20	$+6[1,1,1]_1$	$+3[1,1,1]_2$	$+3[1,1,1]_3$
	.	.	.
	.	.	.
	.	.	.
	.	.	.
	.	.	.
	$+\mathcal{R}[a_1,b_1,c_1]_1$	$+\mathcal{R}[a_2,b_2,c_2]_2$	$+\mathcal{R}[a_3,b_3,c_3]_3$

TABLE 1

2.2 Continuity relations

Differentiating the velocity expansions for u_1, u_2 and u_3 with respect to x_1, x_2 and x_3 respectively, substituting into equation 2 and grouping coefficients of like powers of x_1, x_2 and x_3, the continuity relations can be shown to have the following simple form:

$$[a_1,b_1,c_1]_1 + [a_2,b_2,c_2]_2 + [a_3,b_3,c_3]_3 = 0. \quad (5)$$

where

$$\begin{array}{ll} a_2 = a_1 - 1 & , \ a_3 = a_1 - 1 \\ b_2 = b_1 + 1 & , \ b_3 = b_1 \\ c_2 = c_1 & , \ c_3 = c_1 + 1. \end{array} \quad (6)$$

Hence the continuity relations can be easily generated by considering each coefficient of the u_1 velocity expansion (column 2 of Table 1) and equating them to coefficients of u_2 and u_3 using the above rule (and noting that the indices cannot have negative values). For example, for coefficient $[2,0,1]_1$, the continuity relation is given by

$$[2,0,1]_1 \ + \ [1,1,1]_2 \ + \ [1,0,2]_3 \ = 0 \quad (7)$$

2.3 Navier-Stokes Relations

The Navier-Stokes relations are obtained from equating cross-derivatives of pressure and grouping coefficients of like powers of x_1, x_2 and x_3. The Navier-Stokes equations can be written as:

$$-\frac{\partial P}{\partial x_i} = \frac{\partial u_i}{\partial t} + [I_i] - [V_i] \quad (8)$$

where $\frac{\partial u_i}{\partial t}$ are the time-dependent terms, $[I_i]$ represent inertia terms and $[V_i]$ represent the viscous terms.

The Navier-Stokes relations are obtained by equating the cross-derivatives of pressure, i.e.

$$\frac{\partial}{\partial x_j}\left(\frac{\partial P}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial P}{\partial x_j}\right) \quad (9)$$

and matching the powers of x_1, x_2 and x_3 of the inertia terms and the viscous terms to those coefficients which appear as time-derivatives in the Navier-Stokes relations. The coefficients which appear as time-derivatives can be obtained by differentiating the expansion for u_1, u_2 and u_3 with time and equating cross-derivatives of pressures as given by equation 9, i.e.

$$\frac{\partial u_i}{\partial t} = \mathcal{R}.[a_i, \dot{b_i}, c_i]_i . x_1^{a_i} . x_2^{b_i} . x_3^{c_i} \quad (10)$$

where the *dot* above the coefficient denotes the derivative of the coefficient with time. The relationships between the time-dependent coefficients in the Navier-Stokes relations are given by:

$$[a_1, \dot{b_1}, c_1]_1 - [a_2, \dot{b_2}, c_2]_2 = [\textit{Inertia terms}]_1 + [\textit{Viscous terms}]_1 \quad (11)$$

where

$$\begin{array}{rcl} a_2 & = & a_1 + 1 \\ b_2 & = & b_1 - 1 \\ c_2 & = & c_1 \end{array} \quad (12)$$

$$[a_1, \dot{b_1}, c_1]_1 - [a_3, \dot{b_3}, c_3]_3 = [\textit{Inertia terms}]_2 + [\textit{Viscous terms}]_2 \quad (13)$$

where

$$\begin{array}{rcl} a_3 & = & a_1 + 1 \\ b_3 & = & b_1 \\ c_3 & = & c_1 - 1 \end{array} \quad (14)$$

and

$$[a_2, \dot{b_2}, c_2]_2 - [a_3, \dot{b_3}, c_3]_3 = [Inertia\ terms]_3 + [Viscous\ terms]_3 \tag{15}$$

where

$$\begin{aligned} a_3 &= a_2 \\ b_3 &= b_2 + 1 \\ c_3 &= c_2 - 1 \end{aligned} \tag{16}$$

From equation 11 and 13, it can be seen that all coefficients of u_1, i.e. $[a_1, b_1, c_1]_1$, will appear in a Navier-Stokes relation except when $b_1 = 0$ and $c_1 = 0$. Also, if $b_1 \neq 0$ and $c_1 \neq 0$, then equation 15 will give a redundant relationship.

Once the coefficients which appear as time-derivatives have been obtained, the viscous terms and the inertia terms can be obtained since these must include coefficients such that, after the various diffferentiation and multiplication processes, the powers of x_1, x_2 and x_3 are the same as those coefficients which appear as time-derivatives.

