INVITED PAPER

Feedback Control Under Data Rate Constraints: An Overview

In control systems where feedback data rates are limited, reliability of communications can drastically affect system stability and there is a trade-off between communication rate and optimal performance.

By Girish N. Nair, *Member IEEE*, Fabio Fagnani, Sandro Zampieri, and Robin J. Evans, *Fellow IEEE*

ABSTRACT | The emerging area of control with limited data rates incorporates ideas from both control and information theory. The data rate constraint introduces quantization into the feedback loop and gives the interconnected system a two-fold nature, continuous and symbolic. In this paper, we review the results available in the literature on data-rate-limited control. For linear systems, we show how fundamental trade-offs between the data rate and control goals, such as stability, mean entry times, and asymptotic state norms, emerge naturally. While many classical tools from both control and information theory can still be used in this context, it turns out that the deepest results necessitate a novel, integrated view of both disciplines.

KEYWORDS | Control under communication constraints; feedback data rate; fundamental performance bounds; quantized control

I. INTRODUCTION

Communications and control have traditionally been areas with little common ground. Communications theory is mainly concerned with the reliable transmission of information from one point to another, and is relatively indifferent to the specific purpose of the transmitted information and whether it is eventually fed back to the source. Control theory, in contrast, is concerned mainly with using information in a feedback loop to achieve some performance objective, and usually assumes that limitations in the communication links do not affect performance significantly.

In engineering systems with large communication bandwidth, it makes sense to treat communication and control as independent functions, since the analysis and design of the overall system is simplified. However, recent emerging applications, such as sensor networks, microelectromechanical systems, mobile telephony, and industrial control networks, have begun to challenge the validity of this modular approach. In these applications, the aim is to control one or more dynamical systems, using multiple sensors and actuators transmitting and receiving information over a digital communication network.

Although the total communication capacity in bits per second may be large, each component is effectively allocated only a small portion. This can introduce large quantization errors that impinge on control performance, due to the low resolution of the transmitted data.

Quantization errors are not a new topic in control theory, and there exists a significant body of work in which quantization is modeled as extra additive white noise, thereby allowing the standard solutions of stochastic control to still be applied; see, e.g., [16]. Though this approach is reasonable if the quantizer resolution is high, it is invalid if the resolution is coarse and the open-loop dynamics are unstable. In particular, it fails to capture the fact, discovered only recently in [5], [81], that there exists a critical positive data rate below which there does not exist *any* quantization and control scheme able to stabilize an unstable plant. This phenomenon strongly implies that low communication capacity has a significant negative

Manuscript received August 12, 2005; revised September 3, 2006. The work of G. N. Nair was supported by the Australian Research Council under Grant DP0345044.
G. N. Nair is with the Department of Electrical and Electronic Engineering, University of Melbourne, VIC 3010, Australia (e-mail: gnair@ee.unimelb.edu.au).
F. Fagnani is with the Department of Mathematics, Politecnico di Torino, 10129 Torino, Italy (e-mail: fabio.fagnani@polito.it).
S. Zampieri is with the Department of Information Engineering, Universita di Padova, I-35131 Padova, Italy (e-mail: zampi@dei.unipd.it).
R. J. Evans is with the NICTA Victoria Research Laboratory, Department of Electrical and Electronic Engineering, University of Melbourne, VIC 3010, Australia (e-mail: rob.evans@nicta.com.au).

Digital Object Identifier: 10.1109/JPROC.2006.887294

effect on the attainable control performance. Clearly, a much more rigorous analysis is required, in which the communication and control aspects are considered jointly rather than in isolation.

Though large networked systems with multiple sensors and actuators are the driving motivation for the results described here, the first step towards understanding them is to analyze the simplest possible network topology, consisting of one controller and one dynamical system connected by a feedback loop with a given data rate in bits per unit time. Real digital communications channels of course offer a variety of other challenges, such as bit errors, random delays, erasures, etc., but in this paper we focus exclusively on explaining the limitations imposed by the constrained data rate.

Within this perimeter, the most fundamental question which can be asked is: what is the smallest feedback data rate above which a given dynamical system can somehow be stabilized? This is analogous to Shannon's *source coding* theory, which seeks to determine the smallest data rate above which a given random process can be reliably communicated, i.e., with arbitrarily small probability of error [14], [73]. The crucial difference however, is that in control systems the data are not just transmitted from one point to another, but are used in a feedback loop.

Moving beyond stability, the next question is: given a dynamical system, how can one characterize the fundamental tradeoff that must exist between the communication rate and the optimal attainable control performance? This is the control-theoretic version of Shannon's *rate-distortion theory* for digital communications [7], [74]. The main aim of this paper is to formulate these questions for linear dynamical systems, and to explain some of the answers offered in the literature. However, before doing so, we first provide a brief overview of some of the major contributions to data-rate-limited control in the literature. We then sketch the contents of this paper in more detail.

A. Overview of the Literature

As discussed above, there are striking analogies between the goals of data-rate-limited control, and those of source coding and rate-distortion in information theory [7], [73]. Despite this, information theory has been of limited value in real-time networked control systems, since the bounds it yields rely on coders with arbitrarily long *block lengths* and delays. This can have a particularly severe effect if the system has unstable dynamics.

Somewhat more progress on this topic has been made in the control literature, especially in recent years. As mentioned above, quantizers in control systems were traditionally modeled as sources of extra additive white noise. The shortcomings of this approach were made very clear in the seminal paper [20], in which it was shown that a noiseless and unstable linear plant with eigenvalues less than 2 in magnitude could still be asymptotically controlled to the origin using memoryless quantization of the state, but if an eigenvalue magnitude was larger than 2 then chaotic trajectories resulted. This result was completely beyond the reach of the white noise model, and underlined the importance of a pursuing a rigorous analysis of coding and quantization in feedback systems.

Subsequently, the first results on minimum data rates for stabilizability appeared in [4], [81], where it was shown that a noiseless scalar plant with parameter $|a| > 1$ can be kept bounded by memoryless quantized control if and only if the available data rate exceeds $\log_2 |a|$ bits per sample. These results were the first instances of the *Data Rate Theorem*. Similar tight bounds were subsequently obtained for the asymptotic stabilizability of noiseless autoregressive moving average systems [58], and linear state-space systems [34], [59], [77], using different formulations and techniques.

The improvement from boundedness to asymptotic stability above becomes possible by permitting the quantizer or encoder to possess memory, and follow an adaptive *zooming-in/zooming out* strategy [12], [44], [65]. This is based on dynamically adjusting the range of the quantizer so that it increases as the plant state approaches the target (zooming-in phase), and decreases if the state diverges from the target (zooming-out phase). The underlying intuition is that, in order to drive the state to the target, the quantizer resolution should be high close to the target but coarse far from it. We remark that these techniques can be easily adapted to synthesize controllers yielding guaranteed rates of state convergence. Indeed, with encoder memory the infinite-horizon quadratic regulation cost for a noiseless linear plant may be brought as close as pleased to the to the classical optimal linear quadratic regulation (LQR) cost, provided that the average data rate exceeds the *intrinsic entropy rate H* (defined in Section II) of the plant [72]. The same problem but with a limited *instantaneous* data rate is explicitly solved for scalar plants in [63]. In this case, the optimal cost is, as expected, strictly greater than the classical cost, but approaches it asymptotically with increasing rate.

The idea of increasing quantizer resolution close to the origin can also be applied to memoryless quantizers. It has been shown that, if the number of quantization levels is not constrained *a priori*, then the most efficient quantizer for obtaining stability with respect to a quadratic Lyapunov function is *logarithmic* [22]. The design of logarithmically and uniformly quantized controllers that achieve specifed levels of *quadratic attractivity* is considered in [35]–[37]. In the recent article [30] a sector bound approach is used to study logarithmically quantized systems, in terms of quadratic stability as well as H_2 and H_∞ performance criteria.

The issue of robustness has also been considered, both with respect to variations in the plant [66], and in the effective data rate of the channel [42]. In the latter article, it is proved that if a noiseless linear plant in continuous time is quantized and controlled without memory, then the strategy which is most robust to changes in the data packet transfer time is binary quantization with short sampling intervals. We also remark that stability in the presence of additive disturbances with known bound is considered in [34], [46], and [77]. In particular, the last-mentioned article presents a zooming quantized control policy which achieves *input-to-state stability* for linear systems, without knowledge of a disturbance bound.

Various extensions of the techniques above have been proposed for nonlinear systems [6], [9], [18], [19], [45], [48], [62]. The article [45] applies the zooming strategy to *input-to-state-stable* nonlinear systems to guarantee asymptotic stability. The same techniques are used in [18], but with the ISS assumption on the plant relaxed to asymptotic stabilizability. In [62], the data rate necessary to stabilize a nonlinear system is connected to the concept of the *topological feedback entropy* of a nonlinear plant. This notion generalizes the well-known topological entropy of a nonlinear system without inputs [1], [41], [68]. The article [48] extends the logarithmic quantization strategy to affine nonlinear systems, and in [19] bit-rate bounds for stabilizing nonlinear systems with a feedforward structure are derived. See also the papers [33], [83] for related results on the adaptive stabilization of uncertain plants.

All the results above concern plants that are deterministic apart from a possibly random initial condition. With regard to stochastic plants, the major contributions have been in [10], [50], [52], [57], [60], [79], [80]. In [10], data-rate-limited control of partially observed linear Gaussian systems is considered under a quadratic cost. It is shown there that if the measurements are passed through a minimum variance filter, and the input to the quantizer is chosen to be the innovations of the filter process, then the design of the coding and control laws can be performed separately. Separation and certainty equivalence for linear Gaussian plants are addressed in a more general setting in [79], which also presents rate-distortion-theoretic lower bounds on performance over additive white Gaussian noise channels and high rate noiseless digital channels. The article [80] gives necessary and sufficient conditions for stabilizing single-input, single-output, linear time-invariant (LTI) Gaussian plants, using uniform quantizers and *variable length coding*, under the restriction that the controller must also be LTI. In the paper [60], the mean square stabilizability of linear plants with possibly non-Gaussian noise is considered. By exploiting the properties of *differential entropy power*, a universal lower bound is obtained on the time-asymptotic mean square state norm. In particular, this bound implies that as the data rate approaches the intrinsic entropy rate H of the plant, the mean square state becomes arbitrarily large, regardless of the coding and control scheme. The recent article [50] also has the same flavour of result, showing that as the *Shannon capacity* C of the feedback channel decreases towards H, the ability of the controller to shape the plant input power spectrum diminishes.

The possibility of obtaining tight bounds on the data rate necessary to stabilize a system is based on the use of dynamic encoders and controllers with unlimited memory. The analysis is much more intricate if we restrict to memoryless or finite memory schemes. Under memoryless, finite-level quantization, the set of reachable points is discrete or at most dense [3], [9], and only practical stability can be achieved, namely states in some initial set can be driven to a smaller target set, and not asymptotically to the origin [5], [20], [24], [67], [81]. The main difficulty in this case is due to the fact that performance should be evaluated through a pair of indices, one depending on the steady-state properties of the closed-loop system, the second on the quality of the transient [25], [26]. This prevents the existence of a unique optimal controller, since it would generally depend on the weights associated with the two indices. Even though memoryless quantizers and controllers have been studied for longer, since the seminal paper of Delchamps [20], results on the achievable performance in this case are still quite arduous to obtain and difficult to interpret [25], [26]. The simplest contribution in this field shows that, according to the relative weights assigned to the steady state and to the transient, there are three different optimal strategies: the first based on the uniform quantizer, widely used in applications, the second on the logarithmic quantizer [22], similar to the *μ- and A-law companders* of communications [69], and the third on the chaotic quantizer, recently studied in [23] and [24]. In the framework of memoryless quantization, a recent paper by Delvenne [21] is very promising, because it seems to provide a new perspective which makes the problem much more treatable.

As the data rate approaches the intrinsic entropy rate *H* of the plant, the mean square state becomes arbitrarily large, regardless of the coding and control scheme.

Though the focus of this paper is on control over noiseless digital channels, we remark here that a number of results on noisy channels have recently been proposed. Various models of the channel have been treated, e.g., the digital erasure channel [47], [54], [78], the binary symmetric channel [53], [71], [75], and the *truncation* channel [51]. If the communication channel between the sensor and controller is erroneous, the stabilization problem becomes very complicated, and the results can depend critically on the particular notion of stability, and whether or not the transmitter has *side-information* on the channel errors that have occured [47], [51], [54], [75], [78]. The main point is that we cannot generally adopt the Shannon concept of capacity C as the relevant figure-of-merit for noisy channels in feedback control systems. Although the inequality $C > H$ is both necessary and sufficient for almost sure asymptotic stabilizability without side-information [53], this bound does not hold true for other stability objectives, e.g., mean square stability. The reason is that in Shannon coding theory, reliable transmission at a rate close to C is possible only at the price of significantly increasing the coding delay. However, as this delay is increased, the plant state will be driven further from the origin, and so more information again will be needed to stabilize it. In [71], the novel concept of *anytime capacity* is introduced, a notion which takes into consideration the recursive structure of incoming data. It is argued that if the control objective is moment stability, then anytime, not Shannon, capacity is the correct figure of merit for the noisy channel. Unlike Shannon capacity however, anytime capacity does not in general have a simple expression amenable to computation; this, at the moment, is its main drawback. Notwithstanding the preceding discussion, it turns out that when the channel is analog with additive Gaussian noise, and the plant is linear Gaussian, the inequality $C > H$ is necessary and sufficient for mean square stabilizability, with no quantization or coding required [29], [79]. See also [17], [70] for related results on noisy bandlimited analog channels.

All the articles described above focus on systems with one sensor and one actuator. However, as remarked at the start of this section, the main application for this research is in networked control systems with multiple sensors and actuators. Steps have been recently taken towards deriving fundamental necessary and sufficient rate regions for the stabilizability of such systems, for the case of multiple sensors with a single actuator [55], [76], and with multiple actuators [56], [61], assuming no noise. Channel noise and coding strategies for networked systems are considered in [43], and in [32] a model predictive approach is proposed for designing a centralized control strategy for a noisy linear system with multiple, separately quantized inputs and outputs. Finally, we remark that there are two other strands of research on systems with multiple sensors and actuators, in which quantization and data rate limitations are largely ignored. One focuses primarily on scheduling and communication medium access protocol design—see, e.g., [13] and [64]. The other arises from coordination problems in which the main interest is on decentralized strategies and the graph-theoretic aspects of the required information flows. The special issue [2] contains recent articles along these lines, as well as discussions of other communication issues such as variable delay and random dropouts.

B. Overview of Paper

The aim of the present paper is to present some of the main ideas that form the basis of this field of research. The style will be between a tutorial, with an attempt to cover the multiple aspects of the problem, and a technical paper proposing some new results.

Indeed, in the next section we present new universal lower bounds on feedback data rate and performance for linear systems with deterministic bounded disturbances. The elementary nature of our arguments will hopefully permit the essential aspects of the problem of data-rate-limited control to emerge more clearly. It also lets us easily relate both the data rate *and* channel delay to the degradation in attainable performance, an original contribution.

On the other hand, in order to maintain the tutorial character of the paper, in the construction of the encoder and controller in Section II-B, we prefer to focus on scalar systems. The same has been done in Sections IV and V.

In Section III, we discuss what is known about the structure of coding and control schemes for stochastic linear plants. Building on the arguments in [79], we demonstrate that certainty equivalence and a separation-like property apply under a quadratic cost. In other words, the joint coding and control optimization problem can be solved by encoding the plant states so as to minimize a certain distortion metric, and then using the decoded estimate in a certainty equivalent control law. However, complete separation between coding and control is not attained, since the coder distortion metric depends on the control input matrix as well as the plant dynamical matrix.

In Section III, we also discuss the fact that if a linear plant has at least one strictly unstable mode and either the initial condition or process noise distribution has infinite support, then *no* time-invariant coding and control law with a finite-valued internal state can stabilize the plant. This rules out many common coding schemes, such as memoryless quantization and *differential pulse code modulation*, from infinite-horizon stochastic control problems. Instead, coder-controllers with *continuous*-valued internal states must be considered, similar to the zooming quantizers of [12].

In Section IV we restrict to control strategies in which the encoder and the controller can have finite memory, namely their state spaces have finitely many elements. It is not surprising that this restriction complicates the analysis. It is surprising instead that the particular case of memoryless controllers, which is important in applications, basically maintains these difficulties. As mentioned before, in this case only practical stability is obtainable when starting from unstable plants, and so the performance has to be described by two conflicting parameters, one describing the steady-state behavior, the other describing the transient behavior of the closed-loop system. In this analysis we follow the approach recently proposed in [21], with some original extensions. We present both a general bound highlighting the relations between the performance parameters and the complexity of the control scheme, and two control techniques that allow this bound to be attained in some particular cases.

The key ideas—namely, the linear amplification, addition, and partitioning of uncertainty volumes—remain the same, and in fact emerge more clearly.

The analysis becomes even more arduous if we restrict to quatization laws with connected quantization regions. The aim of Section V is to present some fundamental bounds relating the controller complexity and the performance in this case, as well as a brief description of the fundamental synthesis techniques which can be proposed in this case, namely the zooming-in/zooming-out strategy, the uniform quantizer strategy, the logarithmic quantizer strategy and the chaos based strategy. In addition, we briefly present some of the robustness results proposed in [42] that characterize the relation between quantization and sampling period in the control of continuous time systems.

We conclude this paper with a section devoted to conclusions, containing a list of problems which we think should be considered in future research.

Notation: The symbol $\mathbb{R}^n$ will denote the set of the n-dimensional column vectors with real entries while $\mathbb{R}^{n\times m}$ means the set of the $n\times m$ matrices with real entries. If A is a square matrix, $\det A$ denotes its determinant. The 2-norm of a vector $x\in\mathbb{R}^n$ will be denoted by $\|x\|$. Discrete time is denoted by $t\in[0,1,2,\ldots)$, and the symbol a_t is used to indicate an element of a sequence. Where there is no risk of confusion, the entire sequence is also denoted by a_t; otherwise the more explicit notation $\{a_t\}_{t\geq 0}$ is used. If $\mathcal{S}$ is a finite set, $\|\mathcal{S}\|$ represents its cardinality, namely the number of its elements. If X is a measurable subset of $\mathbb{R}^n$, the symbol $\lambda(X)$ means the Lebesgue measure of X.

II. CODING AND CONTROL SCHEMES WITH UNRESTRICTED MEMORY

In this section, we present a formulation of the problem of communication-limited control for linear time-invariant systems with additive disturbances. For the sake of generality, we permit the coding and control policy to have possibly unrestricted memory,[1] and as our emphasis here is primarily on understanding the effects of finite data rate and delay, we assume that the digital channel used for feedback is errorless, with a constant propagation delay. The main results of this section, Theorem 1 and Proposition 1, present fundamental lower bounds on the allowed data rate and time-asymptotic state norm for all stabilizing policies.

