TENTH AUSTRALASIAN FLUID MECHANICS
CONFERENCE - UNIVERSITY OF MELBOURNE
11-15 DECEMBER 1989

CONVECTIVE CIRCULATION IN RESERVOIR SIDEARMS OF SMALL ASPECT RATIO

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ABSTRACT

The unsteady daytime circulation in a reservoir sidearm caused by differential heating is modelled by the natural convection of a fluid contained in a narrow infinite wedge. Internal heating by the absorption of solar radiation is modelled by Beer's law. The principal driving mechanism for the flow is a heat flux applied along the sloping bottom. The bottom heat flux is calculated from the amount of heat that is not absorbed by the water column. Asymptotic solutions valid for small bottom slopes are found for the flow and temperature fields. Only the lowest order solutions are found here so there is no correction for the effect of convection on the temperature field.

§1 INTRODUCTION

During the daylight hours, the shallower regions of a reservoir sidearm absorb more heat per unit volume than the deeper parts. This can generate a circulation in the sidearm that may be significant for the transport of pollutants or other substances introduced at the shallow end of the sidearm. During the night, an opposite effect occurs with the shallow regions losing more heat per unit volume leading to a circulation in the opposite direction. Fluid contained in an infinite wedge subject to an internal source of heat as well as a boundary heat flux is used to model the daytime circulation

Convection in triangular cavities, particularly when driven by internal heating or boundary heat fluxes, has received relatively little attention in the literature. Unsteady natural convection in a rectangular cavity driven by internal heating has been examined by Patterson (1984). Poulikakos and Bejan (1983) investigated the steady flow in an attic space and suggested that their results could be used in the geophysical context. However, the time taken for the circulation in a sidearm to reach steady or quasi-steady state is comparable to the time scale of the forcing (Monismith and Imberger (1986)) so it would appear that the reservoir sidearm problem is intrinsically unsteady. Typical sidearms have small bottom slopes ($\sim 10^{-2}$). This fact can be exploited to obtain asymptotic solutions for the temperature and velocity distributions.

Environmental Dynamics reference ED-295-DF

§2 MATHEMATICAL FORMULATION

The flow in a reservoir sidearm is modelled by the flow contained in an infinite wedge lying between $y = 0$ and $y = -Ax$ in the (x, y) plane. Temperature differences in a reservoir are typically small so the Boussinesq approximation is appropriate. Thus, the equations that are to be solved are

$$u_t + uu_x + vu_y = -p_x/\rho_0 + \nu\nabla^2 u \tag{1}$$

$$v_t + uv_x + vv_y = -p_y/\rho_0 + \nu\nabla^2 v + g\alpha(T - T_0) \tag{2}$$

$$T_t + uT_x + vT_y = \kappa\nabla^2 T + Q(x, y, t) \tag{3}$$

$$u_x + v_y = 0. \tag{4}$$

where u and v are the horizontal and vertical velocities, T is temperature, p is pressure, ν is the kinematic viscosity, κ is the coefficient of thermal conduction, α is the coefficient of thermal expansion, ρ_0 is the reference density, T_0 is the reference temperature, $Q(x, y, t)$ is the temperature source with units of $℃ s^{-1}$ and independent variable subscripts denote differentiation. If the intensity of solar radiation incident on the water surface is I_0 and the extinction coefficient for Beer's Law is η then, assuming no reflection from the bottom boundary, Q is given by

$$Q(x, y, t) = \frac{I_0\eta}{\rho C_p} e^{\eta y} \equiv Q_0\eta e^{\eta y}. \tag{5}$$

Note that Q has no horizontal dependence and so this in itself will not drive any flow. The driving force for the flow comes from the next assumption; the heat not absorbed by the water column is absorbed by the bottom of the sidearm which immediately releases this heat as a boundary temperature flux. This statement is embodied in the boundary condition

$$\frac{dT}{d\hat{n}} = \frac{1}{\sqrt{1 + A^2}}(T_y + AT_x) = -\frac{1}{\kappa}Q_0\eta e^{-A\eta x} \quad \text{on } y = -Ax \tag{6}$$

where $\hat{n}$ is the direction normal to the sloping bottom. For this model it is assumed that heat exchange between the water surface and the air immediately above is negligible hence $T_y = 0$ on $y = 0$. Boundary conditions for the velocities u and v are $u = v = 0$ an $y = -Ax$ and $u_y = \tau_s/\mu$, $v = 0$ on $y = 0$ where τ_s is the (known) stress at the water surface which, for simplicity, is assumed to be zero in this work. Initial conditions are simply that at $t = 0$ the fluid is at rest and is a uniform temperature T_0.