For example, to be compatible with the coefficients which appear as time-derivatives the viscous terms in equation 11 are given by the following simple form:

$$\begin{aligned} & [Viscous\ terms]_1 = \\ \nu\mathcal{F} \ & \{[a_{1V(12)}, b_{1V(12)}, c_{1V(12)}]_1 + [a_{1V(22)}, b_{1V(22)}, c_{1V(22)}]_1 \\ + \ & [a_{1V(32)}, b_{1V(32)}, c_{1V(32)}]_1 - [a_{2V(11)}, b_{2V(11)}, c_{2V(11)}]_2 \\ - \ & [a_{2V(21)}, b_{2V(21)}, c_{2V(21)}]_2 - [a_{2V(31)}, b_{2V(31)}, c_{2V(31)}]_2\} \end{aligned} \tag{17}$$

where

$$\begin{aligned} &a_{1V(12)} = a_1 + 2, \quad b_{1V(12)} = b_1, \quad c_{1V(12)} = c_1 \\ &a_{1V(22)} = a_1, \quad b_{1V(22)} = b_1 + 2, \quad c_{1V(22)} = c_1 \\ &a_{1V(32)} = a_1, \quad b_{1V(32)} = b_1, \quad c_{1V(32)} = c_1 + 2 \\ \\ &a_{2V(11)} = a_1 + 3, \quad b_{2V(11)} = b_1 - 1, \quad c_{2V(11)} = c_1 \\ &a_{2V(21)} = a_1 + 1, \quad b_{2V(21)} = b_1 + 1, \quad c_{2V(21)} = c_1 \\ &a_{2V(31)} = a_1 + 1, \quad b_{2V(31)} = b_1 - 1, \quad c_{2V(31)} = c_1 + 2 \end{aligned} \tag{18}$$

The factor $\mathcal{F}$ can be shown to be given by:

$$\mathcal{F} = (n+1)(n+2) \tag{19}$$

Similar expressions can be derived using the above method for the viscous terms in equation 13 and equation 15.

A similar analysis can be carried out for the inertia terms. Again simple expressions can be obtained between these nonlinear terms. For example the inertia terms in equation 11 must be in the following form:

$$\begin{aligned} & [Inertia\ terms]_1 = \\ - \ & \alpha_{112}.[a_{1L(12)}, b_{1L(12)}, c_{1L(12)}]_1.[a_{1U(12)}, b_{1U(12)}, c_{1U(12)}]_1 \\ - \ & \alpha_{122}.[a_{1L(12)}, b_{1L(12)}, c_{1L(12)}]_1.[a_{2U(12)}, b_{2U(12)}, c_{2U(12)}]_2 \\ - \ & \alpha_{132}.[a_{1L(12)}, b_{1L(12)}, c_{1L(12)}]_1.[a_{3U(12)}, b_{3U(12)}, c_{3U(12)}]_3 \\ + \ & \beta_{111}.[a_{2L(11)}, b_{2L(11)}, c_{2L(11)}]_2.[a_{1U(11)}, b_{1U(11)}, c_{1U(11)}]_1 \\ + \ & \beta_{121}.[a_{2L(11)}, b_{2L(11)}, c_{2L(11)}]_2.[a_{2U(11)}, b_{2U(11)}, c_{2U(11)}]_2 \\ + \ & \beta_{131}.[a_{2L(11)}, b_{2L(11)}, c_{2L(11)}]_2.[a_{3U(11)}, b_{3U(11)}, c_{3U(11)}]_3 \end{aligned} \tag{20}$$

where

$$\begin{aligned} &a_{1U(12)} = a_1 - a_{1L(12)} + 1, \quad b_{1U(12)} = b_1 - b_{1L(12)}, \\ &a_{2U(12)} = a_1 - a_{1L(12)}, \quad b_{2U(12)} = b_1 - b_{1L(12)} + 1, \\ &a_{3U(12)} = a_1 - a_{1L(12)}, \quad b_{3U(12)} = b_1 - b_{1L(12)}, \\ &c_{1U(12)} = c_1 - c_{1L(12)} \\ &c_{2U(12)} = c_1 - c_{1L(12)} \\ &c_{3U(12)} = c_1 - c_{1L(12)} + 1 \\ &a_{1U(11)} = a_1 - a_{2L(11)} + 2, \quad b_{1U(11)} = b_1 - b_{2L(11)} - 1, \\ &a_{2U(11)} = a_1 - a_{2L(11)} + 1, \quad b_{2U(11)} = b_1 - b_{2L(11)}, \\ &a_{3U(11)} = a_1 - a_{2L(11)} + 1, \quad b_{3U(11)} = b_1 - b_{2L(11)} - 1, \\ &c_{1U(11)} = c_1 - c_{2L(11)} \\ &c_{2U(11)} = c_1 - c_{2L(11)} \\ &c_{3U(11)} = c_1 - c_{2L(11)} + 1 \end{aligned} \tag{21}$$