The framework presented in this section is essentially that of [60], in which the disturbances were modeled as possibly non-Gaussian random vectors. For the sake of simplicity, we adopt a deterministic viewpoint here, and regard the disturbances as bounded unknowns. The key ideas—namely the linear amplification, addition, and partitioning of uncertainty volumes—remain the same, and in fact emerge more clearly.

Nonetheless, there is an important aspect of the stochastic problem, namely the instability of all *finite-state* coding and control schemes, which has no parallel in the deterministic framework. This is discussed briefly in Section III-B. In addition, in the deterministic setup, the universal lower bound on the state magnitude is tight for scalar plants. That is, for one-dimensional systems we obtain an exact characterization of the fundamental tradeoffs between data rate, delay, and "cheap" control performance.

Consider the partially observed, discrete-time linear plant

$$\begin{cases} x_{t+1} &= Ax_t + Bu_t + v_t, \\ y_t &= Cx_t + w_t, \end{cases} \qquad \forall t\geq 0 \tag{1}$$

where $x_t\in\mathbb{R}^n$ is the state at time $t\geq 0$, $u_t\in\mathbb{R}^m$ is the control input, $y_t\in\mathbb{R}^p$ is the measured output, $v_t\in\mathbb{R}^n$ is unknown process noise, $w_t\in\mathbb{R}^p$ is unknown measurement noise, and A, B, and C are constant known matrices of appropriate dimensions. For the problem to be well-posed, it is also assumed that the pair (A,B) is reachable, and (C,A), observable.

Without loss of generality, it may be supposed that a similarity transformation has been applied to the state coordinates so that the stable and unstable modes of the plant are decoupled, i.e.,

$$A = \begin{bmatrix} A^{\mathrm{u}} & 0 \\ 0 & A^{\mathrm{s}} \end{bmatrix} \tag{2}$$

where all the eigenvalues of $A^{\mathrm{u}}\in\mathbb{R}^{f\times f}$ have magnitudes ≥ 1, all those of $A^{\mathrm{s}}\in\mathbb{R}^{n\times f}$ have magnitudes <1 and where the 0's denote matrices with zero entries of appropriate dimensions. It turns out that in all our results the stable part does not play any key role. For the sake of

[1]We note in advance that there exist stabilizing policies with finite-dimensional memory.

keeping the notation as simple as possible, we therefore assume from now on that $A = A^u$, namely that all eigenvalues have magnitudes ≥ 1. We also define the *intrinsic entropy rate* of the plant to be

$$H := \log_2 |\det A|. \quad (3)$$

As mentioned above, in order to simplify the analysis we assume in this section that x_0, v_t, w_t are deterministic unknowns, belonging to bounded and Lebesgue-measurable sets $\mathcal{X}_0 \subset \mathbb{R}^n$, $\mathcal{V} \subset \mathbb{R}^n$, $\mathcal{W} \subset \mathbb{R}^p$, respectively.

Suppose that the sensors that measure the plant output are located some distance away from the controller, and communicate with it over a digital channel, onto which one symbol s_t from a finite alphabet $\mathcal{S}$ of cardinality $M \geq 1$, is transmitted during the $(t+1)$th sampling interval. In many practical applications, the channel is inherently noisy because of interfering transmissions, background noise, contention with other users, and so on. However, by assuming that appropriate error correction coding or repeat request protocols are in place at lower levels of the communication protocol, we can view the channel as being noiseless, with possibly a constant propagation delay of d sampling intervals.[2] We define the *data rate* of the channel as

$$R := \log_2 M(\text{bits/sample}). \quad (4)$$

As the symbols in the channel are discrete-valued but the plant measurements are continuous-valued, analog-to-digital conversion, or coding, is required. In practice constraints such as complexity and finite memory may be important but, in the spirit of source coding [73], such limitations will be ignored in this section to concentrate on the communication aspect of the problem. Each transmitted symbol may thus depend on all past and present measurements and past symbols

$$s_t = \gamma_t(y_t, \ldots, y_0; s_{t-1}, \ldots, s_0) \in \mathcal{S}, \quad \forall t \geq 0 \quad (5)$$

where $\gamma_t : \mathbb{R}^{p\times(t+1)} \times \mathcal{S}^t \to \mathcal{S}$ is the coder mapping at time t. Each transmitted symbol experiences a propagation delay of d sampling intervals, so at time t the controller has the symbols $s_0, \ldots, s_{t-d}$ available. It can then apply a control law of the general form

$$u_t = \delta_t(s_{t-d}, \ldots, s_0) \in \mathbb{R}^m, \quad \forall t \geq 0 \quad (6)$$

[2]If a retransmission protocol is in use at a lower level, then the sampling interval cannot be too short.

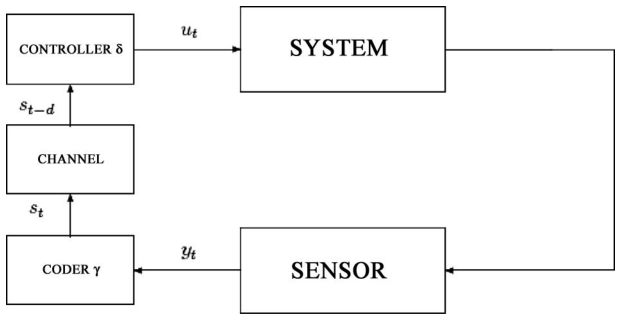

Fig. 1. ***Scheme representing the control under communication constraints.***

where $\delta_t : \mathcal{S}^{t-d+1} \to \mathbb{R}^m$ is the controller mapping at time t.[3] Fig. 1 illustrates the control scheme which results from these considerations.

Let the *coder-controller* be defined as the pair of coder and controller mapping sequences $(\gamma, \delta) := (\{\gamma_t\}_{t\geq 0}, \{\delta_t\}_{t\geq 0})$, and let $\mathcal{C}$ be the set of all such pairs. We quantify the performance of a coder-controller by the asymptotic worst-case state norm

$$J := \overline{\lim_{t\to\infty}} \sup \{\|x_t\| : x_0 \in \mathcal{X}_0, . \\ v_j \in \mathcal{V}, w_j \in \mathcal{W}, j = 0, 1, \ldots\}. \quad (7)$$

In other words, we are interested in how small the state can be made in the long-term worst case scenario, if no cost is placed on the controls and the data rate is fixed. In the sequel, we employ elementary arguments to derive a universal lower bound on the cost of any coder-controller, in terms of the open-loop dynamics, the data rate R, and the channel delay d.

A. Universal Lower Bounds

We now proceed to obtain universal lower bounds that apply to any causal coder-controller, by studying the evolution of state uncertainty volumes. The first result, Theorem 1, presents universal lower bounds on data rate and worst case asymptotic state norm applicable to all stabilizing coder-controllers. The second result, in Proposition 1, is a tighter bound which also captures the effect of channel delay. For the case of scalar systems, it is in fact optimal.

The basic intuition we use to establish these results is that the open-loop growth in subspace uncertainty volume

[3]We adopt the convention that the first d control signals $u_0, \ldots, u_{d-1}$, that cannot be obtained from (6), are known preset inputs. Similarly, in the coder (5) at time $t = 0$, s_0 is taken to be a function only of y_0, namely $s_0 = \gamma_0(y_0)$.

must be counteracted by a reduction in volume due to the coding partitions. As it turns out, volumes are much easier to analyze directly than vector norms because of several convenient mathematical properties not possessed by vector norms. The basic techniques in this section were originally used in a stochastic setting, with *differential entropy power* being the stochastic version of $2/n$th power uncertainty volume, and the *entropy power inequality* playing the role of the *Brunn–Minkowski inequality* below, leading to a mean-square-sense analog of Theorem 1 [60]. However, we adopt a purely deterministic framework in this paper so that the key ideas in the argument may come through more clearly.

Conventions: Throughout the remainder of this section, all system variables are to be understood to be implicit functions on the domain $\{x_0 \in \mathcal{X}_0, v_0, v_1 \ldots \in \mathcal{V}, w_0, w_1 \ldots \in \mathcal{W}\}$. For the sake of notational conciseness this will not be indicated explicitly; only additional restrictions are indicated. In particular, we will simply use the notation $\sup\|x_t\|$ to denote

$$\sup\{\|x_t\| : x_0 \in \mathcal{X}_0, v_0, v_1 \ldots \in \mathcal{V}, w_0, w_1 \ldots \in \mathcal{W}\}.$$

Theorem 1: Let any causal coder-controller (5), (6), with data rate R (4), be applied to the noisy linear system (1), with intrinsic entropy rate H (3). Then the following bounds hold.

1) If $R \leq H$, and the process noise set $\mathcal{V}$ has Lebesgue measure $\lambda(\mathcal{V}) > 0$, then

$$\varliminf_{t\to\infty} \sup\|x_t\| = \infty$$

2) else if $R > H$, then

$$\varliminf_{t\to\infty} \sup\|x_t\| \geq \frac{\beta^{-1/n}\lambda(\mathcal{V})^{1/n}}{1-2^{-(R-H)/n}} \qquad (8)$$

where β is the volume of the n-dimensional sphere with unit radius.

Proof: To begin the volume-based analysis, observe that for any bounded measurable $\mathcal{X} \subset \mathbb{R}^n$, it clearly holds that

$$\lambda(\mathcal{X}) \leq \beta(\sup\{\|x\| : x \in \mathcal{X}\})^n. \qquad (9)$$

Hence, if we put $l_t := \sup\|x_t\|$ for $t = 0, 1, \ldots$, it holds $\beta l_t^n \geq \lambda(\{x_t\})$. Further noting that $\{x_t\} \supseteq \{x_t : \{s_j\}_{j=0}^{t-1} = \{c_j\}_{j=0}^{t-1}\}$ for any symbol values $c_0, \ldots, c_{t-1} \in \mathcal{S}$, we have

$$\beta^{1/n} l_t \geq \max_{c_0,\ldots,c_{t-1}\in\mathcal{S}} \lambda\left(\left\{x_t : \{s_j\}_{j=0}^{t-1} = \{c_j\}_{j=0}^{t-1}\right\}\right)^{1/n} =: m_t. \quad (10)$$

In what follows, we will obtain a recursive lower bound on the nth root maximum state uncertainty volume m_t.

Observe that

$$\begin{aligned} m_{t+1} &\equiv \max_{\{c_j\}_{j=0}^t} \lambda\left(\left\{x_{t+1} : \{s_j\}_{j=0}^t = \{c_j\}_{j=0}^t\right\}\right)^{1/n} \\ &= \max_{\{c_j\}_{j=0}^t} \lambda\left(\left\{Ax_t + B\delta_t\left(\{c_j\}_{j=0}^{t-d}\right)\right.\right. \\ &\qquad\qquad \left.\left. + v_t : \{s_j\}_{j=0}^t = \{c_j\}_{j=0}^t\right\}\right)^{1/n} \\ &= \max_{\{c_j\}_{j=0}^t} \lambda\left(\left\{Ax_t + v_t : \{s_j\}_{j=0}^t \right.\right. \\ &\qquad\qquad \left.\left. = \{c_j\}_{j=0}^t\right\}\right)^{1/n} \end{aligned} \qquad (11)$$

using the fact that the volume of a set remains invariant under a constant translation. Now, for any two sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n$, define the set sum $\mathcal{X} + \mathcal{Y} := \{x + y : x \in \mathcal{X}, y \in \mathcal{Y}\}$. If $\mathcal{X}, \mathcal{Y}$ are Lebesgue-measurable, the *Brunn–Minkowski inequality* (see, e.g., [14, p. 501]) states that

$$\lambda(\mathcal{X} + \mathcal{Y})^{1/n} \geq \lambda(\mathcal{X})^{1/n} + \lambda(\mathcal{Y})^{1/n} \qquad (12)$$

i.e., nth-root volume is super-additive. As v_t does not depend on the symbols $c_0, \ldots, c_t$, we can rewrite the set on the right-hand side (RHS) of (11) as

$$\begin{aligned} &\left\{Ax_t + v_t : \{s_j\}_{j=0}^t = \{c_j\}_{j=0}^t\right\} \\ &\qquad = \left\{Ax_t : \{s_j\}_{j=0}^t = \{c_j\}_{j=0}^t\right\} + \{v_t\}. \end{aligned}$$

Applying the Brunn–Minkowski inequality, we then have

$$\begin{aligned} m_{t+1} &\geq \max_{\{c_j\}_{j=0}^t} \lambda\left(\left\{Ax_t : \{s_j\}_{j=0}^t = \{c_j\}_{j=0}^t\right\}\right)^{1/n} \\ &\quad + \lambda(\{v_t\})^{1/n}, \\ &= |\det A|^{1/n} \max_{\{c_j\}_{j=0}^{t-1}} \\ &\quad \max_{c_t} \lambda\left(\left\{x_t : \{s_j\}_{j=0}^t = \{c_j\}_{j=0}^t\right\}\right)^{1/n} \\ &\quad + \lambda(\{v_t\})^{1/n} \end{aligned} \qquad (13)$$

using the standard formula for volume change under linear transformations. As s_t is a function of previous symbols, and past and present states

$$\left\{x_t : \{s_j\}_{j=0}^{t-1} = \{c_j\}_{j=0}^{t-1}\right\} = \bigcup_{c_t \in \mathcal{S}} \left\{x_t : \{s_j\}_{j=0}^{t} = \{c_j\}_{j=0}^{t}\right\}.$$

Hence

$$\begin{aligned}\lambda&\left(\left\{x_t : \{s_j\}_{j=0}^{t-1} = \{c_j\}_{j=0}^{t-1}\right\}\right)\\ &\leq \sum_{c_t \in \mathcal{S}} \lambda\left(\left\{x_t : \{s_j\}_{j=0}^{t} = \{c_j\}_{j=0}^{t}\right\}\right)\\ &\leq M \max_{c_t \in \mathcal{S}_t} \lambda\left(\left\{x_t : \{s_j\}_{j=0}^{t} = \{c_j\}_{j=0}^{t}\right\}\right).\end{aligned}$$

Substituting this into (13), we obtain

$$\begin{aligned}m_{t+1} &\geq |\det A|^{1/n} \max_{\{c_j\}_{j=0}^{t-1}}\\ &\quad\times M^{-1/n} \lambda\left(\left\{x_t : \{s_j\}_{j=0}^{t-1} = \{c_j\}_{j=0}^{t-1}\right\}\right)^{1/n}\\ &\quad+ \lambda(\{v_t\})^{1/n}\\ &= \left|\frac{\det A}{M}\right|^{1/n} m_t + \lambda(\mathcal{V})^{1/n}\\ &= 2^{-(R-H)/n} m_t + \lambda(\mathcal{V})^{1/n}. \end{aligned} \tag{14}$$

If $R \leq H$, then it follows that $m_t \to \infty$. By (10), this proves item (1) in the theorem.

Assume now that $R > H$. We can easily solve the forced linear recursive inequality (14) to obtain

$$m_t \geq \frac{\lambda(\mathcal{V})^{1/n}}{1 - 2^{-(R-H)/n}} + \left(m_0 - \frac{\lambda(\mathcal{V})^{1/n}}{1 - 2^{-(R-H)/n}}\right) 2^{-(R-H)t/n}. \tag{15}$$

Letting $t \to \infty$ in (15), we immediately obtain (8). ∎

Remarks: Notice that the bounds above are also lower bounds of the cost $J \equiv \overline{\lim}_{t\to\infty} \sup \|x_t\|$ defined in (7). The first part of the theorem states that any coder-controller which yields uniformly bounded worst case states must operate at a data rate R which strictly exceeds the entropy rate H of the plant. In other words, information must be transported as fast as the plant generates it, or else instability occurs. As it turns out, it is possible to attain closed-loop stability at any data rate $R > H$, i.e., the bound $R > H$ is in fact tight. This is sometimes known as the *data-rate theorem*, and has been shown to apply, under different notions of stability, to linear plants that are deterministic [5], [34], [59] and stochastic [60]. In Section II-B, we describe how to construct a stabilizing coder-controller at any rate $R > H$, for the special case of a scalar plant.

The second part of the result above indicates that for any coder-controller, the cost increases as the noise uncertainty volume $\lambda(\mathcal{V})$ increases. Furthermore, as the data rate is reduced to the critical value H, the cost must always become unbounded, implying that a data rate which is too low affects performance significantly, regardless of the coding and control scheme in use. This was also established in a mean-square setting with unbounded non-Gaussian noise in [60].

A major deficiency of (8) is that it is independent of the channel propagation delay d. As performance should naturally deteriorate as d increases, this bound clearly cannot be tight. However, a better one can be obtained in a few more steps, using the same volume analysis ideas.

Proposition 1 (Universal Bound in Terms of Rate and Delay): Let any coder-controller (5), (6) with data rate R (4), and channel propagation delay d, be applied to the noisy linear system (1) with intrinsic entropy rate H (3). If $R > H > 0$, then

$$\begin{aligned}&\underline{\lim}_{t\to\infty} \sup \|x_t\|\\ &\quad\geq \beta^{-1/n}\left[\frac{2^{Hd/n}}{1 - 2^{-(R-H)/n}} + \frac{2^{Hd/n} - 1}{2^{H/n} - 1}\right]\lambda(\mathcal{V})^{1/n}\end{aligned} \tag{16}$$

where β is the volume of the n-dimensional sphere with unit radius, and $\lambda(\mathcal{V})$ is the Lebesgue measure of the process noise set.

Proof: Looking d sample intervals ahead, (10) can be replaced by

$$\begin{aligned}\beta^{1/n} l_{t+d} &\geq \lambda(\{x_{t+d}\})\\ &= \lambda\left(\{A^d x_t + r_t + z_t\}\right), \forall t \geq 0\end{aligned} \tag{17}$$

$$\text{where } r_t := \sum_{j=t}^{t+d-1} A^{t+d-1-j} B \delta_j\left(\{s_i\}_{i=0}^{j-d}\right),$$

$$z_t := \sum_{j=t}^{t+d-1} A^{t+d-1-j} v_j. \tag{18}$$

Observe that with the coder-controller fixed, r_t and x_t are functions of the past noise terms and initial condition, whereas z_t is determined by only the present and future process noise $v_t, \ldots, v_{t+d-1}$. In terms of set addition, we thus have

$$\begin{aligned}\{A^d x_t + r_t + z_t\} &= \{A^d x_t + r_t\} + \{z_t\}\\ &= \left\{A^d x_t + r_t\right\} + \sum_{j=t}^{t+d-1} \{A^{t+d-1-j} v_j\}.\end{aligned} \tag{19}$$

The effects of *d* and *R* do not separate out in a simple way, and with increasing delay the performance deterioration due to a low data rate becomes more severe.