§3 NON-DIMENSIONALISATION

The geometry of the problem imposes no natural length scale and so the vertical coordinate must scale like $y \sim \eta^{-1}$. The geometry then suggests that $x \sim (A\eta)^{-1}$. The flow in the cavity is driven by temperature gradients which suggests that the appropriate time scale is $t \sim (\kappa\eta^2)^{-1}$. Thus, via (3), $T - T_0 \sim Q_0/(\kappa\eta)$. This scale for the temperature difference along with (2) gives a scale for the horizontal pressure gradient which can be used with (1) to give a scale for the horizontal velocity, namely $u \sim AGr\kappa\eta$, where the Grashof number is given by

$$Gr = \frac{g\alpha Q_o}{\kappa^3\eta^4}. \tag{7}$$

Lastly, the continuity equation (4) gives $v \sim A^2 Gr\kappa\eta$

Using the above scales to non-dimensionalise (1)–(4), eliminating p and introducing a stream function ψ yields

$$\begin{aligned}&\psi_{tyy} + A^2\psi_{txx}\\&\quad + A^2 Gr\left(\psi_x\psi_{yyy} - \psi_y\psi_{yyx} + A^2\left(\psi_x\psi_{xxy} - \psi_y\psi_{xxx}\right)\right)\\&\quad = \sigma\left(\psi_{yyyy} + 2A^2\psi_{xxyy} + A^4\psi_{xxxx}\right) + T_x\end{aligned} \tag{8}$$

and

$$\frac{\partial T}{\partial t} + A^2 Gr\left(-\psi_y T_x + \psi_x T_y\right) = A^2 T_{xx} + T_{yy} + e^y \tag{9}$$

where $u = -\psi_y$, $v = \psi_x$, $\sigma = \nu/\kappa$ and all variables are now non-dimensional. The non-dimensional boundary conditions are

$$\psi = \psi_{yy} = 0,\ T_y = 0,\ \text{on } y = 0, \tag{10}$$

$$\psi = \psi_y - 0,\ \frac{T_y + A^2 T_x}{\sqrt{1 + A^2}} = -e^{-x},\ \text{on } y = -x, \tag{11}$$

Equations (8) and (9) with boundary conditions (10) and (11) have no steady state since heat is continuously being added and none is allowed to escape.

§4 ASYMPTOTIC SOLUTION

Equations (8) and (9) are not directly soluble; however for $A \ll 1$ asymptotic solutions are obtainable. Following Cormack *et al.* (1974), T and ψ are expanded as a sum of powers of A^2,

$$T = T^{(0)} + A^2 T^{(2)} + A^4 T^{(4)} + \cdots \tag{12a}$$

$$\psi = \psi^{(0)} + A^2\psi^{(2)} + A^4\psi^{(4)} + \cdots. \tag{12b}$$

Substituting these expressions into equations (8)–(11) and equating like powers of A^2 yields a sequence of linear equations that can, in principle, be solved recursively. For the sake of simplicity only the $O(A^0)$ equations are solved here.