Hence to match all the u_1 coefficients which appear as a time-derivative (i.e. for a given $[a_1, b_1, c_1]_1$), and for each $[a_{1L(12)}, b_{1L(12)}, c_{1L(12)}]_1$ and $[a_{2L(11)}, b_{2L(11)}, c_{2L(11)}]_2$, the above algorithm can be used to generate all the non-linear inertia terms. Simple expressions can be found for the factors α's and β's which appear in the non-linear terms. For example, α_{112} is given by

$$\begin{aligned} \alpha_{112} &= a_{1L(12)} \times (b_{1L(12)} + b_{1U(12)}) \\ &\times \frac{(a_{1L(12)} + b_{1L(12)} + c_{1L(12)})!}{(a_{1L(12)}!b_{1L(12)}!c_{1L(12)}!)} \\ &\times \frac{(a_{1L(12)} + b_{1L(12)} + c_{1L(12)})!}{(a_{1L(12)}!b_{1L(12)}!c_{1L(12)}!)} \\ &\times \frac{(a_1!b_1!c_1!)}{(a_1 + b_1 + c_1)!} \times \frac{1}{b_1} \end{aligned} \tag{22}$$

Similar simple algorithms can be developed for $[Inertia\ terms]_2$ and $[Inertia\ terms]_3$

3 Applications

The above analysis shows that it is possible to develop a simple algorithm for generating the relationships between the coefficients of a Taylor series expansion so that they satisfy continuity and the Navier-Stokes equations. The Navier-Stokes relations are first-order ordinary differential equations for some of the coefficients of the expansion. These can be used to compute the evolution of the coefficients (and hence the flow pattern) in time-dependent problems.

The above technique can be used to obtain local solutions of the Navier-Stokes equations. Examples of the use of the above technique for generating steady three-dimensional separation patterns are given in Perry & Chong (1986) and for generating time-dependent three-dimensional separation patterns in Chong & Perry (1986) and in Chong & Perry (1989). A further improvement of the technique is the extension of the region of accuracy of the solution by

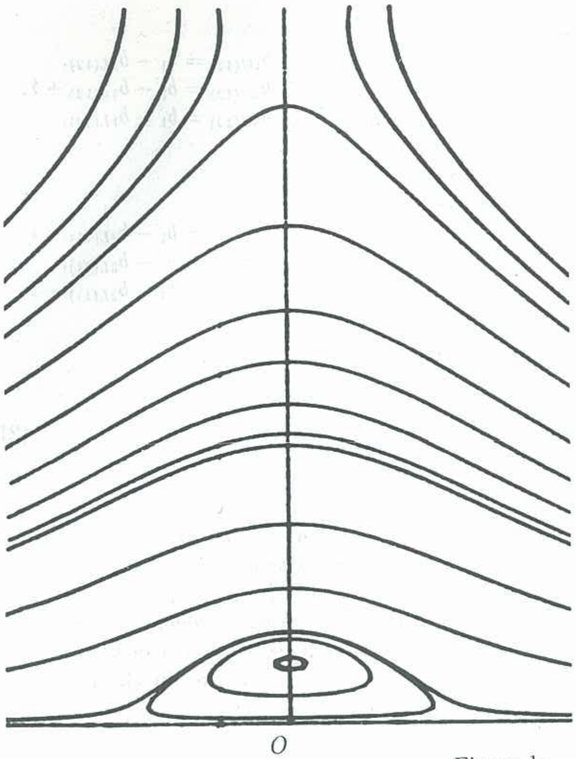

Figure 1a.

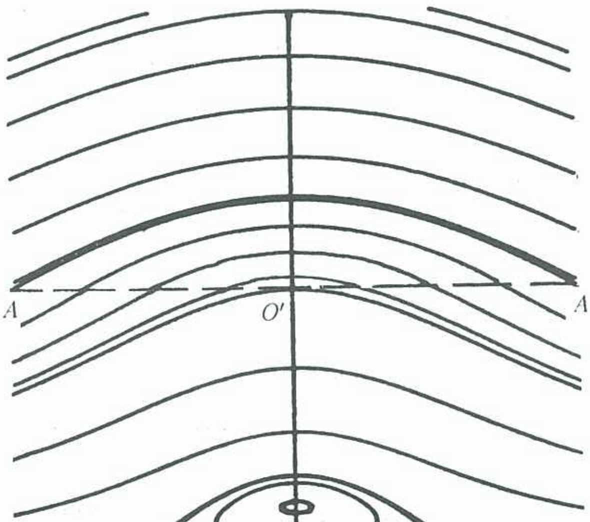

Figure 1b.

Figure 1. Two-dimensional separation bubble generated using 3rd order expansion. (a) Separation bubble obtained from a single expansion at O. (b) Separation bubble obtained from two expansion (at O and O') with matching boundary conditions across $A - A$.

matching boundary conditions for several series expansion. In the example shown in figure 1a, the two-dimensional separation pattern has been obtained by an expansion about the origin O. It can be seen that away from the origin, the flow is unrealistic. However, it is possible to generate another expansion about O' such that the boundary conditions are matched across the boundary of the two expansion, i.e. along $A - A$. This produces a flow pattern, shown in figure 1b, which is more consistent with a two dimensional separation pattern. Further work on the extension of the region of validity is currently being investigated.

4 Conclusion

A disadvantage of using Taylor-series expansion for generating local solutions of the Navier-Stokes has been the difficulty in generating the necessary relations between the coefficients of the series expansion so that they satisfy the Navier-Stokes equations. A simple algorithm for generating these relationships is described in this paper.

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