From (17) and (19), applying the Brunn–Minkowski inequality (12), we obtain

$$\begin{aligned}\beta^{1/n} l_{t+d} \geq& \lambda\left(\{A^d x_t + r_t\}\right)^{1/n} \\ &+ \sum_{j=t}^{t+d-1} \lambda\left(\{A^{t+d-1-j} v_j\}\right)^{1/n}, \\ \geq& \lambda\left(\{A^d x_t + r_t\}\right)^{1/n} \\ &+ \sum_{j=t}^{t+d-1} |\det A|^{(t+d-1-j)/n} \lambda(\mathcal{V})^{1/n}, \\ &\forall t \geq 0. \end{aligned} \tag{20}$$

As additional restrictions on a set cannot increase its size

$$\begin{aligned}\lambda\left(\{A^d x_t + r_t\}\right) &\geq \lambda\left(\left\{A^d x_t + r_t : \{s_j\}_{j=0}^t = \{c_j\}_{j=0}^t\right\}\right), \\ &= \lambda\left(\left\{A^d x_t : \{s_j\}_{j=0}^t = \{c_j\}_{j=0}^t\right\}\right), \\ &\quad \forall c_0, \ldots, c_t \end{aligned}$$

by the translation-invariance of Lebesgue measure, since r_t is constant given the given symbols $\{s_j\}_{j=0}^{t-1}$. Hence

$$\begin{aligned}\lambda\{A^d x_t + r_t\} &\geq \max_{c_0,\ldots,c_t} \lambda\left(\left\{A^d x_t : \{s_j\}_{j=0}^t = \{c_j\}_{j=0}^t\right\}\right), \\ &= |\det A|^d \max_{c_0,\ldots,c_t} \lambda\left(\left\{x_t : \{s_j\}_{j=0}^t = \{c_j\}_{j=0}^t\right\}\right) \\ &\equiv |\det A|^d m_t^n. \end{aligned}$$

Substituting this into (20), and simplifying, we obtain

$$\beta^{1/n} l_{t+d} \geq |\det A|^{d/n} m_t + \frac{|\det A|^{d/n} - 1}{|\det A|^{1/n} - 1} \lambda(\mathcal{V})^{1/n}, \quad \forall t \geq 0.$$

By taking inferior limits, and substituting the second inequality of (8) into the RHS, we obtain (16). ∎

Remarks: The bound (16) provides a method for comparing the relative impacts of delay, data rate, open-loop instability, and process noise on steady-state control performance. The formula states that for a fixed data rate $R > H$ the steady-state norm must always grow at least like $2^{Hd/n}$ with increasing d, and for fixed delay, like $1/(1 - 2^{-(R-H)/n})$ with decreasing R. However, the effects of d and R do not separate out in a simple additive or multiplicative way, and with increasing delay the performance deterioration due to a low data rate becomes more severe. A universal bound growing exponentially in delay d was also derived in [11], for analog channels.

If $H = 0$ then the arguments used in the preceding proof do not hold. However, we can easily obtain in this case

$$\varlimsup_{t\to\infty} \sup \|x_t\| \geq \beta^{-1/n} \left[\frac{1}{1 - 2^{-R/n}} + d\right] \lambda(\mathcal{V})^{1/n} \tag{21}$$

which is just the limiting value of the lower bound (16) as $H \to 0$. In this case, the individual effects of data rate and delay on performance do separate out additively.

We remark briefly that the volume-based techniques used to derive (16) also yield universal lower bounds on worst-case sum-like costs, and similar ideas can also be used in mean-square formulations with stochastic noise. In some applications, it may be desirable to allow the coding alphabet size to vary over time in a predetermined way. It can be shown that (16) still holds in this case, provided that R is taken to be the long-term average data rate.

Observe that in the limit $R \to \infty$, corresponding to the classical situation without communication constraints, (16) becomes

$$\varlimsup_{t\to\infty} \sup \|x_t\| \geq \beta^{-1/n} \frac{2^{H(d+1)/n} - 1}{2^{H/n} - 1} \lambda(\mathcal{V})^{1/n}. \tag{22}$$

As no assumptions but causality were made on the control law, this lower bound holds for *any* possibly nonlinear and time-varying controller with unconstrained data rate.

Finally, note that in the special case of a scalar system with dynamical parameter $|a| > 1$, we have $n = 1$ and $\beta = 2$, and so

$$\begin{aligned}\varlimsup_{t\to\infty} \sup |x_t| &\geq \left[\frac{|a|^d}{1 - |a|/M} + \frac{|a|^d - 1}{|a| - 1}\right] \frac{\lambda(\mathcal{V})}{2} \\ &\equiv \left[\frac{2^{Hd}}{1 - 2^{-(R-H)}} + \frac{2^{Hd} - 1}{2^H - 1}\right] \frac{\lambda(\mathcal{V})}{2}. \end{aligned} \tag{23}$$

In the next section, we will construct a coding and control scheme for scalar systems which actually achieves this lower bound. In other words, the RHS of (23) is in fact the optimal asymptotic worst case state magnitude.

B. Tightness of Bounds

A natural question to ask is whether the universal lower bounds on data rate and performance obtained in Section II-A are tight, i.e., whether it is possible to construct a coder-controller with data rate and/or cost arbitrarily close to them. The tightness of the errorless data rate bound $R > H$ has been established in numerous articles for linear plants with bounded disturbances [34], [77], stochastic disturbances [60], and no disturbances [34]. The essential steps are to 1) transform the plant into real Jordan form, so that all the open-loop dynamical modes of the plant are decoupled and 2) allocate the available data rate among the unstable modes so that each mode receives an average data rate which is slightly larger than its intrinsic entropy rate. We refer the reader to the references above for details. It is important to note that although we have permitted the coder and controller to have possibly infinite memory here, the actual construction of a stabilizing coder-controller typically only requires a finite-dimensional recursive structure. In addition, the channel data rate must be defined in a time-average sense for rates arbitrarily close to H to be attainable in general.

On the other hand, the tightness of the state norm bound (16) is difficult to confirm for state dimensions greater than 1, and geometric considerations suggest that it is not tight. However, for scalar fully-observed plants, the bound is in fact attainable, and we construct the optimal coder-controller below.

Assuming that $M > |a|$, suppose that just before time t the coder can construct, on the basis of the past symbols $s_0, \ldots, s_{t-1}$, an interval $\mathcal{X}_t$ of some length $2l_t$ containing the state x_t. Note that this interval will generally *not* be centered at the origin. At time t, the coder partitions $\mathcal{X}_t$ into M equal subintervals, and transmits the index of the one (call it $\mathcal{I}_t$) which contains x_t.

From the dynamics of the plant, x_{t+1} must then lie in the interval

$$\mathcal{X}_{t+1} = a\mathcal{I}_t + bu_t + \{v_t\}$$

using set addition. As the coder also knows $u_t = \delta_t(s_{t-d}, \ldots, s_0)$, it can calculate the new interval. Observing that the length of $\mathcal{I}_t$ is $2l_t/M$, that $\{v_t\}$ has length $\lambda(\mathcal{V})$, and that u_t has the effect of a simple translation, it follows that the half-length of $\mathcal{X}_{t+1}$ satisfies

$$l_{t+1} = \frac{|a|}{M} l_t + \frac{\lambda(\mathcal{V})}{2}, \ \forall t \geq 0.$$
$$\Rightarrow \lim_{t\to\infty} l_t = \frac{\lambda(\mathcal{V})}{2(1 - |a|/M)}. \tag{24}$$

At the other end of the channel, just after time t the controller would have received $s_0, \ldots, s_{t-d}$. Using the same interval update equations as the coder, it can determine the interval $\mathcal{X}_{t-d+1}$ containing x_{t-d+1}. From the dynamics, it then knows that

$$x_{t+1} \in \mathcal{J}_{t+1} := a^d \mathcal{X}_{t-d+1} + b \sum_{j=t-d+1}^{t} a^{t-j} u_j + \sum_{j=t-d+1}^{t} \{a^{t-j} v_j\} \tag{25}$$

and calculates u_t such that $\mathcal{J}_{t+1}$ is centered on the origin. Denoting the midpoint of $\mathcal{X}_{t-d+1}$ by $\hat{x}_{t-d+1}$, and observing that the noise sets $\{v_j\}$ are symmetrical about the origin, this means that the control at time t is given by

$$u_t = -a^d \hat{x}_{t-d+1}/b - \sum_{j=t-d+1}^{t-1} a^{t-j} u_j.$$

From (25), the half-length of $\mathcal{J}_{t+1} \ni x_{t+1}$ is readily seen to be

$$|a|^d l_{t-d+1} + \sum_{j=t-d+1}^{t} |a|^{t-j} \frac{\lambda(\mathcal{V})}{2} = |a|^d l_{t-d+1} + \frac{|a|^d - 1}{|a| - 1} \frac{\lambda(\mathcal{V})}{2}.$$

As $\mathcal{J}_{t+1}$ is centered on the origin, we immediately obtain

$$\sup |x_{t+1}| \leq |a|^d l_{t-d+1} + \frac{|a|^d - 1}{|a| - 1} \frac{\lambda(\mathcal{V})}{2}, \ \forall t \geq d - 1.$$

As a consequence, using (24)

$$J = \overline{\lim_{t\to\infty}} \sup |x_{t+1}| \leq \left[\frac{|a|^d}{1 - |a|/M} + \frac{|a|^d - 1}{|a| - 1} \right] \frac{\lambda(\mathcal{V})}{2}.$$

This scheme thus achieves the universal lower bound (23), and is globally optimal.

III. STRUCTURAL RESULTS FOR STOCHASTIC PLANTS

In this section we discuss what is known about the structure of coding and control schemes for stochastic linear plants, i.e., if the initial condition and additive disturbances entering the linear plant (1) are treated not as bounded unknowns, but rather as realizations of random vectors with infinite-support distributions. Though the essential nature of the problem does not change, there are nonetheless some significant differences, particularly in regard to the structure of the coding and control scheme.

Beginning with the good news in Section III-A, the advantage of the stochastic viewpoint is that certainty equivalence and a separation-like property apply if the cost is quadratic. In other words, the joint coding and control optimization problem can be solved by encoding the plant states so as to minimize a certain distortion metric D, and then using the decoded estimate in a certainty equivalent control law. If the plant is partially observed with Gaussian noise, then the optimal encoder will act instead on the conditional mean state given the plant outputs. However, complete separation between the coding and control problems is not attained, since the distortion metric D depends on the control input matrix as well as the plant dynamical matrix.

In Section III-B we present the bad news: if a linear plant has at least one strictly unstable mode and either the initial condition or process noise distribution has infinite support, then *no* time-invariant coding and control law with a finite-valued internal state can stabilize the plant. This means that simple and common coding schemes, such as memoryless quantization and *differential pulse code modulation*, cannot be considered for infinite-horizon stochastic control problems. Instead, we are forced to consider coder-controllers with *continuous*-valued internal states, similar to the zooming quantizers of [12]. In contrast, if the disturbances were modeled as bounded unknowns then a memoryless quantizer with sufficiently large range can always stabilize the plant.

Throughout this section, we assume that the linear plant takes the form (1), but with fully observed states $y_t = x_t$, and where the initial condition x_0 and additive disturbances $v_0, v_1, \ldots$ are realizations of mutually independent random variables $X_0, V_0, V_1, \ldots$ with distributions possessing possibly noncompact support. Additional assumptions relevant to each subsection are introduced as necessary.

A. Certainty Equivalence and Quasi-Separation

In the classical situation without communication constraints, it is well known that if the process and observation noise in the system are independent processes, and the cost is quadratic in state and control, then both *certainty equivalence* and the *separation principle* hold (see, e.g., [8, ch. 5]). In other words, the solution is obtained by 1) filtering the plant outputs to generate the conditional mean of the current state given past and present measurements and past controls, and then 2) using this conditional mean in the optimal control law which would apply if the plant were fully observed (certainty equivalence). The first step is equivalent to minimizing the mean square state estimation error conditioned on past and present measurements and past controls. By the linearity of the plant, this conditional mean square error is independent of the control policy used, and hence the optimal filtering law will not depend on the control law or the input matrix B, apart from the control term added at each time step. Hence, it may be assumed that these controls are in fact zero. In the second step, it is clear that the optimal gains do not depend on the solution to the filtering problem or on the output equation, since these gains are derived assuming full state observation. Thus, the classical optimization decomposes into two separate subproblems: an optimal filter problem for the uncontrolled version of the plant, and an optimal control problem assuming full state observations.

We explore here the extent to which these useful properties hold if the measurements are encoded and transmitted over an errorless digital channel to the actuator. In general, the encoder and decoder introduce nonlinearities into the feedback path that invalidate certainty equivalence and the separation principle. However, we show that for recursive encoders that subtract out the effect of past controls prior to coding, certainty equivalence still holds. Furthermore, there is *no* potential loss of optimality in restricting the encoder to this form. The design of the optimal encoder then reduces to minimizing a certain distortion metric. This metric depends on the input matrix, so the strong separation between estimation and control of the classical case no longer applies. Instead, we have an optimal control problem nested within an optimal estimation problem. The early articles [27], [28], [40] presented similar results for memoryless quantizers but are problematic; see [49].[4] In [15], analogous results were derived for quantizers with a *differential* form, but the global optimality of this structure was not considered. Similarly, a separation principle was also derived in [10]

For recursive encoders that subtract out the effect of past controls prior to coding, certainty equivalence still holds.

[4]In [27] and [40], it is assumed that the errors yielded by the memoryless quantizer are independent of past controls, which is generally incorrect. In [27], there is also confusion about the information set available to the controller, and an incorrect assertion that the state conditioned on past symbols is Gaussian. In [28], it is claimed that the optimal encoder is obtained by minimizing *at each time step* the current weighted mean square quantization error, conditioned on past quantizer outputs. The optimality of this *greedy* algorithm is doubtful, since the current quantizer output will generally affect the conditional quantizer error at future times, but the dynamic programming argument is only sketched for the terminal time.

for a class of encoders that quantized the innovations directly.

More recently, certainty equivalence and separation were discussed in [79] for fully observed plants controlled via general, possibly noisy feedback communication channels. Though the results are correct, there is a small but somewhat crucial gap in the proof.[5] In what follows, we build on the ideas from [79] and complete the proof of certainty equivalence and *quasi-separation* for fully observed linear stochastic systems that are regulated via noiseless digital feedback channels.

Let the cost of a given coder-controller $(\gamma, \delta) = (\{\gamma_t\}_{t\geq 0}, \{\delta_t\}_{t\geq 0}) \in \mathcal{C}$ (5), (6) on a finite time interval $t \in [0, \ldots, T]$ be given by the expected quadratic criterion

$$J_2 := \mathbb{E}\left[X'_{T+1}Q_{T+1}X_{T+1}\right] + \sum_{t=0}^{T} \mathbb{E}\left[X'_tQ_tX_t + U'_tR_tU_t\right] \quad (26)$$

where $Q_{T+1}, \ldots Q_0 \geq 0, R_T, \ldots R_0 > 0$ are specified weight matrices that are respectively positive semidefinite and positive definite. Let

$$\phi^{\delta}_t\left(\{s_i\}_{i=0}^{t-d-1}\right) := \sum_{j=0}^{t-1} A^{t-1-j}B\delta_j\left(\{s_k\}_{k=0}^{j-d}\right) \quad (27)$$

i.e., the accumulated effect of past controls on the state x_t, and consider the class $\bar{\mathcal{C}}$ of causal coder-controllers that subtract this out prior to coding

$$s_t = \gamma'_t\left(\left\{x_i - \phi^{\delta}_i\left(\{s_j\}_{j=0}^{i-d-1}\right)\right\}_{i=0}^{t}, \{s_i\}_{i=0}^{t-1}\right) \quad (28)$$

$$u_t = \delta_t\left(\{s_i\}_{i=0}^{t-d}\right). \quad (29)$$

It is trivial that any coder-controller in $\bar{\mathcal{C}}$ is also in $\mathcal{C}$. Furthermore, any coder-controller in $\mathcal{C}$ is also in $\bar{\mathcal{C}}$, since we can write

$$\begin{aligned} s_t &= \gamma_t\left(\{x_i\}_{i=0}^{t}, \{s_i\}_{i=0}^{t-1}\right) \\ &= \gamma_t\left(\left\{\left[x_i - \phi^{\delta}_i\left(\{s_j\}_{j=0}^{i-d-1}\right)\right]\right.\right. \\ &\qquad \left.\left. + \phi^{\delta}_i\left(\{s_j\}_{j=0}^{i-d-1}\right)\right\}_{i=0}^{t}, \{s_i\}_{i=0}^{t-1}\right) \\ &\equiv \gamma'_t\left(\left\{x_i - \phi^{\delta}_i\left(\{s_j\}_{j=0}^{i-d-1}\right)\right\}_{i=0}^{t}, \{s_i\}_{i=0}^{t-1}\right). \end{aligned}$$

[5]Namely, an auxiliary fully observed linear system with uncorrelated process noise is constructed, and it is asserted that the optimal controls for this auxiliary system are still given by the usual, linear law. This claim also occurs elsewhere in the literature, but simple counterexamples can be constructed for horizon-1 scalar plants.

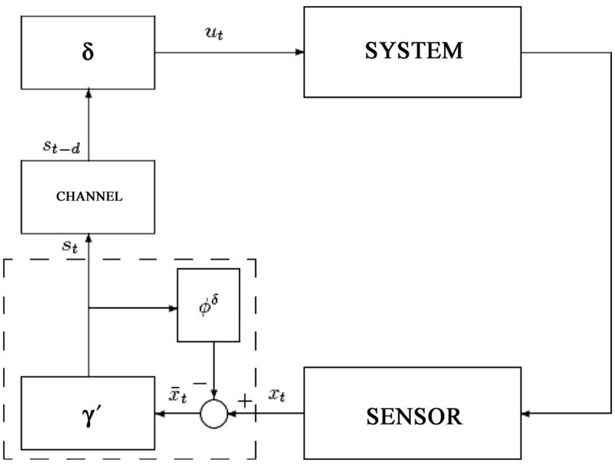

Fig. 2. *Scheme representing the new parametrization of the coder-controller.*

Thus, $\bar{\mathcal{C}} = \mathcal{C}$, and so without loss of generality we may assume that the coding and control equations are given by (28) and (29). This new parametrization of the coder-controller pairs is illustrated in Fig. 2. In effect, we have changed the global optimization coordinates from (γ, δ) to (γ', δ).[6] The reason for doing so is given in the following lemma.

Lemma 1: Let the coder-controller (28), (29) be applied to the fully observed stochastic linear plant (1). Then the statistics of the symbol sequence $\{S_t\}_{t\geq 0}$ are independent of the controller δ.