The $O(A^0)$ equations are

$$\psi^{(0)}_{tyy} = \sigma\psi^{(0)}_{yyyy} + T^{(0)}_x \tag{13}$$

$$T^{(0)}_t = T^{(0)}_{yy} + e^y \tag{14}$$

with boundary conditions

$$\psi^{(0)} = \psi^{(0)}_{yy} = 0,\ T^{(0)}_y = 0 \quad \text{on } y = 0, \tag{15}$$

$$\psi^{(0)} = \psi^{(0)}_y = 0,\ T^{(0)}_y = -e^{-x} \quad \text{on } y = -x \tag{16}$$

and initial conditions

$$\psi^{(0)} = 0,\ T^{(0)} = 0 \text{ at } t = 0 \tag{17}$$

The solution for $T^{(0)}$ is

$$\begin{aligned}T^{(0)} =& t/x - e^y + \frac{1}{2}y^2/x + y + x/3 + (1 - e^{-x})/x\\&- \frac{2}{x}\sum_{n=1}^{\infty} a_n(x) e^{-(n\pi/x)^2 t}\cos(n\pi y/x).\end{aligned} \tag{18}$$

where the $a_n(x)$'s are given in the Appendix. This solution allows for vertical conduction only and thus has no correction for convection. It would be necessary to find $T^{(2)}$ before convective effects on the temperature field would be evident. For simplicity, only the large time solution for $\psi^{(0)}$ is found here. This simply means neglecting the series part of equation (18). The equation for $\psi^{(0)}$ is then

$$\psi^{(0)}_{tyy} = \sigma\psi^{(0)}_{yyyy} + at + by + c \tag{19}$$

where $a = -1/x^2$, $b = -1/(2x^2)$ and $c = 1/3 - (1 - e^{-x} - xe^{-x})/x^2$. Equation (19) with boundary conditions (15) and (16) can be solved by taking Laplace transforms in t. The analytical details are tedious and only the solution is given here. This solution is

$$\begin{aligned}\psi^{(0)} =& -\frac{y(y+x)^2}{2880\sigma^2}\left[60\sigma(at + c)(2y - x)\right.\\&+ 4\sigma b(2y^3 - 4xy^2 + 6x^2y - 3x^3)\\&\left.+ a(4y^3 + xy^2 - 6x^2y + 3x^3)\right]\\&+ \sum_{n=0}^{\infty} f(x, y; \beta_n) e^{-\sigma(\beta_n/x)^2 t}\end{aligned} \tag{20}$$

where $f(x, y; \beta_n)$ is given in the Appendix and the β_n's are the non-zero positive roots of the equation $\beta_n = \tan\beta_n$. Even though $\psi^{(0)}$ as given by equation (20) satisfies the initial condition, it is only a valid solution for t greater than the e-folding time of the neglected terms in $T^{(0)}$.

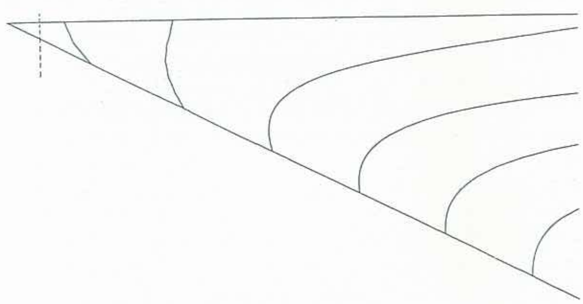

Figure 1. $O(A^0)$ temperature contours at t = 1.0 and x_{max} = 5.0. The contour interval is 4×10^{-1}. The dashed line represents the limit of the conduction dominated tip region as derived in §5.

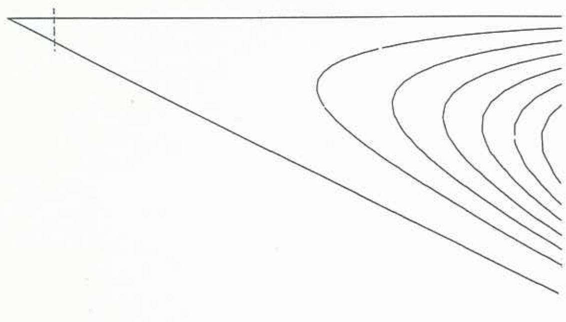

Figure 2. $O(A^0)$ *streamlines at* $t = 1.0$ *and* $x_{max} = 5.0$. *The contour interval is* 2×10^{-2}. *The dashed line represents the limit of the conduction dominated tip region as derived in §5.*

§5 DISCUSSION

Only $O(A^0)$ solution for T and ψ are found here. This has the consequence that the effect of the flow on the temperature in the sidearm cannot be determined. From the scaling in §3 the effect of the flow on the temperature is $O(A^2)$.