Proof: Define

$$\bar{X}_t := X_t - \phi^{\delta}_t\left(\{S_j\}_{j=0}^{t-d-1}\right). \quad (30)$$

It is straightforward to establish that

$$\bar{x}_t = A^t x_0 + \sum_{j=0}^{t-1} A^{t-1-j}v_j \quad (31)$$

i.e., $\{\bar{x}_t\}_{t\geq 0}$ is the state trajectory if no controls were applied. Evidently, the statistics of the $\{\bar{X}_t\}$ process are completely independent of the choice of coder-controller. As $s_t = \gamma'_t(\bar{x}_t, \{s_i\}_{i=0}^{t-1})$, it follows that for fixed $\gamma'_0, \gamma'_1, \ldots,$ the symbols transmitted do not depend on the controller δ. ■

We are now in a position to state the main result of this section. Its first part states that certainty equivalence holds

[6]This cannot be done if the memory of the encoder is finite, nor if we only seek to minimize over δ with γ fixed.

if the effect of past controls is removed prior to coding, as above. The second part states that this structural restriction *does not sacrifice global optimality*, and that a globally optimal coder-controller (γ, δ) can be constructed by finding an encoder γ' for the uncontrolled process (31) which minimizes a certain quadratic distortion criterion, and then applying certainty equivalent controls. As this criterion depends on the control input matrix B, the strong separation of classical linear stochastic control does not apply, although keeping the mean square coding errors small remains the optimization objective.

Theorem 2 (Certainty Equivalence and Quasi-Separation): Let any coder-controller be applied to the fully observed stochastic linear plant (1) under the horizon-T mean quadratic cost J_2 (26). Without loss of generality, represent the coder-controller in the (γ', δ) form (28), (29). Then, for fixed mappings $\gamma'_0, \gamma'_1, \ldots$, the optimal control law is certainty equivalent

$$U_t = \delta^*_t(S_{t-d}, \ldots S_0) := -L_t \mathbb{E}[X_t | S_{t-d}, \ldots, S_0] \tag{32}$$

where $S_{t-d}, \ldots, S_0$ are the symbols received by the controller up to time t, and $L_0, \ldots, L_T$ are the classical optimal gain matrices, defined by the downward Riccati recursion

$$\begin{aligned} P_t &= A' P_{t+1} \big(I - B(B' P_{t+1} B + R)^{-1} B' P_{t+1}\big) A \\ &\quad + Q_t, \;\; 0 \le t \le T, \;\; P_{T+1} = Q_{T+1} \qquad (33) \\ L_t &= (B' P_{t+1} B + R_t)^{-1} B' P_{t+1} A \qquad (34) \end{aligned}$$

(see, e.g., [8, ch. 4]).

Furthermore, the global smallest cost over *all* coder-controllers (γ, δ) (5), (6) decomposes as

$$\inf_{\gamma,\delta} J_2 = \mathbb{E}\big[X'_0 P_0 X_0\big] + \inf_{\gamma'_0, \gamma'_1, \ldots} D$$

where D is the controller-independent distortion criterion

$$\begin{aligned} D := \mathbb{E}\Bigg[&\sum_{t=0}^{T+1} \tilde{X}'_t Q_t \tilde{X}_t \\ &+ \sum_{t=0}^{T} \left(\mathbb{E}\left[\tilde{X}_{t+1} | \{S_i\}_{i=0}^{t+1-d}\right] - \mathbb{E}\left[\tilde{X}_{t+1} | \{S_i\}_{i=0}^{t-d}\right] \right)' \\ &\times P_{t+1} \left(\mathbb{E}\left[\tilde{X}_{t+1} | \{S_i\}_{i=0}^{t+1-d}\right] \right. \\ &\left. - \mathbb{E}\left[\tilde{X}_{t+1} | \{S_i\}_{i=0}^{t-d}\right] \right) \Bigg] \end{aligned} \tag{35}$$

and where $\tilde{X}_t := \bar{X}_t - \mathbb{E}[\bar{X}_t | \{S_i\}_{i=0}^{t-d}]$ is the conditional coding error for the uncontrolled process $\{\bar{X}\}_{t \ge 0}$ (31).

Proof: As the classes $\mathcal{C}$ and $\bar{\mathcal{C}}$ are identical by the discussion following (27), we may represent the coder-controller in the (γ', δ) form (28), (29), without loss of any generality. Letting $\hat{X}_t := \mathbb{E}[X_t | S_{t-d}, \ldots, S_0]$, it is trivial to show by standard arguments that

$$\begin{aligned} &\mathbb{E}\big[X'_t Q_t X_t | S_{t-d}, \ldots, S_0\big] \\ &\quad = \mathbb{E}\big[(X_t - \hat{X}_t)' Q_t (X_t - \hat{X}_t) | S_{t-d}, \ldots, S_0\big] + \hat{X}'_t Q_t \hat{X}_t. \end{aligned}$$

Averaging over the received symbols and substituting into (26), we obtain

$$\begin{aligned} J_2 := \mathbb{E}&\left[\hat{X}'_{T+1} Q_{T+1} \hat{X}_{T+1}\right] \\ &+ \sum_{t=0}^{T} \mathbb{E}\left[\hat{X}'_t Q_t \hat{X}_t + U'_t R_t U_t\right] \\ &+ \sum_{t=0}^{T+1} \mathbb{E}\big[(X_t - \hat{X}_t)' Q_t (X_t - \hat{X}_t)\big]. \end{aligned} \tag{36}$$

Looking at the second sum, observe that since $\phi^\delta_t(\{S_j\}_{j=0}^{t-d-1})$ is fully determined by the symbols $S_0, \ldots, S_{t-d-1}$, we can write

$$\begin{aligned} X_t - \hat{X}_t &\equiv X_t - \mathbb{E}[X_t | S_{t-d}, \ldots, S_0], \\ &= X_t - \phi^\delta_t\left(\{S_j\}_{j=0}^{t-d-1}\right) \\ &\quad - \mathbb{E}\left[X_t - \phi^\delta_t\left(\{S_j\}_{j=0}^{t-d-1}\right) \Big| S_{t-d}, \ldots, S_0\right], \\ &\equiv \bar{X}_t - \mathbb{E}[\bar{X}_t | S_{t-d}, \ldots, S_0] \equiv \tilde{X}_t. \end{aligned} \tag{37}$$

From (31) and Lemma 1, the statistics of $\{\bar{X}_t\}_{t \ge 0}$ and $\{S_t\}_{t \ge 0}$ are independent of the controller δ. By (37), so too is the process $\{X_t - \hat{X}_t\}_{t \ge 0}$, and thus the last sum in (36) may be ignored when optimizing over δ. That is,

$$\inf_\delta \; J_2 = \inf_\delta \{J'_2\} + \sum_{t=0}^{T+1} \mathbb{E}\left[\tilde{X}'_t Q_t \tilde{X}_t\right]$$

where

$$J'_2 := \mathbb{E}\left[\hat{X}'_{T+1} Q_{T+1} \hat{X}_{T+1}\right] + \sum_{t=0}^{T} \mathbb{E}\left[\hat{X}'_t Q \hat{X}_t + U'_t R U_t\right]. \tag{38}$$

Now, define $Z_t := \mathbb{E}[X_{t+1}|S_{t+1-d},\ldots,S_0] - \mathbb{E}[X_{t+1}|S_{t-d},\ldots,S_0]$ and observe that

$$\begin{aligned}\hat{X}_{t+1} &\equiv \mathbb{E}[X_{t+1}|S_{t-d},\ldots,S_0] + Z_t\\ &= \mathbb{E}[AX_t + BU_t + V_t|S_{t-d},\ldots,S_0] + Z_t\\ &= A\hat{X}_t + BU_t + Z_t\end{aligned}\tag{39}$$

since U_t is fully determined by $S_{t-d},\ldots,S_0$, and V_t is independent of $S_t,\ldots S_0$. Furthermore

$$\begin{aligned}Z_t &= \mathbb{E}\left[X_{t+1} - \phi^{\delta}_{t+1}\left(\{S_j\}_{j=0}^{t-d}\right)|S_{t+1-d},\ldots,S_0\right]\\ &\quad - \mathbb{E}[X_{t+1} - \phi^{\delta}_{t+1}\left(\{S_j\}_{j=0}^{t-d}\right)|S_{t-d},\ldots,S_0],\\ &\equiv \mathbb{E}[\bar{X}_{t+1}|S_{t+1-d},\ldots,S_0] - \mathbb{E}[\bar{X}_{t+1}|S_{t-d},\ldots,S_0].\end{aligned}\tag{40}$$

By virtue of (31) and Lemma 1, $\{Z_t\}_{t\geq 0}$ is independent of the controller δ. Thus, for fixed γ', optimizing over the controller reduces to minimizing the expected quadratic cost (38) for the fully observed auxiliary system (39). However, although $\{Z_t\}_{t\,\geq\,0}$ is uncorrelated, it is not independent. Consequently, the standard solution from optimal stochastic control cannot be directly applied to the auxiliary system. Nonetheless, there is sufficient statistical structure in Z_t to permit the classical argument to go through, with some care.

To show this, we use a completion-of-squares technique. Let $P_{T+1} := Q_{T+1}$ and let $P_T,\ldots,P_0 \geq 0$ be arbitrary positive semidefinite matrices. We have

$$\begin{aligned}J'_2 - \mathbb{E}\left[X'_0 P_0 X_0\right] &= \sum_{t=0}^{T}\mathbb{E}\Big[\hat{X}'_t Q_t \hat{X}_t + U'_t R_t U_t\\ &\qquad\qquad + \hat{X}'_{t+1}P_{t+1}\hat{X}_{t+1} - \hat{X}'_t P_t \hat{X}_t\Big],\\ &= \sum_{t=0}^{T}\mathbb{E}\Big[\hat{X}'_t(Q_t - P_t)\hat{X}_t + U'_t R_t U_t\\ &\qquad\qquad + (A\hat{X}_t + BU_t)'P_{t+1}(A\hat{X}_t + BU_t)\Big]\\ &\quad + 2\mathbb{E}\left[Z'_t P_{t+1}(A\hat{X}_t + BU_t)\right]\\ &\quad + \mathbb{E}\left[Z'_t P_{t+1} Z_t\right].\end{aligned}\tag{41}$$

Looking at the second expectation on the RHS, observe that $A\hat{X}_t + BU_t$ is completely determined by the symbols $S_{t-d},\ldots S_0$, since $\hat{X}_t \equiv \mathbb{E}[X_t|S_{t-d},\ldots,S_0]$ and $U_t = \delta_t(S_{t-d},\ldots,S_0)$. Thus

$$\begin{aligned}&\mathbb{E}\left[Z'_t P_{t+1}(A\hat{X}_t + BU_t)|S_{t-d},\ldots S_0\right]\\ &\qquad = \mathbb{E}[Z_t|S_{t-d},\ldots S_0]'P_{t+1}(A\hat{X}_t + BU_t).\end{aligned}\tag{42}$$

However

$$\begin{aligned}&\mathbb{E}[Z_t|S_{t-d},\ldots S_0]\\ &\quad\equiv \mathbb{E}[\mathbb{E}[X_{t+1}|S_{t+1-d},\ldots,S_0]|S_{t-d},\ldots,S_0]\\ &\qquad - \mathbb{E}[X_{t+1}|S_{t-d},\ldots,S_0],\\ &\quad= \mathbb{E}[X_{t+1}|S_{t-d},\ldots,S_0] - \mathbb{E}[X_{t+1}|S_{t-d},\ldots,S_0] = 0.\end{aligned}$$

Substituting this into (42) and taking an average over the symbols, (41) simplifies to

$$\begin{aligned}J'_2 &= \mathbb{E}\left[X'_0 P_0 X_0\right] + \sum_{t=0}^{T}\mathbb{E}\Big[\hat{X}'_t(Q_t - P_t)\hat{X}_t + U'_t R_t U_t\\ &\quad + (A\hat{X}_t + BU_t)'P_{t+1}(A\hat{X}_t + BU_t)\Big] + \sum_{t=0}^{T}\mathbb{E}\left[Z'_t P_{t+1} Z_t\right].\end{aligned}\tag{43}$$

As $\{Z_t\}_{t\,\geq\,0}$ is independent of δ, only the first sum needs be considered when optimizing over the control policy with fixed γ', and this sum is simply what appears in the classical linear quadratic regulation problem. It is easy to confirm that if $P_T,\ldots,P_0$ are defined by the standard Riccati difference equation (33), then

$$\begin{aligned}J'_2 &= \mathbb{E}\left[X'_0 P_0 X_0\right]\\ &\quad + \sum_{t=0}^{T}\mathbb{E}\Big[\left((R_t + B'P_{t+1}B)U_t + B'P_{t+1}A\hat{X}_t\right)'.\\ &\qquad\qquad \times (R_t + B'P_{t+1}B)^{-1}\\ &\qquad\qquad \times \left((R_t + B'P_{t+1}B)U_t + B'P_{t+1}A\hat{X}_t\right)\Big]\\ &\quad + \sum_{t=0}^{T}\mathbb{E}\left[Z'_t P_{t+1} Z_t\right].\end{aligned}$$

This is obviously minimized by the certainty equivalent policy (32), yielding

$$\begin{aligned}\min_{\delta} J'_2 &= \mathbb{E}\left[X'_0 P_0 X_0\right] + \sum_{t=0}^{T}\mathbb{E}\left[Z'_t P_{t+1} Z_t\right].\\ \Rightarrow \min_{\delta} J_2 &\equiv \mathbb{E}\left[X'_0 P_0 X_0\right] + \sum_{t=0}^{T}\mathbb{E}\left[Z'_t P_{t+1} Z_t\right]\\ &\quad + \sum_{t=0}^{T+1}\mathbb{E}\left[(X_t - \hat{X}_t)'Q_t(X_t - \hat{X}_t)\right]\end{aligned}$$

The proof is completed by substituting (37) and (40) into this, taking an infimum over γ', and noting that the coder-controller classes $\bar{\mathcal{C}}$ and $\mathcal{C}$ are the same. ■

B. Instability of Finite-State Coding for Unstable Stochastic Plants

The results of the previous section suggest that the hardest design task is not the design of the controller, which optimally is a certainty equivalent law, but of the encoder, which must minimize a rather complicated distortion metric. This observation is reinforced here, where we explain why many simple coding and control schemes are unable to stabilize unstable stochastic plants, regardless of how high a data rate is used.

Assume that the control law is of the form $u_t = L\hat{x}_t$, where L is any gain matrix such that $A + BL$ is strictly stable, and $\hat{x}_t$ here is the controller's reconstruction of the current state x_t of (1) given the received symbols $s_{t-d}, \ldots, s_0$. The simplest type of coding scheme possible is memoryless static quantization

$$\hat{x}_t \equiv q(x_t) \in \{q_1, \ldots, q_M\} \subset \mathbb{R}^n. \tag{44}$$

in which the index of the selected point q_{s_t} is transmitted across the channel. Another option is a *finite-state, predictive quantizer* (see, e.g., [31]), in which the latest coded state estimate is stored and the *prediction error* is recursively coded according to a finite-valued internal variable ι_t having values in some finite set $\mathcal{I}$,

$$\begin{aligned} q_{s_t} &\equiv q(x_t - (A + BL)\hat{x}_{t-1}; \iota_t) \\ &\in \{q_1 \ldots, q_M\} \subset \mathbb{R}^n \qquad (45) \\ \hat{x}_t &\equiv (A + BL)\hat{x}_{t-1} + q_{s_t}, \\ \iota_{t+1} &\equiv g(\iota_t, s_t) \in \mathcal{I} \qquad (46) \end{aligned}$$

for some function g. Examples are *differential pulse code modulation* and *delta modulation* in speech processing.

For noise distributions with compact support, it can be shown that either type of coder can achieve boundedness. It may seem as if this should also hold in the case of infinite support, since if stability has been achieved then the states and prediction errors remain with high probability in some bounded region, which could then be quantized without memory. However, this circular argument fails if the plant is strictly unstable and either the initial state or a process noise term has infinite support in all directions.

Proposition 2: Suppose that the plant (1) has at least one open-loop eigenvalue with magnitude strictly > 1, and that, for any nonzero $h \in \mathbb{R}^n$, either

$$\begin{aligned} &\mathbb{P}[h' X_0 > x] > 0, \ \forall x \in \mathbb{R}, \ \text{or} \\ &\qquad \exists t \geq 0 \ s.t. \ \mathbb{P}[h' V_t > v] > 0, \ \forall v \in \mathbb{R}. \end{aligned} \tag{47}$$

Then for any static memoryless coder (44) or finite-state predictive quantizer (45), (46)

$$\overline{\lim_{t\to\infty}} \ \mathbb{E}[\|X_t\|^r] = \infty, \forall r > 0$$

regardless of the number M of quantization points.

Proof: See [60].

This distinguishes the stochastic, communication-limited stabilization problem from the deterministic, bounded disturbance version, for which either memoryless or finite-state quantization suffice. The reason for the difference is basically that the finite range of the quantizer causes controller saturation. If the initial state or process noise has infinite support, there is consequently a finite chance that at some time t, the propagated state Ax_t is beyond reach of the control signal. The unstable plant dynamics then amplify this short-fall, causing the same phenomenon to occur with increasing probability at subsequent times, and inevitably leading to instability.

An obvious solution is to use an adaptive quantizer with possibly unbounded range, thereby allowing the control signal to "catch up" with the state. One simple approach is to use a predictive scheme with a scaling factor $l_t > 0$ which is recursively adjusted according to the symbols transmitted

$$\begin{aligned} q_{s_t} &\equiv q\left(\frac{x_t - (A + BL)\hat{x}_{t-1}}{l_t}\right) \in \{q_1, \ldots, q_M\} \subset \mathbb{R}^n. \\ \hat{x}_t &= (A + BL)\hat{x}_{t-1} + l_t q_{s_t}, \\ l_{t+1} &\equiv g(l_t, s_t) > 0. \end{aligned}$$

This approach is essentially that of [12], and is also the basis of *adaptive delta modulation*, and related schemes in communications (see, e.g., [69]). We refer the reader to [60] for details on how the quantizer q and scaling factor update function g can be constructed while maintaining mean square stability at any average data rate $R > H$.

IV. FINITE STATE CODER AND CONTROLLER

In the previous sections we did not assume any limitation on the state space complexity of the coder or controller. In applications in which large numbers of cheap sensors and/or actuators are involved, these complexity parameters have to be kept as low as possible. For this reason, in this section we will assume that the state space of the coder and of the controller is a finite set. This leads us to consider recursive representations of the coder and controller.

For simplicity we will here restrict to the scalar state case, fully observed, with no noise and no delay. In this case the system is described by the equation

$$x_{t+1} = ax_t + u_t \tag{48}$$

where $a \in \mathbb{R}$, and the coder-controller will take the following form:

$$s_t = \gamma_t(x_t; s_{t-1}, \dots, s_0) \tag{49}$$
$$u_t = \delta_t(s_t, s_{t-1}, \dots, s_0) \tag{50}$$

Notice that these two equations together with (48) will produce, from any initial state $x_0 \in \mathbb{R}$, a state evolution $x_t \in \mathbb{R}$, and a symbol evolution $s_t \in \mathcal{S}$. We recall that $\mathcal{S}$ is a finite set of cardinality M. Let $\mathcal{S}^*$ be the language generated by the alphabet $\mathcal{S}$ and let $\mathcal{L} \subseteq \mathcal{S}^*$ be the language constituted by the words associated with the possible symbol evolutions $s_0 s_1 \cdots$.

Consider the following equivalence relation $\sim$ on $\mathcal{L}$

$$\begin{aligned} & s'_0 \cdots s'_{t-1} \sim s''_0 \cdots s''_{t-1} \\ & \Updownarrow \\ & \gamma_{t+n}\left(x; s_{t+n-1}, \dots, s_t, s'_{t-1}, \dots, s'_0\right) \\ & \quad = \gamma_{t+n}\left(x; s_{t+n-1}, \dots, s_t, s''_{t-1}, \dots, s''_0\right) \\ & \delta_{t+n}\left(s_{t+n-1}, \dots, s_t, s'_{t-1}, \dots, s'_0\right) \\ & \quad = \delta_{t+n}\left(s_{t+n-1}, \dots, s_t, s''_{t-1}, \dots, s''_0\right) \\ & \quad \forall n \in \mathbb{N},\ x \in \mathbb{R},\ s_t, \dots, s_{t+n-1} \in \mathcal{S} \end{aligned}$$

Notice that, if $s'_0 \cdots s'_{t-1} \sim s''_0 \cdots s''_{t-1}$, then $s'_0 \cdots s'_{t-1} s_t \cdots s_{t+n-1} \sim s''_0 \cdots s''_{t-1} s_t \cdots s_{t+n-1}$ for all $n \in \mathbb{N}$ and $s_t, \dots, s_{t+n-1} \in \mathcal{S}$. Let $\Xi := \mathcal{L}/\sim$. Then we can define the maps:

- $Q_t : \Xi \times \mathbb{R} \to \mathcal{S}$;
- $F_t : \Xi \times \mathcal{S} \to \Xi$;
- $K_t : \Xi \times \mathcal{S} \to \mathbb{R}$;

such that, given $\xi_t \in \Xi$ and $x_t \in \mathbb{R}$, then for any representative $s_0 \cdots s_{t-1}$ on ξ_t in $\mathcal{L}$ we have that

$$Q_t(\xi_t, x_t) := \gamma_t(x_t; s_{t-1}, \dots, s_0)$$

and, letting $s_t := Q_t(\xi_t, x_t)$ and ξ_{t+1} the equivalence class in Ξ containing $s_0 \cdots s_{t-1} s_t$

$$\begin{aligned} F_t(\xi_t, s_t) &:= \xi_{t+1} \\ K_t(\xi_t, s_t) &:= \delta_t(s_t, s_{t-1}, \dots, s_0). \end{aligned}$$

It can be seen that these are well-defined functions and that system described by (49) and (50) can be described equivalently by the equations

$$\begin{cases} \xi_{t+1} = F_t(\xi_t, s_t) \\ s_t = Q_t(\xi_t, x_t) \\ u_t = K_t(\xi_t, s_t) \end{cases}. \tag{51}$$

The set Ξ is called the coder-controller state space, and the previous equations provide a state space representation of the coder-controller. More precisely, the state representation of the coder is

$$\begin{cases} \xi_{t+1} = F_t(\xi_t, Q_t(\xi_t, x_t)) \\ s_t = Q_t(\xi_t, x_t) \end{cases} \tag{52}$$

while the the state representation of the controller is

$$\begin{cases} \xi_{t+1} = F_t(\xi_t, s_t) \\ u_t = K_t(\xi_t, s_t) \end{cases}. \tag{53}$$

Notice the recursive structure of these representations which are particularly useful for implementation. Moreover, we can interpret the cardinality of Ξ as a computational complexity parameter. In this section we will try to understand what restrictions are imposed by limiting to coder-controllers that are time-invariant with finite state space, namely which are described by the equations

$$\begin{cases} \xi_{t+1} = F(\xi_t, s_t) \\ s_t = Q(\xi_t, x_t) \\ u_t = K(\xi_t, s_t) \end{cases} \tag{54}$$

where Ξ has cardinality N. We assume that the coder-controller initial state ξ_0 is initialized to a known fixed state $\bar{\xi}$. This implies that any initial state $x_0 \in \mathbb{R}$ of (48) will produce evolutions (x_t, ξ_t) uniquely determined by x_0.

Remarks: Notice that the case $N = 1$ corresponds to the memoryless situation in which the coder is described by the function $s_t = Q(x_t)$ and the controller by the function $u_t = K(s_t)$ so that the control law will be described by the quantized feedback $u_t = k(x_t)$ where the function $k = K \circ Q$ is quantized and can take only M different values.

Notice moreover that in what we have done so far we assumed that the coder and the controller have the same state spaces. This is a restriction, since we can imagine situations where, for instance, the coder is memoryless and the controller has memory, and vice versa. The case in which the coder is memoryless and the controller has memory describes situations in which there is more computational capability at the actuator than at the sensor (see [20]), and the other way around for the case in which the coder has memory while the controller is memoryless.

Example: Zooming in/Zooming Out: (See [12]) Consider the system (48) in which we assume for simplicity that a is an integer ≥ 2. Fix

$$\mathcal{S} = \{0, 1, \ldots, a-1\}^2 \cup \{\texttt{Alarm}\}$$

where Alarm is a supplementary alarm symbol. In this way we have a set $\mathcal{S}$ with $M = a^2 + 1$ symbols. Assume, initially, that $\Xi = \mathbb{Z}$ and that $x_0 \geq 0$. Define

$$Q(0, x) := \begin{cases} (\alpha_1, \alpha_2), & \text{if } x \in [0, 1[\\ \texttt{Alarm}, & \text{otherwise} \end{cases}$$

where $\alpha_i \in \{0, 1, \ldots, a-1\}$ provide the a-ry expression of x, namely $x = \sum_{i \in \mathbb{Z}} \alpha_i a^{-i}$. Let

$$Q(\xi, x) := Q(0, a^{-\xi} x)$$

Moreover, let

$$K(0, s) := \begin{cases} -\alpha_1 - \alpha_2 a^{-1}, & \text{if } s \neq \texttt{Alarm} \\ 0, & \text{otherwise} \end{cases}$$

and

$$K(\xi, s) := a^{\xi} K(0, s).$$

Finally, let

$$F(\xi, s) := \begin{cases} \xi - 1 & \text{if } s \neq \texttt{Alarm} \quad \text{zooming in phase} \\ \xi + 2 & \text{otherwise} \quad\quad \text{zooming out phase} \end{cases}$$

It can be seen that this technique yields convergence to zero [12], [44], at the price of requiring a coder-controller with infinite memory $N = \infty$. However, if we modify the map $F(\cdot, \cdot)$ as follows:

$$F(\xi, s) := \begin{cases} \max\{\xi - 1, -n\}, & \text{if } s \neq \texttt{Alarm} \\ \xi + 2, & \text{otherwise} \end{cases}$$

and we know that the initial state $x_0 \in [0, a^m[$, $m \geq 0$, then it can be shown that the state will grow utmost till $x_m \in [0, a^{2m}[$ during the zooming out phase and then it will contract during the zooming in phase till $x_t \in [0, a^{-n}[$ for all $t \geq 3m + n$.

In theprevious example it is clear that limitation in the memory is paid with the impossibility of asymptotic stability. This is always true.

Proposition 3: Assume that $|a| > 1$ and assume we are using a coder-controller as (5), (6). Assume finally that the controller can take only finitely many input values. Then there is an at most countable number of initial conditions x_0 corresponding to a state evolution x_t converging to zero.

Proof: Assume that the controller can take the values $\bar{u}_1, \ldots, \bar{u}_k$. Observe first that, since

$$x_t = a^t x_0 + \sum_{j=0}^{t-1} a^{t-1-j} u_j$$

and since u_j can take only finitely many values, then there exists only an at most countable number of initial conditions x_0 which correspond to a state evolution x_t such that $x_{\bar{t}} = 0$ for some $\bar{t}$. Suppose now that x_0 does not have this property. We want to show that the corresponding state evolution x_t does not converge to zero. Assume by contradiction that $x_t \to 0$. Let

$$m := \min\{|u_i| : u_i \neq 0, i = 1, \ldots, k\} > 0$$

and take $\epsilon < (1 + |a|)^{-1} m$. Then there is a T such that $|x_t| \leq \epsilon$ for all $t \geq T$. Consider now the input sequence $u_T, u_{T+1}, \ldots$. As $x_T \neq 0$, this input sequence cannot be identically zero. Let $\tau \geq T$ be such that $u_\tau \neq 0$. Then

$$m \leq |u_\tau| = |x_{\tau+1} - a x_\tau| \leq |x_{\tau+1}| + |a x_\tau| \leq (1 + |a|)\epsilon < m$$

which is a contradiction. ∎

The previous result shows that asymptotic stability cannot be achieved by a finite memory coder-controller scheme: it must be replaced by some sort of practical stability. More precisely, given two nested intervals $J \subseteq I$, it is only possible to find a coder-controller such that, for all $x_0 \in I$, the evolution it produces will be such that $x_t \in J$ for all $t \geq T$. We define the *entrance time* as

$$T(x_0, J) := \min\{t \in \mathbb{N} | x_s \in J \, \forall s \geq t\}. \tag{55}$$

It makes sense to measure the performance of the coder-controller by the two indices

$$T = T(I,J) := \mathbb{E}[T(x_0,J)] \tag{56}$$
$$C = C(I,J) := \frac{1}{\mathbb{P}[J]} \tag{57}$$

where $\mathbb{P}[\cdot]$ is the uniform probability measure over I, and $\mathbb{E}[\cdot]$ is the expectation with respect this probability measure. The index C is called the *contraction rate*, and describes the steady-state performance of the control strategy, while T is called the *expected time*, and describes its transient performance.

The previous definition of entrance time is based on the perfect knowledge of the state available at the coder. There is another possible definition of entrance time, based only on the partial knowledge of the state available at the decoder. Indeed the decoder knows at time t only the sequence of the symbols $s_0, s_1, \ldots, s_t$ from which it can produce the sequence of states $\xi_0, \xi_1, \ldots, \xi_t$. For any finite word $s_0 s_1 \cdots s_t \in \mathcal{S}^*$, let

$$\begin{aligned} I(s_0 s_1 \cdots s_t) &: \\ = \{x_0 \in I | x_0 &\text{ produces the sequence } s_0, s_1, \ldots, s_t\}, \end{aligned} \tag{58}$$

and given an infinite sequence $s_0, s_1, \ldots$, let

$$\begin{aligned} T_D(s_0, s_1, \ldots, J) := \min\{t \mid & \text{ any } x_0 \\ & \in I(s_0 s_1 \cdots s_{t-1}) \text{ produces } x_t \in J\} \end{aligned} \tag{59}$$

If the set on the right-hand side (RHS) is empty, we define $T_D(s_0, s_1, \ldots, J) := +\infty$. In words, $T_D(s_0, s_1, \ldots, J)$ is the minimum time t in which the decoder is sure, from the symbols till time $t-1$, that the initial state was such that the state x_t achieved the target set. Define finally the *decoder expected entrance time* as

$$T_D = T_D(I,J) := \mathbb{E}[T_D(s_0, s_1, \ldots, J)] \tag{60}$$

It is easy to see that, if x_0 produces the sequence $s_0, s_1, \ldots$, then $T(x_0, J) \le T_D(s_0, s_1, \ldots, J)$. This implies

$$T \le T_D$$

For any $\xi \in \Xi$ and $s \in \mathcal{S}$, let

$$I(\xi, s) := \{x \in \mathbb{R} | Q(\xi, x) = s\},$$

and define $\mathcal{I}(\xi) := \{I(\xi, s) | s \in \mathcal{S}\}$, which constitutes a partition of $\mathbb{R}$. We know that the cardinality of each of these partitions is M. If for any $\xi \in \Xi$, the target interval J is the union of quantization regions $I(\xi, s)$, then we have that

$$T_D \le T + 1 \tag{61}$$

In the sequel, we will present several bounds which will highlight the trade-off relation between the parameters influencing the control design. Let $\mathcal{L}$ be the sublanguage of $\mathcal{S}^*$ obtained as follows

$$\begin{aligned} \mathcal{L} := \{s_0 s_1 \cdots s_{t-1} \in \mathcal{S}^* | \ & \exists x_0 \text{ generating} \\ & s_0, s_1, \ldots \text{ and } t = T_D(s_0, s_1, \ldots, J)\} \end{aligned}$$

The language $\mathcal{L}$ is prefix (the verification of this fact, and the definition of a prefix language can be found in the Appendix). Moreover we have that

$$T_D(s_0, s_1, \ldots, J) = \text{length}(w)$$

Then the decoder expected entrance time T_D can be computed as the average length of a word in $\mathcal{L}$

$$T_D = \mathbb{E}[\text{length}(\mathcal{L})] = \sum_{w \in \mathcal{L}} \text{length}(w) \mathbb{P}[I(w)]. \tag{62}$$

Any bound on T_D has necessarily have to obtained through some approximation of the distribution probability $\mathbb{P}[I(w)]$. We now present a simple but fundamental estimation of this quantity:

Lemma 2: For any word $w \in \mathcal{L}$, it holds

$$\mathbb{P}[I(w)] \le C^{-1} |a|^{-\text{length}(w)},$$

where C is the contraction rate (57).

Proof: Fix any $w \in \mathcal{L}$ and let $t := \text{length}(w)$. Any $x_0 \in I(w)$ produces an evolution such that $x_t \in J$. Moreover for all $x_0 \in I(w)$ we have that

$$x_t = a^t x_0 + u$$

with a u which is independent of x_0. This implies that

$$|a|^t \mathbb{P}[I(w)] \le \mathbb{P}[J] = 1/C.$$

■

The following performance bound is essentially based on the work by Delvenne [21] on this topic. By Shannon's noiseless coding theorem we have that, since $\mathcal{L}$ is prefix, then

$$T_D = \mathbb{E}[\text{length}(\mathcal{L})] \geq \frac{H(\mathcal{L})}{\log M} \tag{63}$$

where

$$H(\mathcal{L}) := -\sum_{w\in\mathcal{L}} \mathbb{P}[I(w)] \log \mathbb{P}[I(w)]$$

is the entropy of the language $\mathcal{L}$.

Theorem 3: The contraction rate C (57) and decoder entrance time T_D (60) must satisfy the inequality

$$\log C \leq T_D \log \frac{M}{|a|}. \tag{64}$$

Proof: It follows from Lemma 2 that

$$-\log \mathbb{P}[I(w)] \geq \text{length}(w) \log |a| + \log C.$$

Substituting this inequality in the definition of the entropy of $\mathcal{L}$ and defining $\mathcal{L}_t$ to be the set of words in $\mathcal{L}$ of length t, we obtain

$$\begin{aligned} H(\mathcal{L}) &= -\sum_{t=0}^{+\infty}\sum_{w\in\mathcal{L}_t} \mathbb{P}[I(w)] \log \mathbb{P}[I(w)] \\ &\geq \sum_{t=0}^{+\infty}(t\log|a| + \log C)\sum_{w\in\mathcal{L}_t}\mathbb{P}[I(w)] \\ &= \log C + \log|a| \sum_{t=0}^{+\infty} t \sum_{w\in\mathcal{L}_t}\mathbb{P}[I(w)] \\ &= \log C + T_D \log|a| \end{aligned}$$

where we used the fact that $\sum_{t=0}^{+\infty} t \sum_{w\in\mathcal{L}_t}\mathbb{P}[I(w)]$ coincides with the expected length of the language. Putting together the previous inequality and (63) we have the thesis. ■

The previous theorem shows that there is a trade-off relation between the steady-state performance index C and the transient performance index T. This is illustrated in Fig. 3.

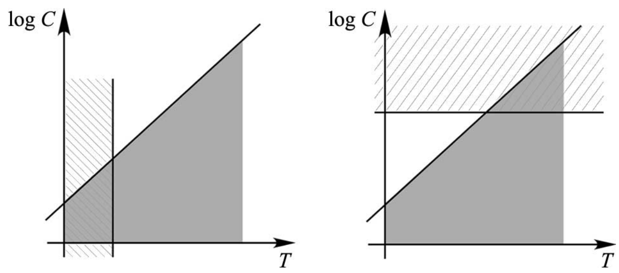

Fig. 3. ***The region of allowed performance indices*** $\log C$ ***and*** T***. The first graph shows that, if we want small*** T ***(good transient performance), we are forced to have small*** C ***(bad steady-state performance). The second graph shows that, if we want big*** C ***(good steady-state performance), we are forced to have big*** T ***(bad transient performance).***

Observe the following facts.

- Surprising enough, the complexity parameter N, namely the dimension of the state space of the coder-controller, does not play a role in the tradeoff relation between C and T.
- If we want to obtain a bound involving T instead of T_D, we need to assume that the target set J is the union of quantization regions. Indeed, in this case from (61) we can argue that

$$\log C \leq (T+1) \log \frac{M}{|a|}. \tag{65}$$

Example: Optimal Coder-Controller: (See [21]) Consider the system (48) in which we assume that a is an integer ≥ 2. Assume that $I = [0, 1]$, $J = [0, a^{-n}]$, and consider the following quantized feedback strategy:

Express any real x in base a expansion

$$x = \sum_{i=1}^{+\infty} \alpha_i a^i,$$

and set

$$k(x) = -a(\alpha_1 a^{-1} + \alpha_n a^{-n}), \ \forall x \in \mathbb{R}. \tag{66}$$

This is a memoryless control law with $M = a^2$ symbols and contractipn rate $C = a^n$. Note that the quantizer region corresponding to each possible value of $k(x)$ is a disconnected union of intervals. This control law, starting from an initial state x_0 with the base a expansion

$$x_0 = \sum_{i=1}^{+\infty} \alpha_i a^i$$

produces a state evolution x_t, whose base a expansion is given in the table at the bottom of the page. Notice that $T_D = n$ and so this example attains the bound (64).

We want to compute the expected entrance time T. Notice that

$$\mathbb{P}[T(x_0, J) \le i] = \begin{cases} a^{i-n}, & \text{if } i = 0, 1, \ldots, n-1 \\ 1, & \text{if } i \ge n. \end{cases}$$

This implies that

$$T = \sum_{i\ge 1} \mathbb{P}[T(x_0, J) \ge i] = \sum_{i\ge 1} 1 - \mathbb{P}[T(x_0, J) \le i-1]$$
$$= \sum_{i=0}^{n-1} 1 - \mathbb{P}[T(x_0, J) \le i] = n - \frac{1 - a^{-n}}{a-1}.$$

Notice, however, that in this case J is not the union of quantization regions and so no estimate involving T can be argued from (64). This coder-controller can be extended [21] so to obtain a class of optimal coder-controller pairs with $M = a^{h+1}$, $T_D = n$ and $C = a^{nh}$ for all $h, n \in \mathbb{N}$.