Figures 1 and 2 show the large time, $O(A^0)$ temperature distribution and flow field. Note that the heat flux at the sloping bottom of the sidearm decreases exponentially with x and so some distance from $x = 0$ the bottom is effectively insulated. Thus the flow in the sidearm is not only driven by the mean horizontal temperature gradient which is only significant near $x = 0$ but also by the insulated sloping bottom in conjunction with the stable stratification in the deeper parts of the sidearm; the turning over of the horizontal isotherms to satisfy the boundary condition results in a flow up the slope (Phillips (1969)).

It can be seen from Figure 1 that near $x = 0$ the fluid is warmer at the bottom than at the surface, leading to the possibility of instabilities. This is currently being investigated.

It is clear from equation (18) that $T^{(0)}$ is singular at $x = 0$. This is not surprising since heat is not allowed to escape at the boundaries and horizontal conduction is an $O(A^2)$ effect. However, it can be seen from Figure 2 that the velocities become small in the tip region. This suggests that there is a region near the tip where the heat transfer is dominated by conduction and the isotherms are nearly vertical. By using equations (1)-(3) and (5) it is possible to derive a scale for the horizontal extent of this conduction dominated region. Suppose that the conduction region is horizontally confined by $x \sim l$. Balancing conduction with the source term in equation (3) gives a scale for the temperature difference in this region

$$\Delta T \sim \frac{1}{\kappa} Q_0 \eta l^2. \tag{21}$$

Using this scale and balancing buoyancy and pressure in equation (2) gives rise to a scale for the horizontal pressure gradient $p_x/\rho_0 \sim Ag\alpha Q_0 l^2/\kappa$. This scale for the the horizontal pressure gradient with equation (1) yields a scale for he horizontal velocity u,

$$u \sim Gr\kappa^2 A^3 l^4 \eta^5/\nu \tag{22}$$

where Gr is given by equation (7). For conduction to dominate heat transfer, vertical conduction must act more rapidly than horizontal convection to give rise to vertical isotherms, that is

$$\frac{\text{vertical conduction}}{\text{horizontal convection}} \sim \frac{\kappa\Delta T/(Al)^2}{u\Delta T/l} > 1. \tag{23}$$

Substituting the velocity scale (22) into equation (23) and rearranging yields a scale for l, the horizontal extent of the conduction dominated region, namely

$$l\eta \sim A^{-1}\left(\frac{\sigma}{Gr}\right)^{\frac{1}{5}} \tag{24}$$

Using $A \sim 10^{-2}$ and the usual typical values for the other parameters gives $l\eta \sim 0.25$. This region is shown in Figures 1 and 2 and is consistent with the behaviour of the temperature and flow there.

The author acknowledges the helpful guidance provided by his supervisor Dr. John Patterson in the carrying out of this work.

APPENDIX

The coefficients in equation (18) for $T^{(0)}$ are given by

$$a_n(x) = \frac{x^2}{(n\pi)^2} - \frac{1-(-1)^n e^{-x}}{1+(n\pi/x)^2}.$$

The coefficients in equation (20) for $\psi^{(0)}$ are

$$f(x,y;\beta_n) = \frac{-2\sigma}{x^2\sin\beta_n}[(g(-x,\beta_n)+xg_y(-x,\beta_n))\sin(\beta_n y/x) - (\beta_n\cos\beta_n\, g(-x,\beta_n)/x + \sin\beta_n\, g_y(-x,\beta_n))y] \tag{A1}$$

where

$$g(y,\beta_n) = \frac{x^2}{\beta_n^2}\left[\frac{cx^4}{\sigma^2\beta_n^4} - (2\sigma b + a)\frac{x^6}{\sigma^3\beta_n^6}\right](\cos(\beta_n y/x) - 1) \left(\frac{by^2}{12} + c/2\right)\frac{y^2x^4}{\sigma^2\beta_n^4} - (\sigma b + a/2)\frac{y^2x^6}{\sigma^3\beta_n^6} \tag{A2}$$

and a, b and c are defined in §4.

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