Example: Zooming in/Zooming Out: (See [25]) Consider the previous zooming in/zooming out example and assume that $I = [0, 1]$ and $J = [0, a^{-n}]$. In this case we have only zooming-in, namely $\xi_t = -t$ for $t \le n-1$ and $\xi_t = -n$ for $t \ge n$. Moreover, only a^2 symbols are transmitted. Therefore, we have $N = n$, $M = a^2$ and $C = a^n$. This control, starting from an initial state x_0 which the following base a expansion

$$x_0 = \sum_{i=1}^{+\infty} \alpha_i a^i$$

will produce a state evolution x_t whose base a expansion is given in the table at the bottom of the next page. Notice that in this case $T_D = n$ and so also this example attains the bound (64). We want to compute the expected entrance time T. Notice that

$$\mathbb{P}[T(x_0, J) \le i] = \begin{cases} a^{2i-2n+1}, & \text{if } i = 0, 1, \ldots, n-1 \\ 1, & \text{if } i \ge n. \end{cases}$$

This implies that

$$T = \sum_{i\ge 1} \mathbb{P}[T(x_0, J) \ge i] = \sum_{i\ge 1} 1 - \mathbb{P}[T(x_0, J) \le i-1]$$
$$= \sum_{i=0}^{n-1} 1 - \mathbb{P}[T(x_0, J) \le i] = n - \frac{a}{a^2-1}(1 - a^{-2n}).$$

	0	1	2	3	$\cdots$	$n-2$	$n-1$	n	$n+1$	$n+2$	$\cdots$
x_0	0	α_1	α_2	α_3	$\cdots$	α_{n-2}	α_{n-1}	α_n	α_{n+1}	α_{n+2}	$\cdots$
x_1	0	α_2	α_3	α_4	$\cdots$	α_{n-1}	α_n	0	α_{n+2}	α_{n+3}	$\cdots$
x_2	0	α_3	α_4	α_5	$\cdots$	α_n	0	0	α_{n+3}	α_{n+4}	$\cdots$
x_3	0	α_4	α_5	α_6	$\cdots$	0	0	0	α_{n+4}	α_{n+5}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{n-3}	0	α_{n-2}	α_{n-1}	α_n	$\cdots$	0	0	0	$*$	$*$	$\cdots$
x_{n-2}	0	α_{n-1}	α_n	0	$\cdots$	0	0	0	$*$	$*$	$\cdots$
x_{n-1}	0	α_n	0	0	$\cdots$	0	0	0	$*$	$*$	$\cdots$
x_n	0	0	0	0	$\cdots$	0	0	0	$*$	$*$	$\cdots$
x_{n+1}	0	0	0	0	$\cdots$	0	0	0	$*$	$*$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Notice however that also in this case J is not the union of quantization regions, so no estimate on T can be argued from (64).

V. QUANTIZERS WITH CONNECTED QUANTIZATION REGIONS

In the previous section, we imposed no limitations on the structure of the quantization regions, and saw that, under certain conditions, the universal performance bound (64) can be achieved by a control law with disconnected quantization regions (66). However, the complexity of the shape of these regions can in practice be a costly parameter. Typically, the complexity and cost of analog-to-digital (A/D) converters depends on the number of inequalities that must be checked in order to determine the digital output associated with a real input. This roughly coincides with the logarithm of the number of intervals into which the A/D converter partitions the real line, as well as the number of different digital values. For the case of a scalar quantizer, each quantization region may in general be a union of several disjoint intervals associated with the same output. In this case, the A/D complexity is better described by the number of intervals than by the number of different values that the output can take.

Mathematically, this is equivalent to forcing the quantizer regions to be intervals, but dropping the constraint that the associated outputs be distinct. In Theorem 5 in this section, we show that this dramatically decreases the attainable performance. As before, we first establish universal bounds, and then present some examples.

As a first step, we need a different estimate of T_D with respect to the one suggested by Theorem 3. We start again from the prefix language $\mathcal{L}$ and we introduce the sequence of numbers

$$\Gamma_t = |\{w \in \mathcal{L} | \text{length}(w) = t\}|$$

namely Γ_t is the number of words in $\mathcal{L}$ of length t. We assume by convention that $\Gamma_0 = 1$. We have the following preliminary results.

Lemma 3: Given any $t \in \mathbb{N}$, we have that

$$\mathbb{P}[T_D(w,J) = t] \le C^{-1}\frac{\Gamma_t}{|a|^t} \tag{67}$$

$$\mathbb{P}[T_D(w,J) \ge t] \ge 1 - C^{-1}\sum_{k=0}^{t-1}\frac{\Gamma_k}{|a|^k} \tag{68}$$

where C is the contraction rate C (57), and $T_D(w,J)$ is the minimum decoder entrance time (59).

Proof: As w varies in $\mathcal{L}$, the various subsets $I(w)$ are relatively disjoint. Hence, using Lemma 2, we easily obtain

$$\mathbb{P}[T_D(w,J) = t] = \mathbb{P}[\text{length}(w) = t] \le \Gamma_t C^{-1}|a|^{-t}$$

which proves the first assertion.

We prove now the second assertion by induction on t. The assertion is trivial for $t = 1$. Assume by induction that

	0	1	2	3	$\cdots$	$n-2$	$n-1$	n	$n+1$	$n+2$	$\cdots$
x_0	0	α_1	α_2	α_3	$\cdots$	α_{n-2}	α_{n-1}	α_n	α_{n+1}	α_{n+2}	$\cdots$
x_1	0	0	α_3	α_4	$\cdots$	α_{n-1}	α_n	α_{n+1}	α_{n+2}	α_{n+3}	$\cdots$
x_2	0	0	0	α_5	$\cdots$	α_n	α_{n+1}	α_{n+2}	α_{n+3}	α_{n+4}	$\cdots$
x_3	0	0	0	0	$\cdots$	α_{n+1}	α_{n+2}	α_{n+3}	α_{n+4}	α_{n+5}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{n-3}	0	0	0	0	$\cdots$	α_{2n-5}	α_{2n-4}	α_{2n-3}	α_{2n-2}	α_{2n-1}	$\cdots$
x_{n-2}	0	0	0	0	$\cdots$	0	α_{2n-3}	α_{2n-2}	α_{2n-1}	α_{2n}	$\cdots$
x_{n-1}	0	0	0	0	$\cdots$	0	0	α_{2n-1}	α_{2n}	α_{2n+1}	$\cdots$
x_n	0	0	0	0	$\cdots$	0	0	0	α_{2n+1}	α_{2n+2}	$\cdots$
x_{n+1}	0	0	0	0	$\cdots$	0	0	0	α_{2n+2}	α_{2n+3}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

the assertion holds for t and let us prove it for $t+1$. We can now write

$$\begin{aligned}\mathbb{P}[T_D(w,J) \geq t+1] &= \mathbb{P}[T_D(w,J) \geq t] - \mathbb{P}[T_D(w,J) = t] \\ &\geq \mathbb{P}[T_D(w,J) \geq t] - C^{-1}\frac{\Gamma_t}{|a|^t} \\ &\geq 1 - C^{-1}\sum_{k=0}^{t-1}\frac{\Gamma_k}{|a|^k} - C^{-1}\frac{\Gamma_t}{|a|^t} \\ &= 1 - C^{-1}\sum_{k=0}^{t}\frac{\Gamma_k}{|a|^k}.\end{aligned}$$

■

From this we can obtain the following result.

Proposition 4: For any $\bar{t} \in \mathbb{N}$, it holds

$$T_D \geq \bar{t} - C^{-1}\sum_{t=1}^{\bar{t}}\sum_{t=0}^{t-1}\frac{\Gamma_k}{|a|^k} \tag{69}$$

where C is the contraction rate C (57), and T_D is the decoder entrance time (60).

Proof: Immediate consequence of Lemma 3 and of the fact that

$$T_D = \sum_{t=1}^{+\infty}\mathbb{P}[T_D(w,J) \geq t] \geq \sum_{t=1}^{\bar{t}}\mathbb{P}[T_D(w,J) \geq t].$$

■

If we can establish upper bounds on Γ_t, through (69) we can thus achieve lower bounds on T_D. The following theorem provides a bound on the growth of Γ_t depending on the number of quantization intervals M and the number of states N. The proof of this theorem is very long and will not be reported in this paper (see [26]).

Theorem 4: Assume that $|a| > 2$. Then

$$\frac{\Gamma_t}{|a|^t} \leq 2\left[\sum_{s=1}^{r\wedge t}\binom{t-1}{s-1}\binom{r}{s}\left(\frac{s}{r}\right)^s\right]\left(\frac{MNK}{t\wedge\frac{MNK}{e}}\right)^{t\wedge\frac{MNK}{e}} \quad \forall t \geq 1 \tag{70}$$

where $r \in \{1,\ldots,MN\}$ is independent of t, but may depend on the specific system, while K depends only on $|a|$, and where $a \wedge b$ means the minimum between a and b.

From bounds (69) and (70), we obtain a lower bound on T_D. However, this bound is very implicit. It depends on the contraction rate C and the complexity parameters N and M (that enter only through their product), and also on the choice of the parameter $\bar{t}$. Of course we would like a simpler expression, similar to (64). The key idea to obtain this is to study various possible regimes and choose appropriate $\bar{t}$ in the above inequality. The next result, whose derivation is long but straightforward, distinguishes three different regimes. The first is when $NM/\log C$ is sufficiently small: this in particular covers the case when we keep N and M fixed and we let C grow. We know from previous considerations that there are examples where T_D grows only logarithmically with respect to C. We will see that this can not happen if the quantization regions have to be intervals: the corresponding expected entrance time T_D in this case exhibits a superlogarithmic growth in C. The second case is a sort of dual of the first one: it is when $T_D/\log C$ is sufficiently small. It contains the case when $T_D/\log C \to 0$, namely the regime of sublogarithmic growth of T_D in C: this time the corresponding complexity parameter NM exhibits a superlogarithmic growth in C. The third situation is the logarithmic regime, which is when both NM and T_D exhibit a logarithmic growth: it is a consequence of the first two bounds.

Theorem 5: For any contraction rate C (57) and decoder entrance time T_D (60), the following bounds hold.

1) There exist $K_1 > 0$, $\beta_1 > 0$ and $C_1 > 1$ such that

$$C \geq C_1 \quad \text{and} \quad \frac{NM}{\log C} \leq \beta_1 \Longrightarrow T_D \geq K_1 NMC^{1/NM}. \tag{71}$$

2) There exist $K_2 > 0$, $\beta_2 > 0$ and $C_2 > 1$ such that

$$C \geq C_2 \quad \text{and} \quad \frac{\lceil T_D\rceil}{\log C} \leq \beta_2 \Longrightarrow NM \geq K_2\lceil T_D\rceil C^{\frac{1}{\lceil T_D\rceil}} \tag{72}$$

where $\lceil\cdot\rceil$ denotes the ceiling function.

3) There exist $C_0 > 1$ and two functions $F, G : \mathbb{R}_+ \to \mathbb{R}_+$ which are decreasing and converging to 0 at $+\infty$, such that for all $C > C_0$ we have that

$$\frac{NM}{\log C} \geq F\left(\frac{\lceil T_D\rceil}{\log C}\right) \quad \text{and} \quad \frac{\lceil T_D\rceil}{\log C} \geq G\left(\frac{NM}{\log C}\right). \tag{73}$$

Remarks: Notice that in the case when both N and M are assumed to be fixed, we obtain a tradeoff between the two performance parameters C and T_D. In this case, (71) shows that T_D has to grow, with respect to C, at least as a power $C^{1/NM}$. This is in sharp contrast with the logarithmic growth obtained in the example presented after Theorem 3. It is important to remark again that this degradation of performance is due to the requirement of having intervals as quantization regions. Examples of memoryless feedback quantization strategies employing a fixed number of quantization intervals (not depending on the contraction rate) are the chaotic stabilizers presented Section V-D. Notice that these quantizers exhibit a growth rate of type $T_D \sim C$, hence worse than bound (71) since $NM > 1$. However, this bound can be refined in certain cases. It is shown in [23] that for a memoryless quantized feedback having $\lceil |a| \rceil$ quantization intervals outside J (which is exactly the case in the example), T_D has to grow at least linearly in C.

Item (3) of the above result shows that in order to achieve entrance times logarithmic in C, it is necessary to consider schemes where the number of quantization intervals M and/or the number N of states of the quantizer also grow at least logarithmically in C. Conversely, schemes for which NM grows at most logarithmically in C yield entrance times that are at least logarithmic in C. Examples of this type of behavior are reported in Section V-C.

Let us comment more on the logarithmic regime (73). Notice that, if we take any linear feedback $u_t = kx_t$, with $k \in \mathbb{R}$, such that $|a+k| < 1$, then the closed-loop system $x_{t+1} = (a+k)x_t$ is exponentially stable. This corresponds to an expected entrance which grows logarithmically with C but decreases with $|a+k|$. Hence, the logarithmic regime corresponds to a kind of exponential stability obtainable by a quantized controller. Therefore, the bound (73), that furnishes a quantitative tradeoff between the two ratios $T_D/\log C$ and $NM/\log C$, can be interpreted as a tradeoff between the rate of exponential stability associated with the quantized controller and cost in terms of the complexity parameters N and M. The constraint provided by (73) are illustrated in Fig. 4 which shows explicitly the region in which the pairs $(NM/\log C, T_D/\log C)$ can not belong to. It can be easily shown that the functions $F(x)$ and $G(x)$ which determines the boundary of this region tend to 0 as the function $f(x) = xe^{1/x}$.

Notice finally that in order obtain faster stabilization strategies, for instance strategies where the entrance time T_D is fixed and does not depend on C (such as the dead-beat controller presented in Section V-B), (72) shows that the product NM has to grow more than logarithmically in C. In particular, in the case when T_D is constant, it has to grow as a power of C. This is in agreement with the examples.

A. Memoryless Quantized Controllers

As observed above, if the coder and the controller are memoryless and time-invariant, then the control system becomes

$$x_{t+1} = ax_t + k(x_t)$$

where $k : \mathbb{R} \to \mathbb{R}$ is a piecewise constant map with at most M levels and is called a quantized controller. Here we are still restricting our attention to quantized controllers

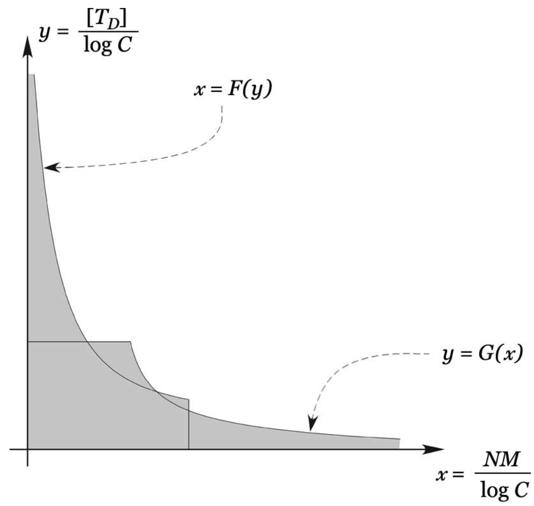

Fig. 4. ***The grey region in this graph represents the set to which the pairs*** $(NM/\log C, T_D/\log C)$ ***cannot belong.***

whose quantization regions are intervals. Differently from what happens without this topological restriction, there are various ways to design a stabilizing quantized controller each providing a different tradeoff choice between steady-state C and transient performance T_D. First observe that the case $C = 1$ corresponds to requiring that the interval $I = J$ is invariant. This can be obtained by $M = \lceil |a| \rceil$ levels.

The Perron–Frobenius Operator for Piecewise Affine Maps: In this subsection we recall some standard results on the ergodic theory of piecewise affine maps which is relevant in the analysis of quantized feedback control systems as first observed in [20].

Let $k(x)$ be a quantized feedback making the interval I invariant and let $\Lambda(x) := ax + k(x)$. This is a piecewise affine map with fixed slope a. Assume here that $|a| > 1$. Let $L^1(I)$ be the set of integrable function and $L^\infty(I)$ the set of bounded functions defined on I. It is a standard fact [41] that Λ induces a linear transformation

$$\mathcal{P}_\Lambda : L^1(I) \to L^1(I)$$

called the Perron–Frobenius operator associated with Λ which is uniquely defined by the following duality relation:

$$\int_I (g \circ \Lambda)(x) f(x) dx = \int_I g(x)(\mathcal{P}_\Lambda f)(x) dx \qquad (74)$$

for all $g \in L^\infty(I), f \in L^1(I)$. It can be shown that the operator $\mathcal{P}_\Lambda$ is bounded with $\|\mathcal{P}_\Lambda\|_1 \leq 1$, and maps probability densities to probability densities. An important interpretation of $\mathcal{P}_\Lambda$ is as follows. Roughly speaking, the Perron–Frobenius operator provides a statistical description of the dynamics associated with the nonlinear autonomous system $x_{t+1} = \Lambda(x_t)$. More precisely, if we see x_t as a random variable defined on I with probability density p_t, then the density of x_{t+1} is $p_{t+1} = \mathcal{P}_\Lambda p_t$. This implies that $p_t = \mathcal{P}^t_\Lambda p_0$, where p_0 is the probability density describing the initial condition x_0.

The relevance of the Perron–Frobenius operator in our investigations is also due to the fact that

$$\mathbb{P}_f[T(x, J) > n] = \int_{I \setminus J} \mathcal{P}^n_\Lambda p_0(x) dx$$

which follows by iterating (74) and by taking $g(x) = \mathbf{1}_{I\setminus J}(x)$. This shows that the asymptotic properties of this operator, and so its spectral properties, will be relevant for our purposes.

The Perron–Frobenius operator has interesting spectral properties if restricted to bounded variation probability densities. Under some technical assumptions it can be shown that there exists a density $\bar{p}(x)$, which is $\mathcal{P}_\Lambda$-invariant, such that for any bounded variation density $p(x)$ we have that $\mathcal{P}^n_\Lambda p(x)$ converges exponentially to $\bar{p}(x)$. This implies that the probability density describing x_t will always converge to $\bar{p}$. This provides a nice statistical description of the asymptotic behavior of the state evolution. Further details on the use of the invariant density and of the Perron–Frobenius operator in the context of quantized control can be found in [20], [24], and [25].

B. Deadbeat Quantized Controllers

This strategy is based on the simple idea of approximating through a uniform quantizer the linear deadbeat controller. More precisely, let $q : \mathbb{R} \to \mathbb{R}$ be the uniform quantizer defined as

$$q(x) = 2k + 1 \quad \forall x \in [2k, 2k+2[, \quad k \in \mathbb{Z}.$$

Observe that $|q(x) - x| \leq 1$. It is easy to verify that, by the quantized controller $k(x) := q(-ax)$ we can obtain the convergence in one step from any interval $I = [-C, C]$ into $J = [-1, 1]$ with a controller with $M = 2\lceil |a|C/2\rceil \simeq |a|C$ levels. So we obtain a contraction C, an expected time $T = 1$ with $M \simeq |a|C$ levels. The closed-loop map $ax + k(x)$ obtained in this way is illustrated in Fig. 5.

This technique can be extended to obtain multistep deadbeat quantized controllers. In this case we obtain a contraction C, an expected time $T = \tau$ with $M \simeq |a|\tau C^{1/\tau}$ levels [26]. This technique can be extended also to multidimensional systems [67].

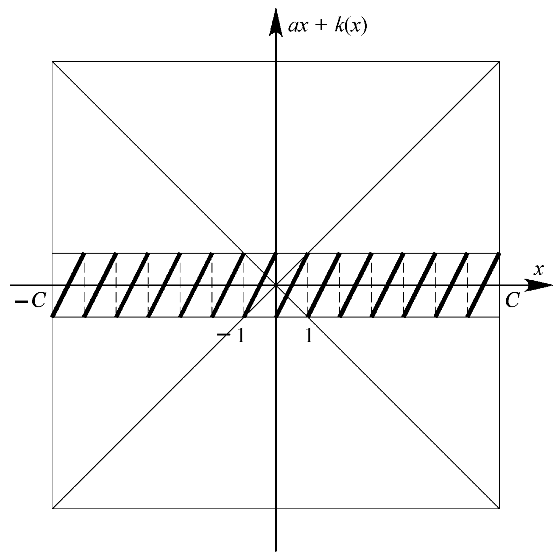

Fig. 5. ***Graph of the closed-loop map*** $ax + k(x)$ ***obtained from a uniform quantizer.***

C. Logarithmic Quantized Controllers

This class of quantized controllers is the most efficient one if we require the closed-loop system to be stable with respect to some Lyapunov function [22]. In our scalar example we can start from the Lyapunov function $V(x) = x^2$. Then we obtain the any quantized controller $k(\cdot)$ such that

$$\Delta V(x) = V(ax + k(x)) - V(x) < 0 \qquad \forall x \in [-C, -1] \cup [1, C] \quad (75)$$

will yield convergence from any interval $I = [-C, C]$ into $J = [-1, 1]$. Condition (75) holds true if and only if $|ax + k(x)| < |x|$ and so if $|ax + k(x)| < \delta|x|$ for some $0 < \delta < 1$. This inequality is equivalent to

$$-\delta|x| \leq ax + k(x) \leq \delta|x| \quad (76)$$

and to

$$-ax - \delta|x| \leq k(x) \leq -ax + \delta|x|. \quad (77)$$

Optimal quantized controllers $k(x)$ satisfying (77) are illustrated in Fig. 6. They correspond to logarithmic quantizers since $k(x)$ can be expressed as

$$k(x) = -(a - \delta)\eta^{q\left(\log_\eta(x)\right)+1}$$

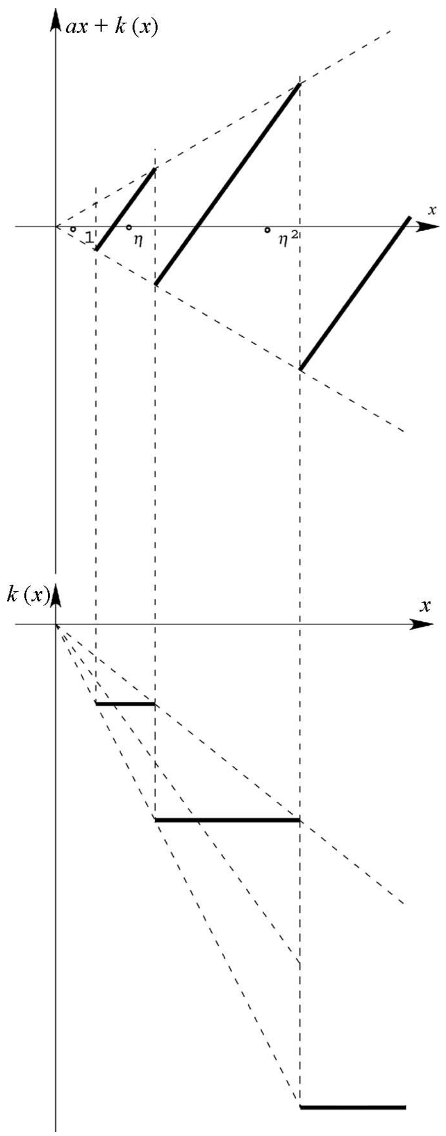

Fig. 6. ***Graphs of the logarithmic quantizer*** $k(x)$ ***and the closed-loop map*** $ax + k(x)$ ***compared with the bounds*** $ax \pm \delta|x|$ ***and*** $\pm\delta|x|$***, for*** $x \geq 0$.

where $\eta = (a + \delta)/(a - \delta)$. So we obtain a contraction C, an expected time $T \leq \log_\delta C$ with

$$M = \lceil |a| \rceil + 2 \left\lceil \frac{\log_\eta C}{2} \right\rceil$$

levels. The logarithmic quantizer $k(x)$ and the closed-loop map $ax + k(x)$ are illustrated in Fig. 6.

D. Chaotic Quantized Controllers

In [24], another possible quantized controller has been proposed. This control strategy exploits the chaotic behavior of the state evolution inside $I = [-1, 1]$ produced by the feedback map $k_0(x) := q(-ax)$ when we have that $|a| \geq 2$. In this way we have that, for almost every initial condition x_0 starting from I, the state evolution x_t is maintained inside the interval I and is dense in this interval. For this reason x_t will visit the interval $J = [-1/C, 1/C]$. Therefore, if we modify this feedback map as follows:

$$k(x) = \begin{cases} k_0(x), & \text{if } x \in I \setminus J \\ k_1(x), & \text{if } x \in J \end{cases} \tag{78}$$

where $k_1(x)$ is any quantized feedback making J invariant, we obtain that the state will move chaotically inside I till it will enter the interval J and there it will be entrapped.

We obtain that in this way a quantized feedback with $M = 2\lceil |a| \rceil \simeq 2|a|$ levels and contraction C. In general it is not easy to determine the expected time. This computation becomes quite easy in the particular cases in which each of the two intervals $[-1, -1/C]$ and $[1/C, 1]$ composing the set $I \setminus J$ can be divided into $\bar{n}$ identical intervals of length $2/|a|$ and the closed-loop map $ax + k(x)$ is affine on these intervals and has the entire I as codomain. The graphs of the closed-loop map $ax + k(x)$ obtained in this way is illustrated in Fig. 7. In this case we have

$$\bar{n} = |a| \frac{C - 1}{2C}$$

and so

$$M = |a| \frac{C - 1}{C} + \lceil |a| \rceil \simeq 2|a|.$$

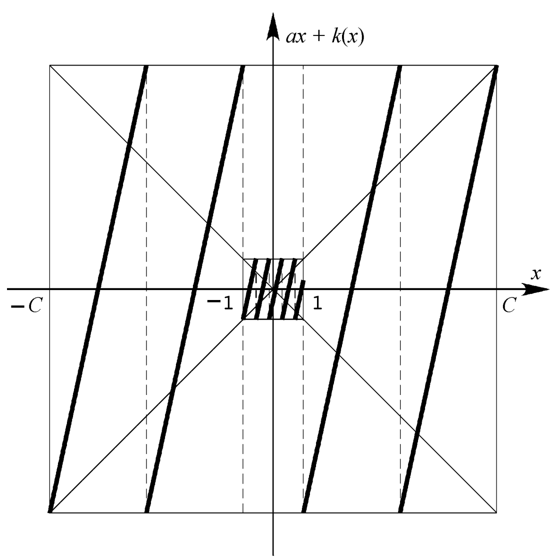

Fig. 7. ***Graph of the closed-loop map*** $ax + k(x)$ ***obtained from the chaotic quantized feedback.***

The evaluation of the expected entrance time can be done as follows. Observe that in this case, if the density function p_t of the random variable x_t is uniform on $I \setminus J$, the same holds true for density function p_{t+1} of x_{t+1}. More precisely it can be shown that, if $p_t(a) = \alpha_t$ for all $a \in I \setminus J$, then $p_{t+1}(a) = \alpha_{t+1} = \alpha_t((C-1)/C)$ for all $a \in I \setminus J$. This implies that

$$p_t(a) = \left(\frac{C-1}{C}\right)^t \frac{1}{2} \quad \forall a \in I \setminus J.$$

Observe finally that

$$\begin{aligned}\mathbb{P}[T(x_0, J) > n] = \mathbb{P}[x_n \in I \setminus J] &= \int_{I \setminus J} p_n(a) da \\ &= 2\frac{C-1}{C}\alpha_n = \left(\frac{C-1}{C}\right)^{n+1}\end{aligned}$$

and so

$$\begin{aligned}T(I, J) = \mathbb{E}[T(x_0, J)] &= \sum_{n=0}^{\infty} \mathbb{P}[T > n] \\ &= \sum_{n=0}^{\infty}\left(\frac{C-1}{C}\right)^{n+1} = C - 1.\end{aligned}$$

Therefore, we obtain expected time $T(I, J) = C - 1 \simeq C$.

Chaotic stabilizers can also be considered for general cases. Some preliminary results on this case have been obtained in [24]. In [23] the following more refined result is proved.

Theorem 6: Let a be such that $|a| > 2$, $I = [-1, 1]$ and $J = [-1/C, 1/C]$. There exists an almost (I, J)-stabilizing quantized feedback $k : I \to \mathbb{R}$ such that

$$\begin{aligned}M &= 2\lceil |a| \rceil + 1 \\ T &\leq KC\end{aligned}$$

where K is a positive constant only depending on a.

E. Robustness in Memoryless Quantized Controllers

Discrete-time systems typically result from sampling continuous-time systems. The synthesis of quantized feedback maps that are robust with respect to the sampling period Δ is proposed in [42]. As in [42] take the unstable continuous time system

$$\dot{x}(t) = \alpha x(t) - u(t), \qquad \alpha > 0$$

and assume that the feedback control law is

$$u(t) = k(x(k\Delta)), \quad \forall t \in [k\Delta, (k+1)\Delta[, \quad k \in \mathbb{Z}$$

where $k(\cdot)$ is a quantized feedback map. It is well-known that the analysis of this system can be done through the discrete time system

$$x_{k+1} = e^{\alpha\Delta}x_k - \frac{e^{\alpha\Delta} - 1}{\alpha}u_k$$

with the feedback

$$u_k = k(x_k).$$

Notice that, if $k(\cdot)$ has M levels, the data rate R involved by this quantized feedback varies with Δ as

$$R = \frac{\log M}{\Delta}.$$

In [42], the authors propose the design of a stabilizing quantized feedback $k(\cdot)$ which is robust with respect to variations of the sampling period Δ. More precisely, a quantized feedback map $k(\cdot)$ is said to be *regular* if, by defining

$$C(\Delta) := \frac{1}{\sup\limits_{x_0 \in [-1,1]} \overline{\lim\limits_{k\to\infty}} |x_k|}$$

we have that

$$\lim_{\Delta \to 0} C(\Delta) = \infty.$$

This means that the contraction has to go to infinity when the data rate goes to infinity.

Theorem 7: [42] A quantized feedback $k(\cdot)$ is regular if and only if

$$\begin{aligned}&\alpha x < k(x) \leq 1 && \text{if } x \in]0, 1] \\ &-1 < k(x) \leq -\alpha x && \text{if } x \in [-1, 0].\end{aligned}$$

In this case we have that $C(\Delta) \geq (\alpha/(e^{\alpha\Delta} - 1))$.

Proof: The proof can be found in [42]. We give here only the sufficiency part. Consider the Lyapunov function

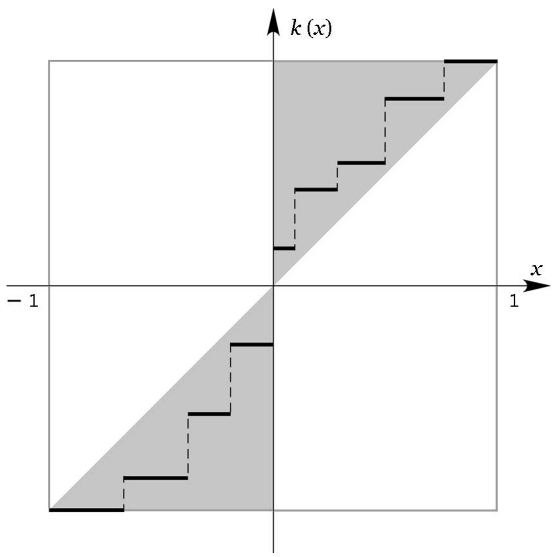

Fig. 8. *Graph of a regular quantized feedback map for the case* $\alpha = 1$.

$V(x) := |x|$. First observe that, if both $x > 0$ and $x^+ := e^{\alpha\Delta}x - ((e^{\alpha\Delta - 1})/\alpha)k(x) > 0$, then

$$\Delta V(x) := V(x^+) - V(x) = x^+ - x$$
$$= (e^{\alpha\Delta} - 1)\left(x - \frac{k(x)}{\alpha}\right) < 0.$$

Similarly we can prove that, if both $x < 0$ and $x^+ < 0$, then $\Delta V(x) < 0$. If instead $x > (e^{\alpha\Delta} - 1)/\alpha$ and $x^+ < 0$, then

$$\begin{aligned}\Delta V(x) &:= -x^+ - x \\ &= -e^{\alpha\Delta}x + \frac{e^{\alpha\Delta} - 1}{\alpha}k(x) - x < \frac{e^{\alpha\Delta} - 1}{\alpha}k(x) - x \\ &\leq \frac{e^{\alpha\Delta} - 1}{\alpha} - x < 0.\end{aligned}$$

Similarly we can prove that, if $x < -(e^{\alpha\Delta} - 1)$ and $x^+ > 0$, then $\Delta V(x) < 0$. In this way we have shown that, if $e^{\alpha\Delta} - 1 < |x| \leq 1$, then $\Delta V(x) < 0$. This implies that for any initial state $x_0 \in [-1, 1]$ we have that x_t converges to the interval $[-(e^{\alpha\Delta} - 1)/\alpha, (e^{\alpha\Delta} - 1)/\alpha]$ proving in this way the assertion. ■

The graph of a regular quantized feedback map is illustrated in Fig. 8. In [42] the optimal regular quantized feedback is determined for any fixed number of levels M.

VI. CONCLUSION

In this paper, we presented an overview of the problem of controlling a linear time-invariant system under a finite data rate constraint on the communication channel between the plant and controller. A classical result in this setting is the existence of a minimum rate (connected to the instability of the plant) below which stability cannot be achieved. To this result, we have added a lower bound on the long term state which captures the deterioration in performance as data rate is reduced and channel delay increased. We have then described the key results that pertain to coding and control schemes for stochastic linear systems, i.e., certainty equivalence, quasi-separation, and the stochastic instability of finite-state coding schemes.

If we impose a finite memory structure on the coder-controller in the noiseless case, further restrictions come into the picture. It turns out that asymptotic stability can not be achieved at all, and has to be replaced with some sort of practical stability. The asymptotic and the transient behavior of the closed-loop system thus become conflicting performance indices to optimize and they are both inherently coupled with the complexity parameters of the scheme (data rate and memory).

The material presented here describes only a limited part of a much broader theory which has been under development in recent years. Many indeed are the research directions that are active at this moment. Below we list a number of those directions that, we believe, will play a prominent role in the next few years.

- The digital communication channel we have considered in this paper is noiseless. Noisy channels pose a number of different problems where the interplay between information and control becomes even stronger. Initial results have been established in [53], [54], [71], [75], [78], and [82], but many questions are still open.
- Optimal control problems in the context of data rate constraints are, in general, hard to formulate and solve except in special cases [10], [52], [63], [72], [79], [82]. Some possible formulations are related to optimal quantization and rate distortion theory, areas that are difficult and generally lack analytically expressible solutions. To find simple but meaningful control cost functionals in this context (possibly different from the classical quadratic one), that can be minimized with a reasonable effort, is certainly an important open issue.
- Centralized feedback control is by no means the only interesting problem one would like to solve in the context of limited data rates. The major applications of networked control deal with a large number of subsystems interacting among each other through a networked communication protocol having a number of complexity constraints (graph connectivity, global data rate, data loss, etc.) and having a global control target. These problems are known as coordinated control

problems and are receiving increasing attention by many researchers in these years. At the moment, most of the effort has been put into designing decentralized control strategies that take into consideration the connectivity constraint of the communication network, while the information theoretic constraints expressed by data rate limitations have received comparatively little attention [56], [61], [76]. Necessarily, a meaningful theory will have to consider both aspects in an integrated way. ■

APPENDIX
PREFIX LANGUAGES

Let $\mathcal{S}$ be a finite set and let $\mathcal{S}^*$ be the language (of possibly infinite words) generated by the alphabet $\mathcal{S}$. Consider a subset $\mathcal{L} \subseteq \mathcal{S}^*$ constituted of infinite words. This is called a prefix language if any word in $\mathcal{L}$ can not be obtained as the concatenation of a word in $\mathcal{L}$ and a word in $\mathcal{S}^*$.

Let $\pi_t : \mathcal{S}^* \to \mathcal{S}^{t+1}$ the usual function which is the identity of words of length $\leq t$ and truncating to length $t+1$ the words of length $\geq t+1$. Consider a subset $\Sigma \subseteq \mathcal{S}^*$ and a function

$$F : \mathcal{S}^* \to \{0, 1\}.$$

Define the stopping time $T : \Sigma \to \mathbb{N}$ as follows:

$$T(w) := \min\{t | F \circ \pi_t(w) = 1\}$$

and the language

$$\mathcal{L} := \left\{\pi_{T(w)}(w) | w \in \Sigma\right\}.$$

Lemma 4: $\mathcal{L}$ is a prefix language.

Proof: Let $w \in \mathcal{L}$ and assume that $\pi_t(w) \in \mathcal{L}$ for some $t \leq \text{length}(w)$. We have to show that $t = \text{length}(w)$. Observe that there exists $\bar{w}, \bar{w}' \in \Sigma$ such that $w = \pi_{T(\bar{w})}(\bar{w})$ and $\pi_t(w) = \pi_{T(\bar{w}')}(\bar{w}')$. This implies that $t = T(\bar{w}')$. Moreover, $\pi_t(\bar{w}) = \pi_t(\bar{w}')$. This fact and the previous one imply that $F \circ \pi_t(\bar{w}) = 1$ and so $T(\bar{w}) \leq t$. ■

REFERENCES

[1] R. Adler, A. Konheim, and M. McAndrew, "Topological entropy," *Trans. Amer. Math. Soc.*, vol. 114, pp. 61–85, 1965.

[2] P. Antsaklis and J. Baillieul, Eds., *IEEE Trans. Autom. Control (Special Issue on Networked Control Systems)*, vol. 49, no. 9, Sep. 2004.

[3] Y. Anzai, "A note on reachability of discrete-time quantized control systems," *IEEE Trans. Autom. Control*, vol. AC-19, no. 5, pp. 575–577, Oct. 1974.

[4] J. Baillieul, "Feedback designs for controlling device arrays with communication channel bandwidth constraints," in *ARO Workshop on Smart Structures*, Pennsylvania State Univ., Aug. 1999.

[5] ——, "Feedback designs in information-based control," in *Stochastic Theory and Control Proceedings of a Workshop Held in Lawrence, Kansas*, B. Pasik-Duncan, Ed. New York: Springer-Verlag, 2001, pp. 35–57.

[6] ——, "Data-rate requirements for nonlinear feedback control," in *Proc. 6th IFAC Symp. Nonlinear Control Syst.*, Stuttgart, Germany, 2004, pp. 1277–1282.

[7] T. Berger, *Rate Distortion Theory: A Mathematical Basis for Data Compression*. Englewood Cliffs, NJ: Prentice-Hall, 1971.

[8] D. P. Bertsekas, *Dynamic Programming and Optimal Control*. Belmont, MA: Athena, 2000.

[9] A. Bicchi, A. Marigo, and B. Piccoli, "On the reachability of quantized control systems," *IEEE Trans. Autom. Control*, vol. 47, no. 4, pp. 546–563, Apr. 2002.

[10] V. S. Borkar and S. K. Mitter, "LQG control with communication constraints," in *Communications, Computation, Control and Signal Processing*. Norwell, MA: Kluwer, 1997, pp. 365–373.

[11] J. Braslavsky, R. Middleton, and J. Freudenberg, "Effects of time delay on feedback stabilization over signal-to-noise ratio constrained channels," in *Proc. 16th IFAC World Congr.*, Praha, Czech Republic, Jul. 2005.

[12] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Autom. Control*, vol. 45, no. 7, pp. 1279–1289, Jul. 2000.

[13] L. G. Bushnell, Ed., *IEEE Control Syst. Mag. (Special Section on Networks and Control)*, vol. 21, no. 1, 2001.

[14] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.

[15] R. E. Curry, "A separation theorem for nonlinear measurements," *IEEE Trans. Autom. Control*, vol. 14, no. 5, pp. 561–564, 1969.

[16] ——, *Estimation and Control With Quantized Measurements*. Cambridge, MA: MIT Press, 1970.

[17] S. Dasgupta, "Control over bandlimited communication channels: Limitations to stabilizability," in *Proc. 42nd IEEE Conf. Decis. Control*, 2003, pp. 176–181.

[18] C. de Persis and A. Isidori, "Stabilizability by state feedback implies stabilizability by encoded state feedback," *Syst. Control Lett.*, vol. 53, pp. 249–258, 2004.

[19] C. de Persis, "n-Bit stabilization of n-dimensional nonlinear systems in feedforward form," *IEEE Trans. Autom. Control*, vol. 50, no. 3, pp. 299–311, Mar. 2005.

[20] D. F. Delchamps, "Stabilizing a linear system with quantized state feedback," *IEEE Trans. Autom. Control*, vol. 35, no. 8, pp. 916–924, Aug. 1990.

[21] J. C. Delvenne, "An optimal quantized feedback strategy for scalar linear systems," *IEEE Trans. Autom. Control*, vol. 51, no. 2, pp. 298–303, Feb. 2006.

[22] N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information," *IEEE Trans. Autom. Control*, vol. 46, no. 9, pp. 1384–1400, Sep. 2001.

[23] F. Fagnani, "Chaotic quantized feedback stabilizers: The scalar case," *Commun. Inf. Syst.*, vol. 4, no. 1, pp. 53–72, 2004.

[24] F. Fagnani and S. Zampieri, "Stability analysis and synthesis for scalar linear systems with a quantized feedback," *IEEE Trans. Autom. Control*, vol. 48, no. 9, pp. 1569–1584, Sep. 2003.

[25] ——, "Quantized stabilization of linear systems: Complexity versus performances," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1534–1548, Sep. 2004.

[26] ——, "A symbolic dynamics approach to performance analysis of quantized feedback systems: The scalar case," *SIAM J. Control Optim.*, vol. 44, no. 3, pp. 816–866, 2005.

[27] T. R. Fischer, "Optimal quantized control," *IEEE Trans. Autom. Control*, vol. AC-27, no. 4, pp. 996–998, Apr. 1982.

[28] ——, "Quantized control with data compression constraints," *Optim. Control Appl. Methods*, vol. 5, no. 1, pp. 39–55, 1984.

[29] J. Freudenberg, J. Braslavsky, and R. Middleton, "Control over signal-to-noise ratio constrained channels: Stabilization and performance," in *Proc. 44th IEEE Conf. Decis. Control*, 2005, pp. 191–196.

[30] M. Fu and L. Xie, "The sector bound approach to quantized feedback control," *IEEE Trans. Autom. Control*, vol. 50, no. 11, pp. 1698–1711, Nov. 2005.

[31] A. Gersho and R. M. Gray, *Vector Quantization and Signal Compression*. Norwell, MA: Kluwer, 1993.

[32] G. C. Goodwin, H. Haimovich, D. E. Quevedo, and J. S. Welsh, "A moving horizon approach to networked control system design,"

IEEE Trans. Autom. Control, vol. 49, no. 9, pp. 1427–1445, Sep. 2004.

[33] T. Hayakawa, H. Ishii, and K. Tsumura, "Adaptive quantized control for nonlinear uncertain systems," in *Proc. Amer. Control Conf.*, Minneapolis, MN, Jun. 2006, pp. 2706–2711.

[34] J. Hespanha, A. Ortega, and L. Vasudevan, "Towards the control of linear systems with minimum bit-rate," in *Proc. 15th Int. Symp. Math. Theo. Netw. Sys.*, Univ. Notre Dame, Aug. 2002.

[35] H. Ishii and B. A. Francis, *Limited Data Rate in Control Systems With Networks*. New York: Springer-Verlag, 2002, vol. 275, *Lecture Notes in Control and Information Sciences*.

[36] ——, "Quadratic stabilization of sampled-data systems with quantization," *Automatica*, vol. 39, no. 10, pp. 1793–1800, 2003.

[37] H. Ishii, T. Basar, and R. Tempo, "Randomized algorithms for quadratic stability of quantized sampled-data systems," *Automatica*, vol. 40, pp. 839–846, 2004.

[38] H. Ishii and T. Başar, "Remote control of LTI systems over networks with state quantization," *Syst. Control Lett.*, vol. 54, no. 1, pp. 15–31, 2005.

[39] J. C. Kieffer and J. G. Dunham, "On a type of stochastic stability for a class of encoding schemes," *IEEE Trans. Inf. Theory.*, vol. IT-29, no. 6, pp. 703–717, Nov. 1983.

[40] R. E. Larson, "Optimum quantization in dynamic systems," *IEEE Trans. Autom. Control*, vol. AC-12, no. 2, pp. 162–168, Feb. 1967.

[41] A. Lasota and M. C. Mackey, *Chaos, Fractals, and Noise*. New York: Springer-Verlag, 1994.

[42] K. Li and J. Baillieul, "Robust quantization for digital finite communication bandwidth (DFCB) control," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1573–1584, Sep. 2004.

[43] ——, "Problems in decentralized sensor-actuator networks," in *Proc. 44th IEEE Conf. Decis. Control*, 2005, pp. 3207–3212.

[44] D. Liberzon, "On stabilization of linear systems with limited information," *IEEE Trans. Autom. Control*, vol. 48, no. 2, pp. 304–307, Feb. 2003.

[45] D. Liberzon and J. P. Hespanha, "Stabilization of nonlinear systems with limited information feedback," *IEEE Trans. Autom. Control*, vol. 50, no. 6, pp. 910–915, Jun. 2005.

[46] D. Liberzon and D. Nesic, "Input-to-state stabilization of linear systems with quantized feedback," in *Proc. 44th IEEE Conf. Decis. Control*, 2005, pp. 8197–8202.

[47] Q. Ling and M. D. Lemmon, "Stability of quantized control systems under dynamic bit assignment," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 734–740, May 2005.

[48] J. Liu and N. Elia, "Quantized feedback stabilization of nonlinear affine systems," *Int. J. Control*, vol. 77, no. 3, pp. 239–249, 2004.

[49] R. S. Marleau and J. E. Negro, "Comments on 'optimum quantization in dynamic systems'," *IEEE Trans. Autom. Control*, vol. AC-17, no. 2, pp. 273–274, Feb. 1972.

[50] N. C. Martins and M. A. Dahleh, "Fundamental limitations of performance in the presence of finite capacity feedback," in *Proc. Amer. Control Conf.*, 2005, pp. 79–86.

[51] N. C. Martins, M. A. Dahleh, and N. Elia, "Feedback stabilization of uncertain systems in the presence of a direct link," *IEEE Trans. Autom. Control*, vol. 51, no. 3, pp. 438–447, Mar. 2006.

[52] A. S. Matveev and A. V. Savkin, "The problem of LQG optimal control via a limited capacity communication channel," *Sys. Control Lett.*, vol. 53, no. 1, pp. 51–64, 2004.

[53] ——, "An analogue of Shannon information theory for networked control systems. Stabilization via a noisy discrete channel," in *Proc. 43rd IEEE Conf. Decis. Control*, 2004, pp. 4491–4496.

[54] ——, "Comments on 'control over noisy channels' and relevant negative results," *IEEE Trans. Autom. Control*, vol. 50, no. 12, pp. 2105–2110, Dec. 2005.

[55] ——, "Multirate stabilization of linear multiple sensor systems via limited capacity communication channels," *SIAM J. Control Optim.*, vol. 44, no. 2, pp. 584–617, 2005.

[56] ——, "Decentralized stabilization of linear systems via limited capacity communication networks," in *Proc. 44th IEEE Conf. Decis. Control*, 2005, pp. 1155–1161.

[57] G. N. Nair, S. Dey, and R. J. Evans, "Infimum data rates for stabilising Markov jump linear systems," in *Proc. 42nd IEEE Conf. Dec. Control*, 2003, pp. 1176–1181.

[58] G. N. Nair and R. J. Evans, "Stabilization with data-rate-limited feedback: Tightest attainable bounds," *Sys. Control Lett.*, vol. 41, no. 1, pp. 49–56, Sep. 2000.

[59] ——, "Exponential stabilizability of finite-dimensional linear systems with limited data rates," *Automatica*, vol. 39, no. 4, pp. 585–593, Apr. 2003.

[60] ——, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM J. Control Optim.*, vol. 43, no. 2, pp. 413–436, Jul. 2004.

[61] G. N. Nair, R. J. Evans, and P. E. Caines, "Stabilizing decentralised linear systems under data rate constraints," in *Proc. 43rd IEEE Conf. Decis. Control*, 2004, pp. 3992–3997.

[62] G. N. Nair, R. J. Evans, I. M. Y. Mareels, and W. Moran, "Topological feedback entropy and nonlinear stabilization," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1585–1597, Sep. 2004.

[63] G. N. Nair, M. Huang, and R. J. Evans, "Optimal infinite horizon control under a low data rate," in *Proc. 14th IFAC Symp. Sys. Ident.*, Newcastle, Australia, Mar. 2006.

[64] D. Nesic and A. R. Teel, "Input-output stability properties of networked control systems," *IEEE Trans. Autom. Control*, vol. 49, no. 10, pp. 1650–1667, Oct. 2004.

[65] I. R. Petersen and A. V. Savkin, "Multi-rate stabilization of multivariable discrete-time linear systems via a limited capacity communication channel," in *Proc. 40th IEEE Conf. Dec. Control*, 2001, pp. 304–309.

[66] V. Phat, J. Jiang, A. V. Savkin, and I. R. Petersen, "Robust stabilization of linear uncertain discrete-time systems via a limited capacity communication channel," *Sys. Control Lett.*, vol. 53, no. 5, pp. 347–360, 2004.

[67] B. Picasso, F. Gouaisbaut, and A. Bicchi, "Construction of invariant and attractive sets for quantized-input linear systems," in *Proc. 41st IEEE Conf. Dec. Control*, 2002, pp. 824–829.

[68] M. Pollicot and M. Yuri, *Dynamical Systems and Ergodic Theory*. Cambridge, U.K.: Cambridge Univ. Press, 1998, vol. 40, London Math. Soc. Student Texts, ch. 3.

[69] J. G. Proakis and M. Salehi, *Communication Systems Engineering*. Englewood Cliffs, NJ: Prentice-Hall, 2002.

[70] A. Rojas, J. H. Braslavsky, and R. H. Middleton, "Control over a bandwidth limited signal-to-noise ratio constrained communication channel," in *Proc. 44th IEEE Conf. Decis. Control*, 2005, pp. 197–202.

[71] A. Sahai, "The necessity and sufficiency of anytime capacity for control over a noisy communication link," in *Proc. 43rd IEEE Conf. Dec. Control*, 2004, pp. 1896–1901.

[72] A. V. Savkin, "Analysis and synthesis of networked control systems: Topological entropy, observability, robustness and optimal control," *Automatica*, vol. 42, pp. 51–62, 2006.

[73] C. E. Shannon, "A mathematical theory of communication," *Bell Syst. Tech. J.*, 1948, Reprinted in 'Claude Elwood Shannon Collected Papers', IEEE Press, 1993.

[74] ——, "Coding theorems for a discrete source with a fidelity criterion," *Inst. Radio Eng. Int. Conv. Rec.*, 1959, Reprinted in 'Claude Elwood Shannon Collected Papers', IEEE Press, 1993.

[75] T. Simsek, R. Jain, and P. Varaiya, "Scalar estimation and control with noisy binary observations," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1598–1603, 2004.

[76] S. Tatikonda, "Some scaling properties of large distributed control systems," in *Proc. 42nd IEEE Conf. Decis. Control*, 2003, pp. 3142–3147.

[77] S. Tatikonda and S. K. Mitter, "Control under communication constraints," *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 1056–1068, Jul. 2004.

[78] ——, "Control over noisy channels," *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 1196–1201, Jul. 2004.

[79] S. Tatikonda, A. Sahai, and S. K. Mitter, "Stochastic linear control over a communication channel," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1549–1561, Sep. 2004.

[80] K. Tsumura and J. Maciejowski, "Stability of SISO control systems under constraints of channel capacities," in *Proc. 42nd IEEE Conf. Decis. Control*, 2003, pp. 193–198.

[81] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints II: Stabilization with limited information feedback," *IEEE Trans. Autom. Control*, vol. 44, no. 5, pp. 1049–1053, May 1999.

[82] S. Yuksel, O. C. Imer, and T. Basar, "Constrained state estimation and control over communication networks," in *Proc. 38th Annu. Conf. Inf. Sci. Syst.*, Princeton Univ., Princeton, NJ, 2004.

[83] G. Z. Zhang, G. N. Nair, R. J. Evans, and B. Wittenmark, "A data-rate limited view of adaptive control," in *Proc. 14th IFAC Symp. Sys. Ident.*, Newcastle, Australia, Mar. 2006.

ABOUT THE AUTHORS

Girish N. Nair (Member, IEEE) was born in Petaling Jaya, Malaysia. He received the B.E. (elec., first class honors) degree in 1994, the B.Sc. (math.) degree in 1995, and the Ph.D. (elec. eng.) degree in 2000, on scholarships from the Australian government and the University of Melbourne, Melbourne, Australia.

He is currently a Senior Lecturer in the Department of Electrical and Electronic Engineering at the University of Melbourne, and has also held visiting positions at the University of Padova and Boston University. His research interests lie in the intersection of communications, information theory, and control. He serves as an Associate Editor for *SIAM Journal on Control and Optimization*.

Dr. Nair has received several prizes, including the SIAM Outstanding Paper Prize 2006, the Best Theory Paper Prize at the UKACC Int. Conf. Control, Cambridge, 2000 (both with Rob Evans), and the L. R. East Medal in electrical engineering, 1994.

Fabio Fagnani received the Laurea degree in mathematics from the University of Pisa, Pisa, Italy, and Scuola Normale Superiore, Pisa, in 1986, and the Ph.D. degree in mathematics from the University of Groningen, Groningen, The Netherlands, in 1991.

From 1991 to 1998, he was an Assistant Professor at the Scuola Normale Superiore. Since 1998, he has been with the Politecnico di Torino where he is currently Full Professor of Mathematical Analysis. His research activities are on the fundamental mathematical aspects of systems and control theory, and of coding theory, on which he is an author of about 30 papers in international journals. Specific themes of his current interest are control under communication constraints, coordinated control, and their connection with graph theory and symbolic dynamics; inverse problems, and recursive deconvolution techniques; codes over groups, and their use in high-performance schemes.

Sandro Zampieri received the Laurea degree in electrical engineering and the Ph.D. degree in system engineering from the University of Padova, Padova, Italy, in 1988 and 1993, respectively.

In 1991–92, 1993, and 1996, he was Visiting Scholar at the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA. He has also held visiting positions at Delft Technical University, The Netherlands, at the Institut fuer Mathematik der Universitaet Innsbruck, Austria, and the Department of Mathematics at the University of Groningen, The Netherlands. Since 1992, he has been with the Department of Information Engineering, University of Padova, where he is presently Full Professor in Automatic Control. He served as an Associate Editor of the *Siam Journal on Control and Optimization* during 2002–2004, and is presently chair of the IFAC technical committee on "Networked Systems." His research interests include automatic control, dynamical systems theory, networked control, and the relations between control and communications.

Robin J. Evans (Fellow, IEEE) was born in Melbourne, Australia, in 1947. He received the B.E. (electrical) degree from the University of Melbourne, in 1969 and the Ph.D. degree from the University of Newcastle, Newcastle, Australia, in 1975. He completed postdoctoral studies at the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA, and the Control and Management Department, Cambridge University, Cambridge, U.K.

After completing the B.E. degree, he spent five years as an engineering officer with the Royal Australian Air Force, working on radar systems. In 1977 he took up a position at the University of Newcastle, where he was Head of the Department of Electrical and Computer Engineering from 1986 to 1991, and Co-director of an ARC Special Research Center on Industrial Control Systems. In 1992 he moved to the University of Melbourne, where he headed the Department of Electrical and Electronic Engineering until 1996, and also led the Cooperative Research Center for Sensor Signal and Information Processing and the Center for Networked Decision Systems. He is currently Director of the Victoria Research Laboratory of National ICT Australia. His research has ranged across many areas, including control theory, radar systems, signal processing, and computer systems.

Dr. Evans is a Fellow of the Australian Academy of Science, and the Australian Academy of Technological Sciences and Engineering.